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NON-DESTRUCTIVE TESTING OF OPTICAL-FIBRE PREFORMS

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Abstract

When optical fibres are made by first constructing optical-fibre preforms, the fibre which is pulled from the heated preform is simply a scaled down version of the original preform structure. The expansion coefficient profile $\alpha(r)$ of the preform, which relates directly to the fabrication variables, can be determined from non-destructive optical retardation measurements $\delta(r)$ performed on the preform. In addition, the residual elastic stress distributions in a fabricated preform, which can be used to compare different fabrication procedures, have simple definitions as linear functionals of the expansion coefficients $\alpha(r)$. Thus, through the use of optical retardation data, an examination of different manufacturing procedures for preform fabrication is reduced to a problem in non-destructive testing and analysis. The underlying numerical problem of evaluating the stress distributions reduces to solving an Abel-type integral equation for $\alpha(r)$, which involves an indeterminacy, followed by the evaluation of linear functionals defined on $\alpha(r)$. It is shown how the known inversion formulae for the Abel-type integral equation can be used formally to reduce the numerical problem of evaluating the radial stress to the evaluation of a linear functional defined on the data $\delta(r)$ which bypasses the indeterminacy. When only the radial stress is required, the problem of actually solving the Abel-type integral equation is avoided. Methods for evaluating the non-radial stresses, which avoid the indeterminacy, are also derived.

1. Introduction

For the monitoring and control of optical fibre fabrication, it is necessary to relate the properties of the fibres to specific characteristics in the manufacturing process. This can be done using non-destructive testing and analysis when the production of fibres is based on the construction of optical fibre preforms which are then heated and pulled to form the fibres.

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In this manufacturing process, successive layers of doped silica are deposited within a rotating silica tube to form the cladding, that is, the inner layers. After the construction of a suitably large number of layers, the process is terminated with a doped silica essentially different from that used for the cladding since it eventually forms the core. The tube is then collapsed to form the preform which thereby consists of axially symmetric cladding layers around a central core. Presby *et al.* [9] have found that the final properties of the fibre can be related to the properties of the preform, except possibly for minor effects of the pulling on the core.

As a result, the properties of the fibres can be monitored either in terms of the refractive index profile of the preform (see Chu and Whitebread [5], Marcuse [7] and Sasaki *et al.* [10]), or its thermal expansion coefficient profile $\alpha(r)$ (see Shibata *et al.* [11]). The use of the latter has the added advantage that the stress distributions, which are important in the manufacture of single-mode, low-birefringence fibres (see Norman *et al.* [8]), have simple definitions in terms of $\alpha(r)$.

Since the $\alpha(r)$ profile is related to the intrinsic stress in the preform, laser light passing through the preform normal to the axial direction experiences a certain retardation $\delta(r)$ and this can be measured as a function of the radial distance r from the centre of the preform. This retardation $\delta(r)$ can be expressed in terms of $\alpha(r)$ by the following Abel-type integral equation (see Shibata *et al.* [11, Section 2.3]):

$$f(x) = \int_x^R r\alpha(r)/(r^2 - x^2)^{1/2}\,dr \qquad \text{for } 0 \leqslant x \leqslant R, \tag{1}$$

with

$$f(x) = (\delta(x)/D) + 2M_1(R^2 - x^2)^{1/2}/R^2 \quad \text{and} \quad D = 4\pi CE\Delta T/(\lambda(1-\sigma)), \tag{2}$$

where R denotes the outer radius of the preform, C, E and σ the photoelastic constant, Young's modulus and Poisson's ratio, respectively, of the medium, λ the wave length of the laser light, ΔT the effective temperature difference between room temperature and the softening temperature of doped fused silica used in the cooling process of preform fabrication, and M_1 the first moment of the expansion coefficient $\alpha(r)$, namely

$$M_1 = \int_0^R r\alpha(r)\,dr. \tag{3}$$

What makes this Abel integral equation formalism non-standard is the presence of the term M_1 in the definition of the data $f(x)$, since, formally, $\alpha(r)$ is needed to determine M_1 and M_1 is needed to determine $\alpha(r)$. To-date, rather crude methods, based on assuming that $\alpha(r)$ has a piecewise constant structure, have been used to cope with this indeterminacy (see Shibata *et al.* [11, Section 2.3]).

The required intrinsic stress components are then defined as linear functionals on the expansion coefficient $\alpha(r)$ (see Shibata *et al.* [11, Section 2.1]):

$$\sigma_r(r) = (E\,\Delta T/(1-\sigma))\left\{(1/R^2)\int_0^R \tau\alpha(\tau)\,d\tau - (1/r^2)\int_0^r \tau\alpha(\tau)\,d\tau\right\}, \tag{4}$$

$$\sigma_\theta(r) = (E\,\Delta T/(1-\sigma))\left\{(1/R^2)\int_0^R \tau\alpha(\tau)\,d\tau + (1/r^2)\int_0^r \tau\alpha(\tau)\,d\tau - \alpha(r)\right\}, \tag{5}$$

and

$$\sigma_z(r) = (E\,\Delta T/(1-\sigma))\left\{(2/R^2)\int_0^R \tau\alpha(\tau)\,d\tau - \alpha(r)\right\}, \tag{6}$$

where σ_r, σ_θ and σ_z denote the stress components in the cylindrical coordinate directions r, θ and z, respectively.

The aim of this paper is to show how the linear functional (4) for the radial stress distribution $\sigma_r(r)$ can be redefined as a linear functional on the retardation data $\delta(r)$ which bypasses the indeterminacy discussed above. Thus, when only the radial stress $\sigma_r(r)$ is required, the necessity of solving (1) and (2) numerically for $\alpha(r)$ is avoided. The idea of replacing linear functionals defined on the solution of some problem by equivalent functionals defined on the data is not new, and has been exploited in a number of contexts (see, for instance, Anderssen [2], Goldberg [6] and Anderssen [3]). Methods for evaluating the stresses $\sigma_\theta(r)$ and $\sigma_z(r)$, which avoid the indeterminacy, are also derived. Methods for estimating M_1 are discussed and compared.

2. Preliminaries

For the Abel equation (1), the inversion formula which will be used is

$$\alpha(r) = -\frac{2}{\pi}\frac{1}{r}\frac{d}{dr}\left\{\int_r^R \frac{xf(x)}{(x^2-r^2)^{1/2}}\,dx\right\}. \tag{7}$$

As an immediate consequence, it follows that

$$M_1 = \int_0^R r\alpha(r)\,dr = (2/\pi)\int_0^R f(x)\,dx, \tag{8}$$

which illustrates how, in the present context, functionals defined on the unknown $\alpha(r)$ can be identified as functionals on the data $f(x)$. The indeterminacy associated with M_1 and $\alpha(r)$, however, has not been removed; it has only been transfered to an indeterminacy between M_1 and $f(x)$ (see the definition of $f(x)$ in (2)).

3. The radial stress

To replace the evaluation of the radial stress $\sigma_r(r)$, defined in (4), by a functional defined on the data $f(x)$, it is necessary only to substitute the right-hand side of (7) for $\alpha(r)$ in (4) to obtain

$$\sigma_r(r) = (E\,\Delta T/(1-\sigma))\Big\{(M_1/R^2) + 2\Big[\int_r^R xf(x)/(x^2 - r^2)^{1/2}\,dx - \pi M_1/2\Big]/(\pi r^2)\Big\}. \quad (9)$$

At first sight, it might appear that the indeterminacy could be removed by simply substituting the definition of $f(x)$ given by (2) in the right-hand side of (8), but this yields the independent condition

$$\int_0^R \delta(x)\,dx = 0, \quad (10)$$

which the retardation observations of $\delta(x)$ are known to satisfy. It is a consequence of the fact that the total axial stress in the preform must be zero, namely

$$\int_0^{2\pi}\int_0^R \sigma_z(r)\,r\,dr\,d\theta = 0, \quad (11)$$

which follows almost automatically from the definition of $\sigma_z(r)$ given by (6). The definition of $f(x)$, however, can be used to simplify (9). In fact, if it is substituted for $f(x)$ on the right-hand side of (9), it follows after minor manipulations that

$$\sigma_r(r) = (\lambda/(2Cr^2\pi^2))\int_r^R x\delta(x)/(x^2 - r^2)^{1/2}\,dx. \quad (12)$$

Although (because of its dependence on r) this linear functional is indefinite, its dependence on M_1 has been removed. As a result, the *ad hoc* strategies developed by previous authors (see Shibata *et al.* [11, Section 2.3]) to cope with the indeterminacy are not, in fact, needed.

4. The non-radial stresses

Because they involve $\alpha(r)$ explicitly, it would at first sight appear necessary to solve (1) and (2) numerically (see Anderssen [1]), with respect to given observational data for the optical retardation $\delta(x)$ and an estimate of M_1, before the stress components $\sigma_\theta(r)$ and $\sigma_z(r)$ could be evaluated. The need to do this, however, is circumvented the moment it is observed that (1) and (2) can be rewritten as

$$K\delta(x)/D = -\int_x^R r\sigma_z(r)/(r^2 - x^2)^{1/2}\,dr, \qquad K = (E\Delta T)/(1-\sigma)$$

since it follows, on using the inversion formula (7), that

$$\sigma_z(r) = \frac{2K}{\pi D}\frac{1}{r}\frac{d}{dr}\left\{\int_r^R x\delta(x)/(x^2 - r^2)^{1/2}\,dx\right\}.$$

This result, when combined with (12), yields

$$\sigma_z(r) = (1/r)d\big(r^2\sigma_r(r)\big)/dr. \tag{13}$$

A further simple calculation then yields

$$\sigma_\theta(r) = \sigma_z(r) - \sigma_r(r) = d(r\sigma_r(r))/dr. \tag{14}$$

As well as showing how to avoid the indeterminacy in the calculation of $\sigma_z(r)$ and $\sigma_\theta(r)$, these two results also verify that the basic information about the axial structure of the preform is contained in $\sigma_r(r)$.

5. The indeterminacy

The indeterminacy reflects the fact that the formulation (1) and (2) determines $\alpha(r)$ only up to an arbitrary constant. In fact, if $\delta(x) \equiv 0$, then (1) and (2) combine to yield

$$\int_x^R r\alpha(r)/(r^2 - x^2)^{1/2}\,dr = 2M_1(R^2 - x^2)^{1/2}/R^2.$$

It is then a simple matter to show that, if $\alpha(r)$ is a solution, then so is $\alpha(r) + K$. It therefore follows that M_1 cannot be determined from (1), (2), (3), (4), (5) and (6) without the introduction of some additional assumption.

If an estimate of M_1 is required, one approach is to use the definition of $\sigma_r(r)$ in (4). On assuming a value for $\alpha(0)$ (without loss of generality we take $\alpha(0) = 0$), it follows from (4) that $\sigma_r(r)$ itself can be used to estimate M_1 as

$$\hat{M}_1 = \frac{R^2(1-\sigma)}{E\,\Delta T}\sigma_r(0) = \frac{\lambda R^2(1-\sigma)}{2E\,\Delta T\,C\pi^2}\lim_{r\to 0}\left\{\frac{1}{r^2}\int_r^R \frac{x\delta(x)}{(x^2 - r^2)^{1/2}}\,dx\right\}. \tag{15}$$

It is clear that the numerical estimation of

$$\lim_{r\to 0}\left\{\frac{1}{r^2}\int_r^R \frac{x\delta(x)}{(x^2 - r^2)^{1/2}}\,dx\right\}$$

from observational data will have to be accomplished with some care. Because of the singular integrands involved, the advantages of product integration should be exploited (see Anderssen and Jakeman [4] and Anderssen [3]). In passing, it is noted that (10) represents a necessary condition for the existence of this limit.

An alternative approach to the estimation of M_1 has been proposed and used in the applied optics literature. It is assumed that $\alpha(r)$ is constant and known in the substrate region adjacent to the surface of the preform, that is,

$$\alpha(r) = \begin{cases} \alpha(r) & \text{for } 0 \leqslant r \leqslant a \text{ (core and cladding)}, \\ \bar{\alpha} & \text{for } a \leqslant r \leqslant R \text{ (substrate)}, \end{cases}$$

where $\bar{\alpha}$ is const. It follows from (1) and (2) that, for $a \leqslant x \leqslant R$,

$$\delta(x) = D(\bar{\alpha} - (2M_1/R^2))(R^2 - x^2)^{1/2};$$

that is, M_1 can be estimated as

$$\tilde{M}_1 = (R^2/2)\left[\bar{\alpha} - \delta(x)/\left(D(R^2 - x^2)^{1/2}\right)\right] \qquad \text{for } a \leqslant x \leqslant R,$$

or, more generally, as

$$\tilde{M}_1 = \lim_{x \to R} (R^2/2)\left[\alpha(x) - \delta(x)/\left(D(R^2 - x^2)^{1/2}\right)\right]. \tag{16}$$

Clearly, this procedure is limited by the necessary and sufficient condition for the existence of the limit (16), namely, $\delta(x) = K_0(R^2 - x^2)^{1/2} + o((R^2 - x^2)^{1/2})$, where K_0 = constant. The conditions on $\hat{M}_1$ are less restrictive. This will be shown in Section 6.

6. Exemplification

Consider the synthetic datum

$$\delta(r) = A(1 - r^2)^\alpha + B(1 - r^2)^\beta \quad \text{for } 0 < r < 1 \text{ and } \alpha < \beta. \tag{17}$$

It automatically satisfies the condition $\delta(1) = 0$. The constants A and B are determined by condition (10) and an arbitrary end condition $\delta(0) = -1$. Using the fact that

$$\int_r^1 \frac{x(1 - x^2)^{\nu - 1}}{(x^2 - r^2)^{1/2}} dx = \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(\nu)}{\Gamma(\nu + \frac{1}{2})}(1 - r^2)^{\nu - 1/2}, \tag{18}$$

it follows that

$$A = \Gamma(\alpha + 3/2)\Gamma(\beta + 1)/C_0 \text{ and } B = -\Gamma(\beta + 3/2)\Gamma(\alpha + 1)/C_0, \tag{19}$$

where $C_0 = \Gamma(\beta + 3/2)\Gamma(\alpha + 1) - \Gamma(\alpha + 3/2)\Gamma(\beta + 1)$. Again using (18), it is not difficult to show that

$$\sigma_r(r) = (\lambda/(4C\pi^2 r^2))\bar{C}\left\{(1 - r^2)^{\alpha+1/2} - (1 - r^2)^{\beta+1/2}\right\}, \tag{20}$$

where

$$\bar{C} = \frac{\Gamma(\frac{1}{2})\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\beta + 3/2)\Gamma(\alpha + 1) - \Gamma(\alpha + 3/2)\Gamma(\beta + 1)}.$$

It then follows naturally from (15) that

$$\hat{M}_1 = \bar{C}(\beta - \alpha)/\pi D \tag{21}$$

and, from (2), that

$$f(x) = \Big[\Gamma(\alpha + 3/2)\Gamma(\beta + 1)(1 - x^2)^\alpha - \Gamma(\beta + 3/2)\Gamma(\alpha + 1)(1 - x^2)^\beta + \frac{2}{\pi}\Gamma(\tfrac{1}{2})\Gamma(\alpha + 1)\Gamma(\beta + 1)(\beta - \alpha)(1 - x^2)^{1/2}\Big]/[DC_0].$$

Further use of (18) now yields, for $-1/2 < \alpha < \beta$,

$$\alpha(r) = (2\bar{C}/(\pi D))\left\{(\alpha + \tfrac{1}{2})(1 - r^2)^{\alpha+1/2-1} - (\beta + \tfrac{1}{2})(1 - r^2)^{\beta+1/2-1}\right\} + 2\hat{M}_1. \tag{22}$$

Estimating M_1 from the constant substrate assumption, it follows from (16) that, for the synthetic datum (17),

$$\tilde{M}_1 = \lim_{r\to 1}\frac{1}{2}\left\{\alpha(r) - \left(\delta(r)/\left(D(1 - r^2)^{1/2}\right)\right)\right\}, \tag{23}$$

$$\lim_{r\to 1}\frac{\delta(r)}{(1 - r^2)^{1/2}} = \begin{cases} 0 & \text{for } 1/2 < \alpha < \beta, \\ \bar{K} & \text{for } 1/2 = \alpha < \beta, \end{cases} \tag{24}$$

and

$$\lim_{r\to 1}\alpha(r) = \begin{cases} 2\hat{M}_1 & \text{for } 1/2 < \alpha < \beta, \\ 2\hat{M}_1 + \bar{K}/D & \text{for } 1/2 = \alpha < \beta, \end{cases} \tag{25}$$

where

$$\bar{K} = \Gamma(\beta + 1)/\{\Gamma(\beta + 3/2)\Gamma(3/2) - \Gamma(\beta + 1)\}.$$

Combining (23), (24) and (25), we obtain

RESULT. *For the synthetic datum* (17), $\tilde{M}_1 = \hat{M}_1$, *if* $1/2 < \alpha < \beta$.

The above results show that $\hat{M}_1$ exists for $-1/2 < \alpha < \beta$, whereas $\tilde{M}_1$ only exists for $1/2 \leqslant \alpha < \beta$, because $\alpha(r)$ is singular at the surface $r = 1$, when $\alpha < 1/2$ and, as a consequence, $\lim_{r \to 1} \alpha(r)$ fails to exist. This tends to imply that $\tilde{M}_1$ will yield a poor estimate of M_1 when $\alpha(r)$ changes rapidly in the neighbourhood of the surface $r = 1$.

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