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A NEW QUANTITY IN RIEMANN-FINSLER GEOMETRY*

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Abstract. In this note, we study a new Finslerian quantity $\hat{C}$ defined by the Riemannian curvature. We prove that the new Finslerian quantity is a non-Riemannian quantity for a Finsler manifold with dimension $n = 3$. Then we study Finsler metrics of scalar curvature. We find that the $\hat{C}$-curvature is closely related to the flag curvature and the H-curvature. We show that $\hat{C}$-curvature gives, a measure of the failure of a Finsler metric to be of weakly isotropic flag curvature. We also give a simple proof of the Najafi-Shen-Tayebi' theorem.

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1. Introduction. In Finsler geometry, there are several important geometric quantities: the flag curvature, the (mean) Cartan torsion and the (mean) Berwald curvature, etc. (cf. [**6, 10**]). In [**1**], H. Arbar-Zadeh considered a Finslerian quantity H, which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. Arbar-Zadeh proved that for a Finsler metric of scalar flag curvature with dimension ≥ 3, the flag curvature is constant on the manifold if and only if $H = 0$.

Recently, a great progress has been made in studying Finsler metrics of weakly isotropic flag curvature. These Finsler metrics are of scalar curvature whose flag curvature is in a special form $K = \theta/F + \sigma$ where θ is a 1-form and σ is a scalar function on M. Finsler metrics of weakly isotropic flag curvature not only include Finsler metrics of constant flag curvature, but also include Finsler metrics of (almost) isotropic S-curvature and of scalar flag curvature [**4, 6, 13**]. Cheng and Shen have

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classified Finsler metrics of Randers type with weakly isotropic flag curvature via the navigation problem in Riemannian manifolds [**5**, **12**]. Chen-Zhao constructed explicitly Finsler metrics are of scalar curvature such that they are not of weakly isotropic flag curvature [**3**]. A natural task for us is to give a geometric quantity on a Finsler manifold, which characterises the Finsler metrics of weakly isotropic flag curvature.

In this paper, we find the desired quantity (see Section 2 below) and call it the $\hat{C}$-curvature. We show that the $\hat{C}$-curvature gives a measure of the failure of an n-dimensional Finsler metric of scalar curvature to be of weakly isotropic flag curvature if $n \geq 3$. Precisely we prove the following:

THEOREM 1.1. *Let (M, F) be an $n(\geq 3)$-dimensional Finsler manifold of scalar curvature with flag curvature $K(x, y)$. Then K is weakly isotropic if and only if the $\hat{C}$-curvature vanishes.*

Recall that a Finsler metric F is said to be of *scalar curvature* if the flag curvature $K = K(x, y)$ is a scalar function on the slit tangent bundle $TM\backslash\{0\}$. For these Finsler metrics, we establish an equation between the flag curvature K, the $\hat{C}$-curvature $\hat{C}$ and the H-curvature H (see Proposition 3.2 below), therefore the $\hat{C}$-curvature is subtly related to the flag curvature and the H-curvature.

Recently, Najafi-Shen-Tayebi extended Arbar-Zadeh's characterisation for Finsler metrics of constant flag curvature and proved the following [**8**, **11**, **13**]:

THEOREM 1.2. *Let F be a Finsler metric of scalar flag curvature on an $n(\geq 3)$-dimensional manifold M. For a 1-form θ, the flag curvature is weakly isotropic given by (2.10) if and only if the H-curvature satisfies the following:*

$$H_{ij} = \frac{n+1}{6}\theta F_{y^i y^j}, \tag{1.1}$$

where $\theta = \theta_i(x)y^i$ is a 1-form on M.

See Section 2 for the definition of the H-curvature. Say a Finsler metric F has *almost vanishing* H-curvature if its H-curvature is given by (1.1) [**11**, **13**]. By using Theorem 1.1 and Proposition 3.2, we obtain a new and simple proof of Theorem 1.2 (see Section 5).

Riemannian metrics are a special case of Finsler metrics, namely Finsler metrics with the quadratic restriction. Call a geometric quantity on a Finsler manifold *non-Riemannian* if it vanishes for a Riemannian metric. For instance, the (mean) Cartan torsion, the S-curvature and the H-curvature are all non-Riemannian [**6**, **7**]. In Section 6, we show the following:

THEOREM 1.3. *For a Finsler manifold with dimension $n = 3$, the $\hat{C}$-curvature is a non-Riemannian quantity.*

2. Preliminaries. Let (M, F) be a Finsler manifold of dimension $n \geq 3$. In a standard local coordinate system (x^i, y^i) in TM, $F = F(x, y)$ is a function of (x^i, y^i). Let

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$$

and $(g^{ij}) := (g_{ij})^{-1}$. Let $R^i{}_j$ denote the Riemannian curvature of F [7]. The $\hat{C}$-curvature is defined by $\hat{C} = \hat{C}_{ij} dx^i \otimes dx^j$ where

$$\hat{C}_{ij} = R_{ij} + \frac{1}{n-2}\left(y_i Ric_j + y_j Ric_i - g_{ij} Ric - Ric_{ij} F^2\right) + \frac{SF^2}{(n-1)(n-2)} h_{ij}, \quad (2.1)$$

where $R_{ij} := g_{ik} R^k{}_j$ is the flag curvature tensor of F, $h_{ij} := F F_{y^i y^j}$ is the angular metric of F [10] and

$$Ric := R^j{}_j, \quad (2.2)$$

$$Ric_i := \frac{1}{2} Ric_{y^i}, \quad (2.3)$$

$$Ric_{ij} = \frac{1}{2} Ric_{y^i y^j}, \quad (2.4)$$

$$y_i := g_{ij} y^j, \quad (2.5)$$

$$S := g^{ij} Ric_{ij}. \quad (2.6)$$

F is of scalar flag curvature with flag curvature K is equivalent to the following equation (see [6, page 110]):

$$R^i{}_j = K F^2 h^i{}_j, \quad (2.7)$$

where

$$h^i{}_j = \delta^i{}_j - F^{-2} g_{jk} y^k y^i = g^{ik} h_{kj}. \quad (2.8)$$

It is easy to see that (2.7) holds if and only if

$$R_{ij} = K F^2 h_{ij}. \quad (2.9)$$

In Sections 4 and 5, we will consider Finsler metrics of weakly isotropic flag curvature defined as follows:

$$K = \frac{\theta}{F} + \sigma, \quad (2.10)$$

where $\sigma = \sigma(x)$ is a scalar function and $\theta = \lambda_i(x) y^i$ is a 1-form.

The *H-curvature* $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$ is defined by

$$H_{ij} = E_{ij|k} y^k$$

where "$|$" denotes the covariant horizontal derivatives and E_{ij} denote the mean Berwald curvature of F [8, 13]. The H-curvature vanishes for a R-quadratic Finsler metric [7, 9].

3. Finsler metrics of scalar flag curvature. Assume that F is of scalar curvature, that is, the flag curvature $K = K(x, y)$ is a scalar function on $TM \setminus \{0\}$. Using (2.2) and (2.7), we obtain the Ricci scalar Ric is given by

$$Ric = (n-1) K F^2, \quad (3.1)$$

where we have used $F^{-2}g_{ij}y^iy^j = 1$. We use the following notations:

$$K_i = K_{y^i}, \quad K_{ij} = K_{y^iy^j}.$$

By (2.3) and (3.1) we get

$$Ric_j = \frac{n-1}{2}[KF^2]_{y^j} = \frac{n-1}{2}(K_jF^2 + 2Ky_j), \tag{3.2}$$

where we have used the fact

$$y_j = FF_{y^j} = \left(\frac{F^2}{2}\right)_{y^j}. \tag{3.3}$$

Together with (2.4) we obtain

$$Ric_{ij} = \frac{n-1}{2}(K_{ij}F^2 + 2K_iy_j + 2K_jy_i + 2Kg_{ij}), \tag{3.4}$$

where we have used $(y_j)_{y^i} = \left(\frac{F^2}{2}\right)_{y^jy^i} = g_{ij}$. Note that the flag curvature K is homogeneous of degree zero with respect to y. It follows that

$$K_jy^j = 0. \tag{3.5}$$

Using (2.6), (3.4) and (3.5), we obtain the scalar curvature S is determined by

$$S = \frac{n-1}{2}(g^{ij}K_{ij}F^2 + 4K_jy^j + 2nK) = n(n-1)K + \frac{(n-1)^2}{2}F^2\Psi,$$

where $\Psi := \frac{1}{n-1}g^{ij}K_{ij}$. Together with (2.1), (2.9), (3.2) and (3.4), we get

$$\begin{aligned}\hat{C}_{ij} = {} & KF^2h_{ij} + \tfrac{n-1}{2(n-2)}(4Ky_iy_j - K_iy_jF^2 - K_jy_iF^2 - 4KF^2g_{ij} - K_{ij}F^4)\\ & + \tfrac{n-1}{2(n-2)}\Psi F^4h_{ij} + \tfrac{nK}{n-2}F^2h_{ij}.\end{aligned}$$

Note that $y_iy_j - F^2g_{ij} = -F^2h_{ij}$. It follows that

$$\begin{aligned}\hat{C}_{ij} &= KF^2h_{ij} - \tfrac{n-1}{2(n-2)}F^2(4Kh_{ij} + K_iy_j + K_jy_i + K_{ij}F^2)\\ &\quad + \tfrac{n-1}{2(n-2)}\Psi F^4h_{ij} + \tfrac{nK}{n-2}F^2h_{ij}\\ &= \tfrac{n-1}{2(n-2)}\left(\Psi F^2h_{ij} - K_iy_j - K_jy_i - K_{ij}F^2\right)F^2.\end{aligned} \tag{3.6}$$

Hence, we have the following:

LEMMA 3.1. *Let (M, F) be an $n(\geq 3)$-dimensional Finsler manifold of scalar curvature with flag curvature $K(x, y)$. Then F has vanishing $\hat{C}$-curvature if and only if in any standard local coordinate system*

$$\Psi F^2h_{ij} = K_iy_j + K_jy_i + K_{ij}F^2, \tag{3.7}$$

where $\Psi := \frac{1}{n-1}g^{ij}K_{ij}$. In particular, F has vanishing $\hat{C}$-curvature if F has isotropic (or constant) flag curvature.

A Finsler metric F is said to be of *isotropic curvature* if the flag curvature $K(P, y) = K(x)$ is a scalar function on M. In particular, F is said to have *constant (flag) curvature* if the flag curvature $\mathbf{K}(P, y) =$ constant. Now we are going to establish an important equation between the flag curvature, the $\hat{C}$-curvature and the H-curvature.

PROPOSITION 3.2. *Let F be a Finsler metric of scalar flag curvature on an n-manifold M. Then the flag curvature, $\hat{C}$-curvature and the H-curvature satisfy*

$$\hat{C}_{ij} = \frac{n-1}{2(n-2)}\left(\Psi F^3 F_{y^i y^j} + \frac{6H_{ij}}{n+1}\right)F^2, \tag{3.8}$$

where $\Psi := \frac{1}{n-1} g^{ij} K_{ij}$.

Proof. A direct calculation yields (cf [**7**, (3.26)])

$$0 = 6H_{ij} + (n+1)F[(FK)_{y^i y^j} - KF_{y^i y^j}] = 6H_{ij} + (n+1)(K_i y_j + K_j y_i + K_{ij} F^2). \tag{3.9}$$

It follows that

$$-(K_i y_j + K_j y_i + K_{ij} F^2) = \frac{6H_{ij}}{n+1}.$$

Plugging this into (3.6) yields (3.8). □

4. Proof of theorem 1.1. First suppose that F has weakly isotropic flag curvature, i.e. (2.10) holds. Differentiating (2.10) with respect to y^i, we obtain

$$K_i = \frac{\lambda_i}{F} + \frac{\theta l_i}{F^2}, \tag{4.1}$$

where $l_i := F_{y^i} = F^{-1} y_i$. Moreover

$$K_{ij} = -\frac{\lambda_i l_j}{F^2} - \frac{\lambda_j l_i + \theta F_{y^i y^j}}{F^2} + \frac{2\theta l_i l_j}{F^3} = \frac{\theta(3 l_i l_j - g_{ij}) - (\lambda_i l_j + \lambda_j l_i)}{F^3}, \tag{4.2}$$

where we have used the fact

$$g_{ij} = h_{ij} + l_i l_j. \tag{4.3}$$

It follows that

$$\Psi = -\frac{\theta}{F^3}. \tag{4.4}$$

From (4.1), (4.2) and (4.3) we get

$$K_i y_j + K_j y_i + K_{ij} F^2 = \frac{\theta(3 l_i l_j - g_{ij}) - 2\theta l_i l_j}{F} = -\frac{\theta}{F} h_{ij}. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\Psi F^2 h_{ij} = -\frac{\theta}{F} h_{ij} = K_i y_j + K_j y_i + K_{ij} F^2.$$

By Lemma 3.1, F has vanishing $\hat{C}$-curvature.

Conversely, suppose that $\hat{C} = 0$. Then (3.7) holds. Differentiating (3.7) with respect to y^k, we obtain

$$\Psi_{y^k} F^2 h_{ij} + 2\Psi h_{ij} y_k + \Psi F^2 (h_{ij})_{y^k} = g_{ik} K_j + y_i K_{jk} + g_{jk} K_i + y_j K_{ik} + 2y_k K_{ij} + F^2 K_{ijk}, \tag{4.6}$$

where

$$K_{ijk} := (K_{ij})_{y^k} = K_{y^i y^j y^k} \tag{4.7}$$

is totally symmetric. Direct computations yield

$$g^{ij} h_{ij} = n - 1, \tag{4.8}$$

$$K_{ij} y^i = -K_j, \tag{4.9}$$

$$(h_{ij})_{y^k} = 2C_{ijk} - F^{-2} y_j h_{ik} - F^{-2} y_i h_{jk}, \tag{4.10}$$

where C_{ijk} is the Cartan torsion [**6**]. Contracting (4.10) with g^{ij} yields

$$g^{ij} (h_{ij})_{y^k} = 2I_k \tag{4.11}$$

where I_k is the mean Cartan torsion [**6**] and we have used the fact

$$h_{ij} y^i = 0. \tag{4.12}$$

Contracting (4.6) with g^{ij} gives, by (4.8), (4.9) and (4.11),

$$(n-1)\Psi_{y^k} + 2\Psi I_k = g^{ij} K_{ijk}. \tag{4.13}$$

Since Ψ is homogeneous of degree -2 with respect to y, we have $\Psi_{y^k} y^k = -2\Psi$. Together with (2.8) yields

$$\Psi_{y^k} h^k{}_j = \Psi_{y^j} + 2F^{-2} y_j \Psi. \tag{4.14}$$

Contracting (4.10) with g^{ik} yields

$$g^{ik} (h_{ij})_{y^k} = 2I_j - (n-1) F^{-2} y_j, \tag{4.15}$$

where we have made use of (4.8) and (4.12). Contracting (4.6) with g^{ik} gives, by (3.5), (4.9) and (4.12),

$$\Psi_{y^k} F^2 h^k{}_j + \Psi F^2 g^{ik} (h_{ij})_{y^k} = nK_j + (n-1) y_j \Psi - 2K_j + F^2 g^{ik} K_{ijk}. \tag{4.16}$$

Plugging (4.14) and (4.15) into (4.16) yields

$$F^2(\Psi_{y^j} + 2F^{-2} y_j \Psi) + \Psi F^2 [2I_j - (n-1) F^{-2} y_j] = (n-2) K_j + (n-1) y_j \Psi + F^2 g^{ik} K_{ijk}.$$

Taking this together with (4.13) yields

$$(n-2)(F^2 \Psi_{y^j} + K_j + 2y_j \Psi) = 0. \tag{4.17}$$

It follows from (4.17) that

$$(F^2\Psi + K)_{y^j} = F^2\Psi_{y^j} + 2y_j\Psi + K_{y^j} = 0.$$

Thus

$$\sigma := F^2\Psi + K \tag{4.18}$$

is a scalar function on M. Plugging (4.18) into (3.7) yields

$$[\sigma(x) - K]\, h_{ij} = K_i y_j + K_j y_i + K_{ij}F^2. \tag{4.19}$$

By using (3.3), (4.19) and the definition of the angular metric we get

$$F_{y^i}K_{y^j} + F_{y^j}K_{y^i} + FK_{y^iy^j} + (K-\sigma)F_{y^iy^j} = 0.$$

This implies that

$$\begin{aligned}[(K-\sigma)F]_{y^iy^j} &= [K_{y^i}F + (K-\sigma)F_{y^i}]_{y^j} \\ &= K_{y^iy^j}F + K_{y^i}F_{y^j} + K_{y^j}F_{y^i} + (K-\sigma)F_{y^iy^j} = 0.\end{aligned} \tag{4.20}$$

Note that $(K-\sigma)F$ is homogeneous of degree one with respect to y. Together with (4.20) we obtain $(K-\sigma)F$ is a 1-form

$$(K-\sigma)F = \lambda_i(x)y^i = \theta.$$

We get that $K = \theta/F + \sigma$.

5. An alternative proof of the Najafi-Shen-Tayebi' theorem. In this section, we are going to give a new proof of Theorem 1.2 (see Section 1) using Theorem 1.1 and the important identity (3.8).

5.1. Proof of Theorem 1.2. First suppose that H is almost vanishing given by (1.1). Plugging (1.1) into (3.9) yields

$$\theta F_{y^iy^j} + K_i y_j + K_j y_i + K_{ij}F^2 = 0. \tag{5.1}$$

Contracting (5.1) with g^{ij} gives, by (3.5) and (4.8), $(n-1)\left(\frac{\theta}{F} + F^2\Psi\right) = 0$. It follows that

$$\theta = -F^3\Psi. \tag{5.2}$$

Plugging this into (1.1) yields $H_{ij} = -\frac{n+1}{6}\Psi F^3 F_{y^iy^j}$. Substituting this into (3.8) gives $\hat{C} = 0$. By Theorem 1.1,

$$K = \frac{\tilde{\theta}}{F} + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function and $\tilde{\theta} = a_i(x)y^i$ is a 1-form on M. By (4.4) we arrive at the following identity

$$\Psi = -\frac{\tilde{\theta}}{F^3}.$$

Plugging this into (5.2) yields $\tilde{\theta} = \theta$. We conclude that the flag curvature is weakly isotropic given by (2.10).

Conversely, suppose that $K = \frac{\theta}{F} + \sigma$ where $\sigma = \sigma(x)$ is a scalar function and $\theta = \lambda_i(x)y^i$ is a 1-form on M. From (4.4) we deduce that

$$\Psi = -\frac{\theta}{F^3}. \tag{5.3}$$

By Theorem 1.1, F has vanishing $\hat{C}$-curvature. Together with (3.8) we obtain

$$H_{ij} = -\frac{n+1}{6}\Psi F^3 F_{y^i y^j}. \tag{5.4}$$

Plugging (5.3) into (5.4) yields (1.1). □

6. Three-dimensional Finsler manifold. In Finsler geometry, there are several important non-Riemannian quantities: the mean Cartan torsion **I**, the Cartan torsion **C** and the H-curvature **H**, etc [**6**, **7**]. They all vanish for Riemannian metrics, hence they said to be non-Riemannian. In this section, we are going to show the following:

THEOREM 6.1. *For a Finsler manifold with dimension $n = 3$, the $\hat{C}$-curvature is a non-Riemannian quantity.*

Proof. Assume that (M, F) is a Riemannian manifold. Then then the flag curvature tensor R_{ij} is given by

$$R_{ij} = R_{kijl}(x)y^k y^l \tag{6.1}$$

where $R_{kijl}(x)$ is the Riemannian curvature of F. It follows that, from (2.2) and (2.4),

$$Ric = R_{ij}(x)y^i y^j \tag{6.2}$$

and

$$Ric_{ij} = R_{ij}(x), \tag{6.3}$$

where $R_{ij}(x)$ is the Ricci tensor of F. By using (2.6) and (6.1) we have

$$S = R, \tag{6.4}$$

where R is the scalar curvature of Riemannian metric F. Plugging (6.1)–(6.4) into (2.1) yields

$$\begin{aligned}\hat{C}_{ij} &= R_{kijl}(x)y^k y^l + \tfrac{R}{(n-1)(n-2)}(g_{ij}g_{kl}y^k y^l - g_{il}g_{kj}y^k y^l)\\ &\quad - \tfrac{1}{n-2}(R_{ij}F^2 - R_{il}g_{kj}y^k y^l - R_{jl}g_{ki}y^k y^l + R_{kl}g_{ij}y^k y^l) = C_{kijl}y^k y^l,\end{aligned}$$

where C_{kijl} is the Weyl conformal curvature tensor [**2**]. Now our conclusion is an immediate consequence of $C_{kijl} \equiv 0$ for a 3-dimensional Riemannian manifold [**2**, Proposition 3.3.9]. □

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