ON THE ISOMORPHISM OF INTEGRAL GROUP RINGS. II

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1. Introduction. Let $Z(G)$ denote the integral group ring of a group G. Let $\mathscr{C}$ be the class of groups G with the property that for any isomorphism $\theta: Z(G) \to Z(H)$, we have $\theta(g) = \pm h$, $h \in H$, for all $g \in G$. We study this class in § 2 and establish that it contains classes of torsion-free abelian groups, torsion abelian groups, and ordered groups.

In § 4, we prove the following result.

THEOREM. *Let G be a group which contains a normal abelian subgroup A such that $G/A \in \mathscr{C}$. Suppose that $\theta: Z(G) \to Z(H)$ is an isomorphism such that $\theta(\Delta(G, A)) = \Delta(H, B)$ for a suitable normal subgroup B of H. Then $G \simeq H$. (Here $\Delta(G, A)$ is the kernel of the natural map $Z(G) \to Z(G/A)$.)*

Jackson (**3**) and Whitcomb (**6**) proved the special case of this theorem when G is supposed to be finite metabelian. The lemmas needed are given in § 3. In § 5, we extend (to arbitrary finite groups) a result of Passman (**4**), proved by him for finite nilpotent groups. As a corollary, it follows that if two finite groups have isomorphic integral group rings, then they have isomorphic derived series.

In the last section we give an easy proof of a theorem of Banaschewski.

2. The class $\mathscr{C}$. The following lemma was proved in (**5**).

LEMMA 1. *Let G be an abelian group. Suppose that $\gamma \in Z(G)$ is such that $\gamma^n = 1$ for some natural number n. Then $\gamma = \pm t$, where t is a torsion element of G.*

LEMMA 2. *Let G be a torsion-free abelian group. Then $U_{Z(G)}$, the group of units of $Z(G)$, is given by*

$$U_{Z(G)} = \pm G = \{\pm g \mid g \in G\}.$$

Proof. We can suppose, without loss of generality, that G is finitely generated and hence free. Suppose that $G = \langle x_1, \ldots, x_n \rangle$. For $g = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, set $\deg(g) = \sum_1^n \alpha_i$. Let $\gamma, \mu \in Z(G)$ be such that $\gamma\mu = 1$. Let m and t be the maximum degrees of terms appearing in γ and μ, respectively. Due to the fact that $Z(G)$ is free of zero divisors, it follows that $\gamma\mu$ has a term of degree

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$m + t$. Therefore $m + t = 0$. Similarly, supposing s and u to be the minimum degrees of terms appearing in γ and μ, respectively, we conclude that $s + u = 0$. Thus $s \leqq m$ yields $-s \geqq -m$, i.e. $u \geqq t$, and hence $u = t$ and $m = s$. Therefore, all the terms appearing in γ have degree m and those in μ are of degree $-m$.

Let us use induction on n, the number of generators of G. If $n = 1$, then clearly $\gamma = \pm x_1^{\alpha_1}$. If $n > 1$, we use the same argument as above on the degree in x_1, namely $d_1(g) = \alpha_1$, and conclude that $\gamma = x^d\gamma_1$ and $\mu = x^{-d}\mu_1$, where $\gamma_1, \mu_1 \in Z(G_1)$, $G_1 = \langle x_2, \ldots, x_n\rangle$.

Now by induction, $\gamma_1 = \pm g_1$, $g_1 \in G$, and $\gamma = \pm g$, $g \in G$.

Remark. The last lemma follows from the next but we have included the direct proof because of its independent interest.

LEMMA 3. *Let G be an ordered group. Then*

$$U_{Z(G)} = \pm G = \{\pm g \mid g \in G\}.$$

Proof. Let $\gamma = \sum_1^s a_i g_i$, $\mu = \sum_1^t b_j h_j$ be such that $\gamma\mu = 1$. Suppose that

$$g_s = \max_{1 \leqq i \leqq s} \{g_i\}, \qquad g_1 = \min_{1 \leqq i \leqq s} \{g_i\},$$

$$h_t = \max_{1 \leqq i \leqq t} \{h_i\}, \qquad h_1 = \min_{1 \leqq i \leqq t} \{h_i\}.$$

Then $g_s h_t = 1 = g_1 h_1$. Since $h_1 \leqq h_t$, it follows that $h_1^{-1} \geqq h_t^{-1}$, i.e. $g_1 \geqq g_s$, and therefore $g_1 = g_s$. We conclude that $\gamma = \pm g$.

An immediate consequence of these lemmas is the following result.

PROPOSITION 1. *$\mathscr{C}$ contains G if it is of any of the following types:*
(i) *torsion abelian,*
(ii) *torsion-free abelian,*
(iii) *ordered.*

3. Some lemmas.

For any normal subgroup H of G, let p_H: $Z(G) \to Z(G/H)$ be the linear extension of the natural homomorphism $G \to G/H$. Then $\Delta(G, H)$, the kernel of p_H, is the ideal generated by $\{(1 - h) \mid h \in H\}$. We shall write $\Delta(H)$ instead of $\Delta(G, H)$ if no confusion can arise. Furthermore, for $\sum_g a_g g = \gamma \in Z(G)$, we denote by $c(\gamma)$ the integer $\sum_g a_g$. We need the following lemmas.

LEMMA 4. *Suppose that $g \in G$ is such that $g \equiv 1 \bmod (\Delta(G))^2$. Then $g \in G'$, the derived group of G.*

Proof. Let us first suppose that G is abelian. We can assume that G is finitely generated, say, $G = \langle g_1, g_2, \ldots, g_r\rangle$. Let $g = \prod_1^r g_i^{a_i}$. Since

$$g - 1 \equiv \sum_i a_i(g_i - 1) \mod (\Delta(G))^2,$$

we have $\sum_i a_i(g_i - 1) \in (\Delta(G))^2$. Therefore

$$(*) \qquad \sum_i a_i(g_i - 1) = \sum x(g_1^{\alpha_1} \ldots g_r^{\alpha_r} - 1)(g_1^{\beta_1} \ldots g_r^{\beta_r} - 1), \qquad x \in Z(G).$$

Extend the endomorphism $\gamma_j: g_i \to g_j^{\delta_{ij}}$ of G to $Z(G)$. Applying this to both sides of (*) we obtain:

$$a_j(g_j - 1) = \sum x_{st}(g_j^s - 1)(g_j^t - 1), \qquad x_{st} \in Z(\langle g_j \rangle),$$

and therefore,

$$(a_j - y)(g_j - 1) = 0, \text{ where } y(g_j - 1) = \sum x_{st}(g_j^s - 1)(g_j^t - 1).$$

Suppose that $a_j \neq 0$, then since $c(y) \neq 0$, we can say that $(a_j - y) \neq 0$. It follows that g_j has finite order n_j and $a_j - y = m(1 + g_j + \ldots + g_j^{n_j - 1})$, $m \in Z$. Therefore $a_j = mn_j$ and $g = 1$.

Now let G be arbitrary and let $\beta: G \to G/G'$ be the natural map. Then $g^\beta \equiv 1 \bmod (\Delta(G/G'))^2$, therefore $g^\beta = 1$ and $g \in G'$. This completes the proof of the lemma.

COROLLARY 1. $G/G' \simeq \Delta(G)/(\Delta(G))^2$.

Proof. Let $\theta: G \to \Delta(G)/(\Delta(G))^2$ be given by $\theta(g) = (g-1) \bmod (\Delta(G))^2$. Then θ is a homomorphism as $gh - 1 = (g-1) + (h-1) + (g-1)(h-1)$. It is an epimorphism because of the same reason. By Lemma 4, the kernel of θ is G', and the proof is complete.

The next corollary was proved by Higman (**2**) for finite abelian groups.

COROLLARY 2. *Let G be an abelian group such that $Z(G) \simeq Z(H)$. Then $G \simeq H$.*

Proof. Let $\theta: Z(G) \to Z(H)$ be the given isomorphism. Then $c(\theta(g)) = \pm 1$. Normalize θ by defining $\mu: Z(G) \to Z(H)$ by $\mu(g) = c(\theta(g))\theta(g)$ for $g \in G$, and linear extension. It is easy to see that μ is an isomorphism and $c(\mu(g)) = 1$ for all $g \in G$. Now

$$\mu(\Delta(G)) = \Delta(H) \quad \text{and} \quad G \simeq \Delta(G)/(\Delta(G))^2 \simeq \Delta(H)/(\Delta(H))^2 \simeq H.$$

Similarly, we have the following result.

COROLLARY 3. $Z(G) \simeq Z(H) \Rightarrow G/G' \simeq H/H'$.

LEMMA 5. *Let N be a normal subgroup of G. Then*

$$\gamma \in Z(N), \qquad \gamma \equiv 0 \bmod \Delta(G,G)\Delta(G,N) \Rightarrow \gamma \equiv 0 \bmod (\Delta(N,N))^2.$$

Proof. Choose a set of coset representatives $\{g_i\}$ of $G \bmod N$. Define for $g_i n \in G$, $\sigma(g_i n) = n$ and extend this linearly to $\sigma: Z(G) \to Z(N)$. Now $\gamma = \sum_i \gamma_i(n_i - 1)$, $\gamma_i \in \Delta(G,G)$, $n_i \in N$. Therefore

$$\gamma = \gamma^\sigma = \sum_i \gamma_i^\sigma(n_i - 1) \quad \text{and} \quad \gamma \in (\Delta(N,N))^2.$$

COROLLARY 4. *Let N be a normal subgroup of G. Then*

$$g \in G, \qquad g \equiv 1 \bmod \Delta(G,G)\Delta(G,N) \Rightarrow g \in N'.$$

Proof. Since $g - 1 \equiv 0 \bmod \Delta(G,G)\Delta(G,N)$, we have $gN = N$ and $g \in N$.

Now by Lemma 5, $g - 1 \equiv 0 \bmod \Delta(N, N)^2$, and hence $g \in N'$ by Lemma 4.

LEMMA 6. *Let A be a normal subgroup of G. Then*

$$A/A' \simeq \Delta(G, A)/\Delta(G)\Delta(G, A).$$

Proof. Define the map $\theta: A \to \Delta(G, A)/\Delta(G)\Delta(G, A)$ by $\theta(a) = 1 - a$. The kernel of θ is A' by the last corollary. That θ is an epimorphism follows from the fact that

$$g(a - 1) \equiv a - 1 \mod \Delta(G)\Delta(G, A).$$

4. Proof of the main Theorem.

PROPOSITION 2. *Suppose that G has a normal abelian subgroup A such that $G/A \in \mathscr{C}$. Suppose that there is a normalized isomorphism $\theta: Z(G) \to Z(H)$ such that $\theta(\Delta(G, A)) = \Delta(H, B)$ for a suitable normal subgroup B of H. Then B is abelian and $H/B \in \mathscr{C}$.*

Proof. Since $\theta(\Delta(G, A)) = \Delta(H, B)$, we have:

$$Z(G/A) \simeq Z(G)/\Delta(G, A) \simeq Z(H)/\Delta(H, B) \simeq Z(H/B).$$

Let $\lambda: Z(G/A) \to Z(H/B)$ be the isomorphism. Then $\lambda(\bar{g}) = \bar{h} \in H/B$ for all $\bar{g} \in G/A$. Actually, $\lambda(G/A) = H/B$. Now, given an isomorphism $\mu: Z(H/B) \to Z(K)$, we need to prove that for $\bar{h} \in H/B$, $\mu(\bar{h}) = k \in K$. Consider $\mu\lambda: Z(G/A) \to Z(K)$. We know that $\bar{h} = \lambda(\bar{g})$ for some $\bar{g} \in G/A$. Since $G/A \in \mathscr{C}$, $(\mu\lambda)(\bar{g}) = k$, $k \in K$. It follows that $\mu(\bar{h}) = k$ and $H/B \in \mathscr{C}$. Furthermore, B is abelian, since by Lemma 6:

$$A/A' \simeq \Delta(G, A)/\Delta(G)\Delta(G, A) \simeq \Delta(H, B)/\Delta(H)\Delta(H, B) \simeq B/B'.$$

Remark. It is easy to see that if in the above proposition $A = G'$, then $B = H'$ satisfies the condition $\theta(\Delta(G, A)) = \Delta(H, B)$.

Remark. If in the above proposition G/A is ordered (in particular, torsion-free abelian) we do not need to assume the existence of B, since in this case it can be proved that there always exists a B with the property

$$\theta(\Delta(G, A)) = \Delta(H, B).$$

THEOREM 1. *Suppose that G has a normal abelian subgroup A such that $G/A \in \mathscr{C}$. Suppose that there is an isomorphism $\theta: Z(G) \to Z(H)$ such that $\theta(\Delta(G, A)) = \Delta(H, B)$ for a suitable normal subgroup B of H. Then $G \simeq H$.*

Proof. We can assume that θ is normalized; then due to Proposition 2, B is abelian and $H/B \in \mathscr{C}$. For $g \in G$, let $\theta(g) = \gamma$. Then since $H/B \in \mathscr{C}$, we have:

$$\gamma \equiv h_1 \mod(\Delta(H, B)), \qquad h_1 \in H.$$

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Now

$$\begin{aligned}\gamma &= h_1 + \sum_{b\in B} \alpha_b(1-b), \qquad \alpha_b \in Z(H),\\ &\equiv h_1 + \sum_{b\in B} n_b(1-b) \mod(\Delta(H)\cdot\Delta(B)), \quad \text{where } n_b = c(\alpha_b),\\ &\equiv h_1 + 1 - \prod_b b^{n_b} \mod(\Delta(H)\cdot\Delta(B)),\\ &\equiv h_1\prod_b b^{-n_b} \mod(\Delta(H)\cdot\Delta(B)).\end{aligned}$$

Thus for $g \in G$, if $\theta(g) = \gamma$, there exists $h_\gamma \in H$ such that

$$\gamma \equiv h_\gamma \mod(\Delta(H)\cdot\Delta(B)).$$

This h_γ is unique since if $h' \equiv h \mod(\Delta(H)\cdot\Delta(B))$, then $h'h^{-1} \in B' = \{1\}$ by Corollary 4, and hence $h' = h$. Define $\lambda: G \to H$ by $\lambda(g) = h_\gamma \in H$. Then λ is a well-defined homomorphism. Notice that we have:

$$g \xrightarrow{\theta} \gamma \xrightarrow{\phi_1} h_\gamma \xrightarrow{\theta^{-1}} \mu \xrightarrow{\phi_2} g_\mu \qquad (g, g_\mu \in G),$$

where ϕ_2 is the map obtained (as was ϕ_1) by using θ^{-1} instead of θ. We see that $g_\mu = g$ since

$$\begin{aligned}(\phi_2\theta^{-1}\phi_1\theta)(g) &= \phi_2\theta^{-1}(\theta(g)+\delta), &&\text{where } \delta \in \Delta(H)\Delta(B),\\ &= \phi_2(g+\beta), &&\text{where } \theta^{-1}(\delta) = \beta \in \Delta(G)\Delta(A),\\ &= g &&\text{(due to uniqueness of } g_\mu).\end{aligned}$$

Thus $\phi_2\theta^{-1}\phi_1\theta = I_G$, and similarly $\phi_1\theta\phi_2\theta^{-1} = I_H$. It follows that $\lambda = \phi_1\theta$ is an isomorphism.

5. Group rings of finite groups. Suppose that $\theta: Z(G) \to Z(H)$ is a normalized isomorphism, where G (and hence H) is a finite group. We proved in (**5**) that for any normal subgroup A of G, $\theta(\sum_{a\in A} a) = (\sum_{b\in B} b)$ and $A \leftrightarrow B = \Phi(A)$ is a one-to-one correspondence between normal subgroups of G and those of H. This correspondence preserves union, intersection, and order. This is a result of Passman (**4**) who also proved the next proposition for finite nilpotent groups.

PROPOSITION 3. *Suppose that* $\Phi(A) = B$. *Then*
(i) $Z(G/A) \simeq Z(H/B)$,
(ii) $A/A' \simeq B/B'$, *and*
(iii) $\Phi(A') = B'$.

Proof. Since the annihilator of $(\sum_{a\in A} a)$ in $Z(G)$ is $\Delta(G, A)$, it follows that $\theta(\Delta(G, A)) = \Delta(H, B)$. Therefore

$$Z(G/A) \simeq Z(G)/\Delta(G, A) \simeq Z(H)/\Delta(H, B) \simeq Z(H/B).$$

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To prove (ii) we only have to notice that Lemma 6 yields:

$$A/A' \simeq \Delta(G, A)/\Delta(G)\Delta(G, A) \simeq \Delta(H, B)/\Delta(H)\Delta(H, B) \simeq B/B'.$$

We now prove (iii). Suppose that $\Phi(A') = C$. Then we have a sequence of isomorphisms

$$Z(G/A') \to Z(G)/\Delta(A') \to Z(H)/\Delta(C) \to Z(H/C),$$

where $\bar{g} \to g \bmod \Delta(A') \to \theta(g) \bmod \Delta(C) \to \theta(g) \bmod C$. Thus we have the isomorphism λ: $Z(G/A') \to Z(H/C)$ given by $\lambda(\bar{g}) = \theta(g) \bmod C$. Now A/A' is a normal subgroup of G/A' and $\Phi(A/A')$ is a normal subgroup of H/C. From

$$\lambda\left(\sum_{a\in A} \bar{a}\right) = \left(\sum_{a\in A} \theta(a) \bmod C\right) = \left(\sum_{b\in B} b \bmod C\right),$$

we conclude that $\Phi(A/A') = B/C$. Since A/A' is abelian, B/C is abelian and $C \supset B'$. Therefore, due to (ii) and equality of orders of A/A' and B/C, we have $C = B'$. This completes the proof.

THEOREM 2. *Suppose that G is a finite group and $Z(G) \simeq Z(H)$. Then the derived series of G and H are isomorphic. In particular, if G is solvable, then so is H.*

Proof. Suppose that G_i and H_i are ith terms of the derived series of G and H, respectively. Let θ: $Z(G) \to Z(H)$ be the normalized isomorphism and $\Phi(G_i) = H_i$. Then by Proposition 3 (iii), $\Phi(G_{i+1}) = H_{i+1}$. Also $\theta(\Delta(G, G_i)) = \Delta(H, H_i)$, and by Lemma 6,

$$G_i/G_{i+1} \simeq \Delta(G, G_i)/\Delta(G) \cdot \Delta(G, G_i) \simeq \Delta(H, H_i)/\Delta(H) \cdot \Delta(H, H_i) \simeq H_i/H_{i+1}.$$

This completes the proof by induction.

6. A theorem of Banaschewski. For $\gamma = \sum_{g\in G} a_g g \in Z(G)$, let $\gamma^* = \sum_{g\in G} a_g g^{-1}$. Then

(i) $(\gamma + \mu)^* = \gamma^* + \mu^*$,
(ii) $(\gamma\mu)^* = \mu^*\gamma^*$, and
(iii) $(a\gamma)^* = a\gamma^*$ for $a \in Z$.

We say that a map θ: $Z(G) \to Z(H)$ is *-preserving if $\theta(\gamma^*) = (\theta(\gamma))^*$ for all $\gamma \in Z(G)$. We offer a simple proof of the following theorem proved by Banaschewski (**1**) for finite groups.

THEOREM 3. *Suppose that θ: $Z(G) \to Z(H)$ is a *-preserving isomorphism. Then $G \simeq H$.*

Proof. Let $\theta(g) = \gamma = \sum a_h h$; then $1 = \theta(gg^{-1}) = \theta(g)\theta(g^*) = \gamma\gamma^*$. Now

$$\gamma\gamma^* = (\sum a_h^2) \cdot e + \sum_{h\neq e} \beta_h h$$

implies that $\sum a_h^2 = 1$ and $\gamma = \pm h$, $h \in H$. Thus $G \simeq H$.

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