Points at Infinity, etc., in a plane.

By P. Pinkerton, M.A.

1. The object of this paper is to correlate the geometrical and analytical aspects of the elements of the theory of points at infinity, etc., in a plane. It is assumed that the reader is acquainted with the method of tracing the graphs of rational functions of x, by using the artifices of change of origin and approximation by Ascending or Descending Division (see Chrystal's *Introduction to Algebra*, Ch. XXV.).

2. When we write $\tan\frac{\pi}{2} = \infty$, geometrically we mean (1) that a right-angled triangle can be constructed, the ratio of whose sides is equal to n, where n is any pre-assigned positive number, however large, and (2) that the difference between the greater of the acute angles of the triangle and a right angle continually tends to zero as n increases. The full notation is $\underset{\theta=\frac{\pi}{2}}{\text{Lt.}} \tan\theta = \infty$. Algebraically, we have $\tan\frac{\pi}{2} = r \div 0$, where r is the length of the radius vector which, by revolving according to the general definition of the tangent of an angle, traces out the angle $\frac{\pi}{2}$ radians. But $r \div 0$ is not a particular or special case of the algebraic operations which obey the Laws of Algebra; for, if it were, we might reason thus:

$0 \times 8 = 0$ and $0 \times 9 = 0$, $\therefore\ 0 \times 8 = 0 \times 9$, $\therefore\ 0 \times 8 \div 0 = 0 \times 9 \div 0$, $\therefore\ 8 \times (0 \div 0) = 9 \times (0 \div 0)$, $\therefore\ 8 = 9$. Thus $\tan\frac{\pi}{2}$ has no algebraic value, *i.e.*, is not equal to a number which obeys the Laws of Algebra. We call $r \div 0$ a *limiting case* of an algebraic operation, and write, *in this case*, $r \div 0 = \infty$, the full notation being again $r \div 0 = \underset{x=0}{\text{L}}\, r \div x = \infty$. Again a tangent to a circle is a *limiting case* of a secant of a circle; it is not a secant, for it lacks some of the properties of a secant. Correspondingly in analysis; if $x_1 y_1$ and $x_2 y_2$

https://doi.org/10.1017/S0013091500033502 Published online by Cambridge University Press

are the coordinates of the points in which a secant cuts the circle $x^2+y^2=a^2$, then the secant has for equation

$$(y-y_1)/(y_1-y_2)=(x-x_1)/(x_1-x_2).$$

When x_2y_2 coincides with x_1y_1, we get the equation

$$(y-y_1)\div 0=(x-x_1)\div 0,$$

involving limiting cases of algebraic operation; and we have to "evaluate" the equation

$$\underset{y_2=y_1}{\mathrm{L}}\ \frac{y-y_1}{y_1-y_2}=\underset{x_2=x_1}{\mathrm{L}}\ \frac{x-x_1}{x_1-x_2},$$

under the conditions $x_1^2+y_1^2=x_2^2+y_2^2=a^2$, to find the equation to the tangent at x_1y_1. The introduction of limiting cases into Geometry finds its equivalent in the introduction of limiting cases of algebraic operation into Analysis.

3. Let A, B be two fixed points on a given straight line, and let t be a variable number, finite both ways, and positive or negative. Through A and B draw AX, BY parallel straight lines such that $AX/BY=t$, account being taken of the directions of AX, BY as well as of their magnitudes (AX, BY, in fact, being *steps*). The line XY meets the line AB unless $t=+1$. Let P be the variable point of intersection; then, clearly, $AP/BP=AX/BY=t$, AP, BP being *steps*. Also if it be supposed possible to find another point Q such that $AQ/BQ=t$, we shall have $AP/BP=AQ/BQ$, $\therefore$ $(AP-BP)/BP=(AQ-BQ)/BQ$, $\therefore$ $AB/BP=AB/BQ$, $\therefore$ $BP=BQ$; so that Q is the same point as P, BP and BQ being steps. Hence *one and only one* point P can be found such that $AP/BP=t$, provided $t\neq+1$. The nearer t is to $+1$ the further is P from A (or B). To make the possible values of t complete, we introduce *as a convention the limiting case* of a point on the line, corresponding to $t=+1$. Since in the case of actual points on the line, there is one and only one point for each value of t, so we make the convention that there is only one point on the line whose "position-ratio" is equal to $+1$. This point is called *the point at infinity on the line.* It is to be observed that the point at infinity on a straight line is not a point in sober fact on the line, any more than a tangent to a circle is a secant. Indeed we have not defined the point at infinity on a line to have position, but only a "position-ratio," the value of the "position-ratio," viz., $+1$, corresponding to *no* position on the line.

Analytically, let B be taken as an origin of coordinates on the line AB, and let the coordinates of A and P be a and x respectively. Then if $AP/BP = t$, we have, to determine x, the equation $\frac{x-a}{x} = t$, so that $x = \frac{a}{1-t}$, if $t \neq 1$. If $t = 1$, there is *no value* of x; we have the anomalous equation $0 \, . \, x = a$. But making use of limiting cases of algebraic operation, we write $x = a \div 0 = \infty$. If, then, points at infinity are in question, we cannot omit from an equation such a term as mx when $m = 0$, but only when actual (finite) points are in question.

4. We now proceed to endow points at infinity with geometrical and analytical properties, taking care that the process involves us in no contradictions. Geometrically, we may suppose the point at infinity on the line AB joined to a finite point in the plane not having its position on AB. This line cannot meet the line AB again; for we must escape the contradiction that two straight lines have more than one point in common. The straight line must therefore be considered as parallel to AB. A system of such straight lines will be a system of parallel straight lines; that is, parallel straight lines have their respective points at infinity in common, or meet at infinity, as is the usual expression. This result also flows from the geometrical construction of § 3. Analytically: let one of a system of parallel straight lines be chosen as the x-axis of a Cartesian system of reference. Then the equation of any other may be written in the form $y = 0 \, . \, x + c$; the term $0 \, . \, x$ not being omitted, as points at infinity are in question (see § 3). To find the point of intersection of $y = 0 \, . \, x + c$, and the x-axis, we solve the equations $y = 0 \, . \, x$ and $y = 0 \, . \, x + c$, whence arises the anomalous equation $0 \, . \, x + c = 0$, showing (§ 3) that the point of intersection is the point at infinity on the x-axis. Therefore, a system of parallel straight lines "meet at infinity."

5. Associated with any system of parallel straight lines, therefore associated with any direction in a plane, is a point at infinity. Hence associated with a plane is an assemblage of points at infinity. This assemblage of points is met by any straight line in the plane in one and only one point, and therefore is to be regarded as forming a

straight line, called the *line at infinity* in the plane. Referred to a Cartesian system of reference in the plane, the equation to the line at infinity must be written $0 . x + 0 . y + c = 0$. For an equation of the first degree is in question; this equation is satisfied by the coordinates of no actual point in the plane, and the point-pair common to $0.x + 0.y + c = 0$ and $Ax + By + C = 0$ is $(\infty, \infty), (\infty, 0)$ or $(0, \infty)$, according as $A \neq 0$ and $B \neq 0$, or $A = 0$ and $B \neq 0$, or $A \neq 0$ and $B = 0$.

6. The necessity of preserving terms like $0 . x$ in equations, or of understanding that they are present if not actually written, where points at infinity or infinite values of x are in question, may be illustrated geometrically and analytically by considering the coaxal system of circles determined by the equations

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$\text{and} \quad S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

The equation, $S + kS' = 0$, represents the system of coaxal circles except when $k = -1$, when $S + kS' = 0$ takes the form

$$0 . x^2 + 0 . y^2 + 2(g - g')x + 2(f - f')y + c - c' = 0,$$

which may be written

$$(0 . x + 0 . y + 1)\{2(g - g')x + 2(f - f')y + c - c'\} = 0,$$

representing the line at infinity as well as the line given by

$$2(g - g')x + 2(f - f')y + c - c' = 0.$$

This *limiting case* of the system of circles, $S + kS' = 0$, thus appears as *two* straight lines, one lying wholly at infinity, the other being (as usually defined) the radical axis of the coaxal system defined by the circles $S = 0$, $S' = 0$. Now if a secant through a point P cuts a circle in A and B, the power of the point P with respect to the circle is geometrically defined as PA . PB. When the point P lies on the circle PA . PB = 0. But if the radical axis with the line at infinity be included, as a limiting case, in the system of coaxal circles defined by $S = 0$, $S' = 0$, and if P be taken on the radical axis PA . PB assumes the form $0 \times \infty$—an indeterminate form, as is geometrically obvious. For since P lies on the radical axis PA . PB in the limiting case remains equal to the square on the tangent from P to any circle of the system, and this is indeterminate, depending on the position of P.

7. The usefulness of the conventions, point at infinity and line at infinity, in connection with Menelaus' and Ceva's Theorems, Harmonic Ranges and Pencils, etc., is well known. The task of making what is found in text-books clear and intelligible may be left to the reader. The rest of this paper will be occupied with remarks on points at infinity on plane curves.

8. It has already been seen that, when an equation of the first degree takes the anomalous form $0 \, . \, x + c = 0$, it is to be interpreted as having its root infinite. If the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \,..\, + a_rx^{n-r} + \ldots + a_{n-1}x + a_n = 0, \quad - \quad (1)$$

be transformed by putting $x = 1/y$, we obtain the equation

$$a_ny^n + a_{n-1}y^{n-1} + \ldots + a_ry^r + \ldots + a_2y^2 + a_1y + a_0 = 0. \quad - \quad (2)$$

Consider the coefficients a_0, a_1, etc., as varying according to some law; one root of (2) tends to zero as a_0 tends to zero, provided $a_1 \neq 0$; two roots of (2) tend to zero if a_0 and a_1 tend to zero while a_2 does not tend to zero; r roots tend to zero as a_0, a_1, ..., a_{r-1} tend to zero while a_r does not tend to zero. But while y tends to zero, x tends to ∞, if we introduce limiting cases of operation, as we have seen we must do if points at infinity and infinite values of x are in question. Hence if $a_0 = 0$ and $a_1 \neq 0$, we make the convention that *one* root of (1) is infinite; if $a_0 = 0$ and $a_1 = 0$ while $a_2 \neq 0$, *two* roots of (1) are to be considered infinite; and so on.

9. These analytical conventions may be graphically illustrated. The case of the equation $0 \, . \, x + c = 0$ has its equivalent in the geometrical convention that two parallel straight lines have their points at infinity in common. The cases $0 \, . \, x^2 + px + q = 0 (p \neq 0)$, and $0 \, . \, x^2 + 0 \, . \, x + q = 0 (q \neq 0)$, may be illustrated from the graph of $y = (x - 1)/(x + 1)$, fig. 4. The quadratic equation

$$(mx + c)(x + 1) = x - 1 \quad - \quad - \quad - \quad - \quad (3)$$

determines the abscissae of the points of intersection of the graph and the line $y = mx + c$. There are in general two such points. When $m = 0$ and $c \neq 1$, equation (3) takes the form $0 \, . \, x^2 + c(x + 1) = x - 1$; there is one finite point of intersection, whose abscissa is determined by the equation $c(x + 1) = x - 1$, and one point of intersection at infinity. The branches of the graph that stretch to right and left tend to assume the form of a parallel to $y = c(c \neq 1)$, consistently

https://doi.org/10.1017/S0013091500033502 Published online by Cambridge University Press

with $\underset{x=\infty}{\mathrm{L}}(x-1)/(x+1)=1$. The abscissae of the points of intersection of $y=1$ and the curve are given by the equation $0 \,.\, x^2+0 \,.\, x+2=0$. The straight line $y=1$ never meets the curve at any actual points. But we are at liberty to say that it meets the curve at two coincident points—these points coinciding with the point at infinity on the line. We thus have a point at infinity on a curve of the second degree behaving like an actual point of simple concavity towards a straight line as tangent. We thus speak of a point of simple concavity at infinity and call the "tangent" at the point an *asymptote*.

10. Curves of higher degree than the second present similar properties. Consider the graph of $y=(-x^2+3x+2)/(x^2+x+1)$, fig. 5. The abscissae of the points of intersection of the straight line $y=mx+c$ with the curve are given by the equation

$$(mx+c)(x^2+x+1)+x^2-3x-2=0. \qquad - \qquad - \quad (4)$$

A cubic equation is in question. When the straight line $y=mx+c$ is parallel to the x-axis, $m=0$; and equation (4) takes the limiting form

$$0 \,.\, x^3+(c+1)x^2+(c-3)x+(c-2)=0. \qquad - \qquad - \quad (5)$$

If $c+1 \neq 0$, one root is infinite, and there are two actual points of intersection. For x large, the curve tends to assume the form $y=-1$, which explains the infinite root. When $c=-1$, equation (5) becomes

$$0 \,.\, x^3+0 \,.\, x^2+4x+3=0.$$

That is, the point at infinity on the curve in the direction of the x-axis, is a point of simple concavity at infinity, and the equation to the corresponding tangent or asymptote is $y=-1$.

11. In the case of a curve passing through the origin of coordinates, the appearance of the curve at the origin is obtained by making successive approximations of the form $y=ax$, $y=ax+bx^2$, etc., and thus the nature of a simple concavity, a point of inflexion, a cusp, a node, a conjugate point at any actual point is explained. When x is large, the corresponding successive approximations to the equation to the curve will be of the form

$$y=ax+b, \;\; y=ax+b+\frac{c}{x}, \;\; y=ax+b+\frac{c}{x^2}, \;\text{ etc.,}$$

and from such equations are investigated the appearance of a curve at infinity in any chosen direction.

https://doi.org/10.1017/S0013091500033502 Published online by Cambridge University Press

12. To investigate the appearance of a curve at infinity when there is a simple concavity at a point at infinity in the plane of the curve, consider the curve given by the equation $y = ax + b + \frac{c}{x}$ (fig. 6, c + ve). The curve has a simple concavity at the point at infinity on the line $y = ax + b$, and also a simple concavity at the point at infinity on the line $x = 0$. Hence corresponding to a simple concavity at infinity, the curve approaches its asymptote from *opposite sides* of the line at the two extremities.

13. The case of a *point of inflexion* at infinity arises in the curve given by the equation $y = ax + b + \frac{c}{x^2}$, (fig. 7, c + ve). The line $y = ax + b$ meets the curve at points whose abscissae are given by the equation $c = 0$, where a cubic is in question; that is, the line meets the curve at *three* coincident points at infinity. And any other straight line through the point of contact of curve and line, being a parallel straight line, has an equation of the form $y = ax + k(k \neq b)$ and therefore meets the curve in two finite points (the abscissae of which are determined from the equation $(k - b)x^2 = c$) and in one point at infinity. The point at infinity in the direction $y = ax + b$ has therefore precisely the characteristics of a point of inflection at an actual point on a curve of the third degree, say the origin on the curve $y = x^3$, and is a point of inflection at infinity. In such a case we see from the graph that the curve appears at *both ends* of the asymptote $y = ax + b$, but on the *same side of* the *line.*

14. The same curve has a *cusp at infinity* in the direction given by the equation $x = 0$. The line $x = 0$ meets the curve in *three* coincident points at infinity. Any straight line through the point of contact except $x = 0$ is given by the equation $x = k$, and the ordinates of its intersection with the curve are given by the equation $y = ak + b + \frac{1}{k^2}$, a linear equation when a cubic is in question. Hence *two* of the points of intersection are coincident with the point at infinity on $x = k$ or $x = 0$, and the third is the finite point $(k,\ ak + b + 1/k^2)$. Hence this point at infinity is a double point. It is a cusp since the tangents at the point are real and coincident. Any line through the point may be explained geometrically to meet

each of the two branches, the particular line $x=0$ touching one branch and meeting the other. From the graph we see that at a cusp at infinity, the curve appears only at one end of the asymptote, and on both sides at that end.

15. There is a *node at infinity* on the curve given by the equation $y=ax+b+c/(x^2-1)$, fig. 8, c + ve. As in § 13, there is a point of inflection at infinity on $y=ax+b$, and the curve appears on the same side of the line at both ends. But $x+1=0$ also meets the curve in *three* coincident points at infinity, and so does $x-1=0$. Any other line through this point at infinity meets the curve there in two coincident points (one on each branch), and in one finite point. The point is a double point, and $x \pm 1=0$ are real tangents at the point, which is therefore a node. The curve appears at both ends and on opposite sides of each asymptote.

16. There is a *conjugate point* at infinity on the curve whose equation is $y=ax+b+c/(x^2+1)$, fig. 9, c + ve. $x \pm i=0$ are the imaginary asymptotes at the point.

17. To illustrate the method of finding the asymptotes of a given curve, consider that given by the equation $xy(x+y)+x^2+y^2=0$. The lines $x=0$, $y=0$, $x+y=0$ meet the curve in points whose abscissae or ordinates are given by the equations $y^2=0$, $x^2=0$, $2x^2=0$ respectively. These are quadratic equations where cubic equations are in question; therefore each of these lines meets the curve in *one* point at infinity. Hence any straight line parallel to one of these meets the curve in a point at infinity, since parallel lines have a common point at infinity. To find asymptotes, it remains to select (if possible) the particular parallels which meet the curve in two or more points at infinity.

(1) Consider the points of intersection of $x=k$ $(k \neq 0)$ and $xy(x+y)+x^2+y^2=0$. The ordinates of these points are given by $y^2(1+k)+k^2y+k^2=0$. Choosing $1+k=0$, we see that the ordinates of the points of intersection of $x+1=0$, and the curve are given by the linear equation $y+1=0$, where a cubic is in question. Therefore $x+1=0$ meets the curve in two coincident points at infinity and at the finite point $(-1, -1)$. That is, $x+1=0$ is an asymptote and its point of contact is a point of simple concavity at

infinity, and the curve will appear at both ends of $x+1=0$, and on opposite sides. By symmetry, similar remarks hold regarding the asymptote $y+1=0$.

(2) The points of intersection of $x+y=k$ $(k \neq 0)$ and the curve have their abscissae given by the equation $x^2(2-k)+kx(k-2)+k^2=0$. Hence the line $x+y=2$ meets the curve in points whose abscissae are given by the anomalous equation $4=0$, where a cubic is in question. The point at infinity on the line $x+y=2$ is a point of inflexion at infinity on the curve, and the curve will appear on the same side of the asymptote $x+y=2$, at its two ends.

We should therefore (§§ 12, 13) expect the approximate forms of the equation $xy(x+y)+x^2+y^2=0$ at infinity in the directions $x=0$, $y=0$, $x+y=0$ to be

$$x=-1+c/y, \quad y=-1+c/x, \quad y=-x+2+c/x^2 \quad \text{respectively.}$$

The first approximation to the equation to the curve at infinity in the direction $x=0$ is obtained by writing $x=\underset{x=0}{\text{Lt}}\dfrac{x^2+y^2}{-y(x+y)}=-1$.

For the second approximation, write

$$x=\underset{x=-1}{\text{Lt}}\frac{x^2+y^2}{-y(x+y)}=-\frac{1+y^2}{y(y-1)}=-1-\frac{1}{y},$$

for y large, by Descending Division.

Again write $x+y=\underset{y=-x}{\text{Lt}}\dfrac{x^2+y^2}{-xy}=2$, which gives the first approximation at infinity in the direction $x+y=0$.

Next write $x+y=\underset{y=-x+2}{\text{Lt}}\dfrac{x^2+y^2}{-xy}=\dfrac{2x^2-4x+4}{x^2-2x}=2+\dfrac{4}{x^2}$,

for x large, by Descending Division.

Hence, in accordance with §§ 12, 13, the approximate forms of the equation at infinity are $x=-1-1/y$, $y=-1-1/x$, $x+y=2+4/x^2$; see fig. 10.

18. The parabola at infinity is of peculiar interest. Taking the equation in the form $y=ax^2$, we see that the point at infinity in the direction given by $x=0$ is a point on the curve. No straight line in this direction *touches* the curve and a straight line in any other

https://doi.org/10.1017/S0013091500033502 Published online by Cambridge University Press

direction meets the curve in two finite points. Is there, then, no line which meets and touches the curve at infinity, that is, is there no asymptote? Let us find the limiting form of the equation to the tangent at the point (x', y') when $x' = k$, $y' = \infty$. Change the origin to the point (x', y'); the equation becomes $\eta + y' = a(\xi + x')^2$, or $\eta = 2ax'\xi + a\xi^2$. The equation to the tangent at the new origin is therefore $\eta = 2ax'\xi$ or $y - y' = 2ax'(x - x')$, which may be written in the form $y(1/y') + 1 = 2a(x'/y') . x$. In the limiting case we get $0 . y + 1 = 0 . x$, which is the line at infinity. Hence the line at infinity touches every parabola.

19. The line at infinity may touch other curves, so that these curves will touch certain parabolas at their points at infinity. $y(x-2)^2 = x^3(x-1)$, fig. 11, is an example of a curve which has parabolic contact at infinity. Using Descending Division, we may write the equation in the form $y = x^2 + 3x + 8 + \mathrm{R}$, where $\mathrm{R} = (20x - 32)/(x^2 - 4x + 4)$. Now $\underset{x=\infty}{\mathrm{Lt}}\, \mathrm{R} \div 8 = 0$, therefore $y = x^2$ is a first approximation, $y = x^2 + 3x$ a second, and $y = x^2 + 3x + 8$ a third. All these approximate forms at infinity are parabolas with a point at infinity in the direction of the y-axis. The first meets the curve at one point at infinity, the second at two points at infinity, the third at three points at infinity; $y = x^2 + 3x + 8$ is a *parabolic asymptote.*