

CHARACTERISATIONS OF Ω -STABILITY AND STRUCTURAL STABILITY VIA INVERSE SHADOWING

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We give characterisations of Ω -stable diffeomorphisms and structurally stable diffeomorphisms via the notions of weak inverse shadowing and orbital inverse shadowing, respectively. More precisely, it is proved that the C^1 interior of the set of diffeomorphisms with the weak inverse shadowing property coincides with the set of Ω -stable diffeomorphisms and the C^1 interior of the set of diffeomorphisms with the orbital inverse shadowing property coincides with the set of structurally stable diffeomorphisms.

1. INTRODUCTION

Structurally stable systems and Ω -stable systems have been the main objects of interests in the global qualitative theory of dynamical systems over recent 30 years and various attempts have been made to characterise the systems via the notions of hyperbolicity, shadowing, Axiom A, strong transversality condition, no-cycle condition, et cetera.

It was proved in [10] and [11] that the C^1 interior of the set of topologically stable diffeomorphisms [respectively, topologically stable C^1 vector fields] coincides with the set of structurally stable diffeomorphisms [respectively, structurally stable C^1 vector fields]. Sakai [14] showed that the C^1 interior of the set of diffeomorphisms having the shadowing property was characterised as the set of structurally stable diffeomorphisms.

The weak shadowing property which is really weaker than the shadowing property was introduced by Corless and Pilyugin [2], and they proved that the weak shadowing property is generic in the set of homoemorphisms on a C^∞ closed manifold endowed with the C^0 topology. Moreover Sakai [16] showed that every element in the C^1 interior of the set of diffeomorphisms on a C^∞ closed surface having the weak shadowing property is Ω -stable, but the converse does not hold even in dimension 2.

Recently Pilyugin, Rodionova and Sakai [13] proved that the C^1 interior of the set of diffeomorphisms having the orbital shadowing property consists of structurally stable diffeomorphisms.

Received 2nd March, 2006

This work was supported by the KRF Grant funded by the Korean Government (MOEHRD) (KRF-2005-070-C00015).

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The inverse shadowing property which is a “dual” notion of shadowing was introduced by Corless and Pilyugin in [2], and the qualitative theory of dynamical systems with the property was developed by various authors (see [1, 2, 3, 5, 6, 7, 13], for example).

In this paper we introduce the notion of weak [respectively, orbital] inverse shadowing which is a “dual” notion of weak [respectively, orbital] shadowing, and prove that the C^1 interior of the set of diffeomorphisms with the weak [respectively, orbital] inverse shadowing property coincides with the set of Ω -stable [respectively structurally stable] diffeomorphisms. These facts improve the results in [6, 13, 16] which say that every structurally diffeomorphism has the inverse shadowing property with respect to classes of continuous methods, and the C^1 interior of the set of diffeomorphisms satisfying persistency is characterised as the set of diffeomorphisms satisfying Axiom A and the strong transversality condition.

2. PRELIMINARIES

Consider a dynamical system generated by a homeomorphism f of a compact metric space X with a metric d . For a point $x \in X$, we denote by $O(x, f)$ its orbit in the system f : that is, the set

$$O(x, f) = \{f^n(x) : n \in \mathbb{Z}\}.$$

We say that a sequence $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is a δ -pseudo orbit of f if the inequalities

$$d(f(x_n), x_{n+1}) < \delta, n \in \mathbb{Z}$$

hold. A δ -pseudo orbit is a natural model of computer output in a process of numerical investigation of the system f . In this case, the value δ measures errors of the method, round-off errors, et cetera.

Recall that f has the *shadowing property* if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ -pseudo orbit $\xi = \{x_n : n \in \mathbb{Z}\}$ we can find a point $y \in X$ with the property

$$d(f^n(y), x_n) < \varepsilon, n \in \mathbb{Z}.$$

Of course, if f has the shadowing property formulated above, then the results of its numerical study with a proper accuracy reflect its qualitative structure.

It is said that f has the *weak shadowing property* [respectively *orbital shadowing property*] if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ -pseudo orbit $\xi = \{x_n : n \in \mathbb{Z}\}$ of f we can find a point $y \in X$ with the property

$$\xi \subset N(\varepsilon, O(y, f)) \left[\text{respectively } d_H(\xi, \overline{O(y, f)}) < \varepsilon \right],$$

where d_H denotes the Hausdorff distance on the set of compact subsets of X . The weak shadowing property was introduced in [2] and the orbital shadowing property was introduced in [14].

Let $X^{\mathbb{Z}}$ be the space of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ -pseudo orbits of f . A mapping $\varphi : X \rightarrow \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a δ -method for f if $\varphi(x)_0 = x$, where $\varphi(x)_0$ denotes the 0-th component of $\varphi(x)$. Then each $\varphi(x)$ is a δ -pseudo orbit of f through x . For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k \in \mathbb{Z}}$. Say that φ is a *continuous δ -method* for f if the map φ is continuous. The set of all δ -methods [respectively, continuous δ -methods] for f will be denoted by $\mathcal{T}_0(f, \delta)$ [respectively $\mathcal{T}_c(f, \delta)$]. If $g : X \rightarrow X$ is a homeomorphism with $d_0(f, g) < \delta$, where

$$d_0(f, g) = \sup_{x \in X} \left\{ d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)) \right\},$$

then g induces a continuous δ -method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.$$

Let $\mathcal{T}_h(f, \delta)$ denote the set of all continuous δ -methods φ_g for f which are induced by $g \in Z(X)$ with $d_0(f, g) < \delta$, where $Z(X)$ denotes the space of homeomorphisms on X with the C^0 metric d_0 . We define $\mathcal{T}_\alpha(f)$ by

$$\mathcal{T}_\alpha(f) = \bigcup_{\delta > 0} \mathcal{T}_\alpha(f, \delta),$$

where $\alpha = 0, c, h$. Clearly,

$$\mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f).$$

Note that a method in $\mathcal{T}_c(f)$ need not be generated by a single mapping.

We say that f has the *shadowing property* [respectively *inverse shadowing property*] with respect to the class \mathcal{T}_α , $\alpha = 0, c, h$, if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ -method φ in $\mathcal{T}_\alpha(f, \delta)$ and any point $x \in X$ there exists a point $y \in X$ for which

$$d(f^n(y), \varphi(x)_n) < \varepsilon \left[\text{respectively } d(f^n(x), \varphi(y)_n) < \varepsilon \right], \quad n \in \mathbb{Z}.$$

When we study the qualitative theory of differentiable dynamical systems (or diffeomorphisms) the notion of shadowing (or inverse shadowing) with respect to the class \mathcal{T}_c (or \mathcal{T}_h) is sometimes too strong as can be seen in ([6, Theorem 3.3]). Here we give another types of shadowing and inverse shadowing as follows.

For our purpose, let M be a C^∞ closed n -dimensional manifold with metric d induced by a Riemannian metric $\|\cdot\|$ on TM , and let $\text{Diff}(M)$ denote the space of C^1 diffeomorphisms on M with the C^1 metric d_1 . As before, for any $\delta > 0$ and $f \in \text{Diff}(M)$, every $g \in \text{Diff}(M)$ with $d_1(f, g) < \delta$ induces a continuous δ -method $\varphi_g : M \rightarrow M^{\mathbb{Z}}$ for f by defining

$$\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}.$$

Let $\mathcal{T}_d(f, \delta)$ denote the set of all continuous δ -methods φ_g for f which are induced by $g \in \text{Diff}(M)$ with $d_1(f, g) < \delta$. Put

$$\mathcal{T}_d(f) = \bigcup_{\delta > 0} \mathcal{T}_d(f, \delta).$$

Similarly we say that f has the *shadowing property* [respectively, *inverse shadowing property*] with respect to the class \mathcal{T}_d if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any δ -method φ in $\mathcal{T}_d(f, \delta)$ and any point $x \in M$ there exists a point $y \in M$ for which

$$d(f^n(y), \varphi(x)_n) < \varepsilon \left[\text{respectively } d(f^n(x), \varphi(y)_n) < \varepsilon \right], \quad n \in \mathbb{Z}.$$

Now let us recall the notion of weak [respectively, orbital] inverse shadowing which is a “dual” notion of weak [respectively, orbital] shadowing (see [1]).

DEFINITION 2.1: We say that f has the *weak* [respectively, *orbital*] *inverse shadowing property* with respect to the class \mathcal{T}_α , $\alpha = 0, c, h, d$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ -method $\varphi \in \mathcal{T}_\alpha(f, \delta)$ and any point $x \in M$ there is a point $y \in M$ for which

$$\varphi(y) \subset N(\varepsilon, O(x, f)) \left[\text{respectively, } d_H(\overline{O(x, f)}, \overline{\varphi(y)}) < \varepsilon \right].$$

REMARK 2.2. An appropriate choice of the class of admissible pseudo orbits is crucial when we consider the notions of inverse shadowing (see [3, 6, 11]). Suppose that $\mathcal{T}_a(f) \subset \mathcal{T}_b(f)$ for some $f \in \text{Diff}(M)$ and $a, b \in \{0, c, h, d\}$. If f has the weak [respectively, orbital] inverse shadowing property with respect to the class \mathcal{T}_b then it clearly has the weak [respectively, orbital] inverse shadowing property with respect to the class \mathcal{T}_a . We can easily show that every irrational rotation f on the unit circle S^1 has the weak inverse shadowing with respect to the class \mathcal{T}_c (or \mathcal{T}_h), but it does not have the inverse shadowing property with respect to the class \mathcal{T}_c (or \mathcal{T}_h). Recently Choi, Kim and Lee [1] showed that the weak inverse shadowing property for \mathcal{T}_h is generic in the space of homeomorphisms on a compact metric space with the C^0 topology.

3. MAIN THEOREMS

The aim of this paper is to investigate the dynamics of diffeomorphisms belonging to the C^1 interior of the set of diffeomorphisms having the weak and orbital inverse shadowing property with respect to the class \mathcal{T}_d . Through the paper, for simplicity, we shall abbreviate “weak [respectively orbital] inverse shadowing property with respect to the class \mathcal{T}_d ” to “weak [respectively, orbital] inverse shadowing property”.

We say that $f \in \text{Diff}(M)$ is *structurally stable* if there exists a C^1 neighbourhood \mathcal{U} of f in $\text{Diff}(M)$ such that every $g \in \mathcal{U}$ is topologically conjugate to f . Let $\Omega(f)$ denote the set of nonwandering points of f . Recall that f is Ω -stable if there is a C^1

neighbourhood \mathcal{V} of f such that for any $g \in \mathcal{V}$ the restriction of f to $\Omega(f)$ and g to $\Omega(g)$ are topologically conjugate.

We denote by $\text{WIS}(M)$ the set of $f \in \text{Diff}(M)$ having the weak inverse shadowing property, and denote by $\text{OIS}(M)$ the set of $f \in \text{Diff}(M)$ having the orbital inverse shadowing property. The following two theorems are the main results of this paper.

THEOREM 3.1. *The C^1 interior of $\text{WIS}(M)$ in $\text{Diff}(M)$ coincides with the set of Ω -stable diffeomorphisms on M .*

THEOREM 3.2. *The C^1 interior of $\text{OIS}(M)$ in $\text{Diff}(M)$ coincides with the set of structurally stable diffeomorphisms on M .*

REMARK 3.3. It was proved in [16] that every element in the C^1 interior of the set of diffeomorphisms on a C^∞ closed surface having the weak shadowing property is Ω -stable, but the converse does not hold in general. We can see that the above result does not generalise to higher dimensions. In fact, it was proved in [9] that there is a C^1 open set \mathcal{U} of the set of diffeomorphisms on the 3-torus such that every $f \in \mathcal{U}$ is topologically transitive and it is not Anosov. This means that every $f \in \mathcal{U}$ is not Ω -stable but it has the weak shadowing property.

REMARK 3.4. Theorem 3.2 improves the results obtained by [6, 13, 16]. It was proved in [13] that every structurally stable diffeomorphism has the inverse shadowing property with respect to the class θ_c (or θ_s). It was shown in [16] that the C^1 interior of the set of diffeomorphisms with persistency is characterised as the set of structurally stable diffeomorphisms. It was also proved in [6] that the C^1 interior of the set of diffeomorphisms having the inverse shadowing property with respect to the class \mathcal{T}_d coincides with the set of structurally stable diffeomorphisms.

Note that the notion of the class θ_c in [13] is the same as that of the class \mathcal{T}_c in this paper, and the notion of persistency in [16] is the same as that of the inverse shadowing property with respect to the class \mathcal{T}_h in this paper.

If $f \in \text{Diff}(M)$ has the inverse shadowing property with respect to the class \mathcal{T}_h then it clearly has the inverse shadowing property with respect to the class \mathcal{T}_d , but the converse does not hold in general. Moreover if $f \in \text{Diff}(M)$ has the inverse shadowing property with respect to the class \mathcal{T}_d then it has the orbital inverse shadowing property, but the converse does not hold. Indeed, an irrational rotation on the unit circle has the orbital inverse shadowing property but does not have the inverse shadowing property with respect to the class \mathcal{T}_d .

We say that $f \in \text{Diff}(M)$ is *hyperbolic* on a closed invariant set $\Lambda \subset M$ if there is a continuous splitting of the tangent bundle, $TM|_\Lambda = E^s \oplus E^u$, and there are constants $C > 0, 0 < \lambda < 1$, such that

$$\|Df^n|_{E^s}\| < C\lambda^n, \quad \|Df^{-n}|_{E^s}\| < C\lambda^n$$

for any $n > 0$ and $x \in \Lambda$. The *stable* [respectively *unstable*] *manifold* of $x \in \Lambda$ is defined by $W^s(x, f)$ [respectively $W^u(x, f)$], which is the set of points $p \in M$ such that $d(f^k(x), f^k(p))$ tends to 0 as k tends to ∞ [respectively $-\infty$]. We say that f satisfies *Axiom A* if its periodic points are dense in the set of nonwandering points $\Omega(f)$, and f is hyperbolic on $\Omega(f)$. An *Axiom A* diffeomorphism f is said to satisfy the *strong transversality condition* if $W^s(x, f)$ and $W^u(y, f)$ are transversal for any $x, y \in \Omega(f)$.

Let $\mathcal{F}(M)$ be the set of $f \in \text{Diff}(M)$ having a C^1 neighbourhood $\mathcal{U} \subset \text{Diff}(M)$ such that for $g \in \mathcal{U}$, every periodic point of g is hyperbolic. Hayashi [4] proved that $\mathcal{F}(M)$ is contained in the set of diffeomorphisms satisfying *Axiom A* with no-cycle condition. It is well known that f is Ω -stable if and only if it satisfies *Axiom A* and no-cycle condition.

4. PROOF OF THEOREM 3.1

First we show that every element f in the C^1 interior of $\text{WIS}(M)$ in $\text{Diff}(M)$, $\text{WIS}(M)^0$, belongs to $\mathcal{F}(M)$. Then it is Ω -stable by the result of Hayashi [4]. To get a contradiction, we assume that there exists $f \in \text{WIS}(M)^0 \setminus \mathcal{F}(M)$. Choose a C^1 neighbourhood \mathcal{U} of f in $\text{WIS}(M)^0$ and take a diffeomorphism $f' \in \mathcal{U}$ having a non-hyperbolic periodic point p . Let m be the period of the point p . Passing from f' to its C^1 small perturbation f'' , we may assume that the derivative $D(f'')^m(p)$ has an eigenvalue equal to 1.

Standard arguments (as in the proof of [14, Theorem 3.2]) enable us to find a C^1 small perturbation of f in \mathcal{U} (again denoted by f) having the following properties:

- (1) p is a periodic point of f with period m (we denote $p_i = f^i(p)$ for $i = 0, \dots, m$);
- (2) if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the derivative $Df^m(p)$, then $\lambda_1 = 1$ and $|\lambda_j| \neq 1$ for $j = 2, \dots, n$;
- (3) we can introduce local coordinates $y = (y_1, \dots, y_n)$ in disjoint neighbourhoods U_i of the points p_i so that p_i is the origin in U_i , and
- (3.1) if L_0 is the subspace of the tangent space $T_p M$ (identified with \mathbb{R}^n) corresponding to the eigenvalue λ_1 of $Df^m(p)$, then the spaces

$$L^i = Df^m(p)L_0, \quad i = 0, \dots, m - 1,$$

coincide with the subspaces

$$\{y \in \mathbb{R}^n : y_2 = \dots = y_n = 0\}$$

in the coordinates of the corresponding neighbourhoods;

- (3.2) there is a number $a > 0$ such that the mapping $\varphi^{(i)}$, the restriction of f to the set $U'_i = U_i \cap \{|y| < 4a\}$, maps U'_i into U_{i+1} (of course,

$U_m = U_0$) and is given by the formula

$$\varphi^{(i)}(y) = (k_i y_1, B_i y'),$$

where $y' = (y_2, \dots, y_n)$.

Since f is a diffeomorphism, we have $k_i \neq 0$ for $i = 0, \dots, m - 1$. It follows from (2), (3.1) and (3.2) that $k_0 k_1 \dots k_{m-1} = 1$ and the matrix $B = B_{m-1} \dots B_0$ is hyperbolic. Let

$$K_1 = \min_{0 \leq i \leq m-1} |k_i k_{i-1} \dots k_0|,$$

$$K_2 = \max_{0 \leq i \leq m-1} |k_i k_{i-1} \dots k_0|, \text{ and } K = \frac{K_1}{K_2}.$$

It is easy to construct a diffeomorphism $g_\lambda \in \mathcal{U}$ coinciding with f outside U_{m-1} such that $\psi_\lambda(y)$, the restriction of g_λ to the set U'_{m-1} , is given by the formula

$$(4.1) \quad \psi_\lambda(y) = (\lambda y_1, B_{m-1} y')$$

where $0 < k_0 k_1 \dots k_{m-2} \lambda = \mu < 1$ and $1 - \mu$ is arbitrary small. Fix $\varepsilon = aK$, and find a corresponding $\delta > 0$ from the definition of $f \in \text{WIS}(M)^0$. Take a diffeomorphism $g_\lambda \in \mathcal{U}$ with $d_1(f, g_\lambda) \leq \delta$, and let

$$x_0 = \left(\frac{2a}{K_2}, 0, \dots, 0 \right).$$

Note that for any point $f^j(x_0) \in O(x_0, f)$, its first coordinate $f^j(x_0)_1$ is obtained by

$$k_{i-1} \dots k_0 \frac{2a}{K_2}$$

for some i with $j \equiv i \pmod{m}$, and satisfies the inequalities

$$2\varepsilon = K_1 \frac{2a}{K_2} \leq |f^j(x_0)_1| \leq K_2 \frac{2a}{K_2} = 2a.$$

Formula (4.1) implies that for any point (y_1, y') with $y' \neq 0$, its trajectory leaves the set $N(\varepsilon, O(x_0, f))$. Hence the inclusion

$$O(y, g_\lambda) \subset N(\varepsilon, O(x_0, f))$$

shows that $y = (y_1, 0, \dots, 0)$. If $y_1 = 0$, then we get

$$(0, 0, \dots, 0) \notin N(\varepsilon, O(x_0, f)).$$

If $y_1 \neq 0$, there exists $j_0 \in \mathbb{N}$ such that $\mu^{m j_0} < \varepsilon$. This implies

$$g_\lambda^{m j_0}(y_1, 0, \dots, 0) = (\mu^{m j_0} y_1, 0, \dots, 0) \notin N(\varepsilon, O(x_0, f)),$$

and so contradict to the fact that $f \in \text{WIS}(M)^0$.

Next we show that every Ω -stable diffeomorphism has the weak inverse shadowing property. Suppose that $f \in \text{Diff}(M)$ is Ω -stable, and let $\Omega_1, \dots, \Omega_m$ be the basic sets of f with

$$\Omega(f) = \Omega_1 \cup \dots \cup \Omega_m.$$

Let $\alpha = \min\{d(\Omega_i, \Omega_j) : i \neq j\}$ and $0 < \varepsilon < \alpha/4$. Since each Ω_i is a closed hyperbolic invariant set for f , we can choose a C^1 neighbourhood \mathcal{U}_i of f in $\text{Diff}(M)$ and a continuous map $\Phi_i : \mathcal{U}_i \rightarrow C^0(\Omega_i, M)$ satisfying

- (1) $\Phi_i(f)(\Omega_i) = \Omega_i$;
- (2) $\Phi_i(g)(\Omega_i)$ is a hyperbolic invariant set for $g, g \in \mathcal{U}_i$;
- (3) $\Phi_i(g)$ is a homeomorphism of Ω_i onto $\Phi_i(g)(\Omega_i)$ and topologically conjugates the restriction of f to Ω_i to the restriction of g to $\Phi_i(g)(\Omega_i)$;
- (4) there is a constant K_i such that

$$d_0(\Phi_i(g), \Phi_i(f)) < K_i d_0(g, f),$$

where $i = 1, 2, \dots, m$.

Choose $\delta_i > 0$ satisfying $B_{d_1}(\delta_i, f) \subset \mathcal{U}_i$ for each i , and let

$$\delta = \min_{1 \leq i \leq m} \left\{ \delta_i, \frac{\varepsilon}{4K_i} \right\},$$

where

$$B_{d_1}(\delta_i, f) = \{g \in \text{Diff}(M) : d_1(f, g) < \delta_i\}.$$

Let $g \in B_{d_1}(\delta, f)$ and $x \in M$. Suppose $\omega(x, f) \subset \Omega_i$ for some i and take a point $z \in \omega(x, f)$. Let $y = \Phi_i(g)(z)$. Then we have

$$O(y, g) \subset N(\varepsilon, O(z, f)) \subset N(\varepsilon, \omega(x, f)) \subset N(\varepsilon, O(x, f)).$$

This means that f has the weak inverse shadowing property, and so completes the proof of Theorem 3.1.

5. PROOF OF THEOREM 3.2

It is well known that the C^1 interior of the Kupka–Smale diffeomorphisms consists of structurally stable diffeomorphisms, and we see that every structurally stable diffeomorphism has the orbital inverse shadowing property by [12, Theorem 1.1].

To show that every element f in the C^1 interior of $\text{OIS}(M)$ in $\text{Diff}(M)$, $\text{OIS}(M)^0$, is structurally stable, it suffices to show that f has the strong transversality condition by applying the first part of the proof of Theorem 3.1. That is, it is enough to prove that if p and q are periodic points of $f \in \text{OIS}(M)^0$ then their stable manifold $W^s(p, f)$ and unstable manifold $W^u(q, f)$ are transverse. To get a contradiction, we assume that

there is a diffeomorphism $f \in \text{OIS}(M)^0$, having periodic points p and q and a point r of nontransverse intersection of $W^s(p, f)$ and $W^u(q, f)$. Note that the point r is wandering, and p and q are in different basic sets. To simplify presentation, we assume that p and q are fixed points. (The case of periodic point is considered similarly.) By the same techniques described in the proof of [13, Theorem 4.1], for any C^1 neighbourhood \mathcal{U} of f , we can choose a C^1 small perturbation of f (denoted by f) with the following properties:

- (1) p and q are fixed points of f ;
- (2) f is linear in a neighbourhood of p ;
- (3) r is a point of nontransverse intersection of $W^s(p, f)$ and $W^u(q, f)$;
- (4) there exists a small open (with respect to the inner topology of $W^u(q, f)$) disk $C^u(r') \subset W^u(q, f)$ of r' such that $C^u(r) = f(C^u(r')) \subset L^u + r$, where $r' = f^{-1}(r)$ and $L^u = T_r W^u(q, f)$.

Let $\Omega_1, \dots, \Omega_m$ be the basic sets of f with

$$\Omega(f) = \Omega_1 \cup \dots \cup \Omega_m,$$

and suppose that $q \in \Omega_1$ and $p \in \Omega_2$. Since the point r is wandering, there is a neighbourhood V of r such that $f^n(V) \cap f^m(V) = \emptyset$ for all $n \neq m$.

We first claim that there is a neighbourhood $U(\subset V)$ of r such that for each $x \in U$,

$$(5.1) \quad x \notin W^u(\Omega_2) \cap W^s(\Omega_1).$$

Since f satisfies Axiom A and no cycle condition, there exists a continuous Lyapunov function $L : M \rightarrow \mathbb{R}$ such that

- (1) $L(f(x)) < L(x)$ for $x \in M \setminus \Omega(f)$;
- (2) L is constant on $\Omega_i, i = 1, \dots, m$.

Since $L(p) < L(r) < L(q)$ and L is continuous, there is a neighbourhood $U(\subset V)$ of r such that for all $x \in U$,

$$L(p) < L(x) < L(q).$$

Then the set U satisfies (5.1).

Next we show that there exist $\varepsilon' > 0$ and $\delta' > 0$ such that

- (1) $N(\delta', r') \subset U'$, where $U' = f^{-1}(U)$;
- (2) for any $x \in N(\delta', r') \setminus C^u(r')$, we can find $l \geq 1$ satisfying

$$(5.2) \quad d(f^{-l}(x), \overline{O(r, f)}) \geq \varepsilon'.$$

To show this, we take $\varepsilon_1 > 0$ satisfying $\varepsilon_1 \leq d(\Omega_1, \Omega_2)$. Choose $0 < \varepsilon_2 < \varepsilon_1/4$. Let

$$O_q(r) = O(r, f) \cap N(\varepsilon_2, q) \quad \text{and} \quad O_p(r) = O(r, f) \setminus N(\varepsilon_2, q).$$

Note that $d(O_p(r), \Omega_1) > 0$. Choose $\varepsilon_3 < \varepsilon_2$ satisfying

$$2\varepsilon_3 \leq d(O_p(r), \Omega_1).$$

For the constant ε_3 , there exists $\delta_1 > 0$ and $l_1 \geq 1$ such that for any $x \in N(\delta_1, r') \cap W^u(\Omega_1)$,

$$f^{-l}(x) \in N(\varepsilon_3, \Omega_1)$$

for all $l \geq l_1$. For this l_1 , we take $0 < \delta' < \delta_1$ such that for any $x \in N(\delta', r') \setminus C^u(r')$, there exists $l = l(\delta', x) \geq l_1$ satisfying

$$f^{-l}(x) \notin N(2\varepsilon_2, q)$$

(This is from the stable manifold theorem). It is easy to find $\varepsilon_4 > 0$ satisfying

$$d(\overline{O(r, f)}, \Omega_i) > \varepsilon_4$$

for $i \neq 1, 2$.

Let $\varepsilon' = \min\{\varepsilon_4, \varepsilon_3\}$, and take $x \in N(\delta', r') \setminus C^u(r')$. Then we can show that if $x \in W^u(\Omega_i)$ for $i \neq 1, 2$, there exists $l \geq 1$ such that

$$d(f^{-l}(x), \overline{O(r, f)}) \geq \varepsilon_4 \geq \varepsilon';$$

if $x \in W^u(\Omega_1)$, there exists $l \geq l_1$ such that

$$f^{-l}(x) \notin N(2\varepsilon_2, q).$$

So we have

$$d(f^{-l}(x), O_q(r)) > \varepsilon'$$

for some $l \geq l_1$. Since $2\varepsilon_3 < d(\Omega_1, O_p(r))$, we get

$$d(f^{-l}(x), O_p(r)) > \varepsilon'.$$

Moreover we have $f^{-l}(x) \in N(\varepsilon_3, \Omega_1)$. Combining these two statements, we obtain the inequality (5.2).

Let $C^s(r)$ be the connected component of $W^s(p, f)$ contained in U which contains r . By the same techniques as above, we can find $\varepsilon > 0$ and $\delta > 0$ such that

- (1) $N(\delta, r) \subset U$;
- (2) for any $x \in N(\delta, r) \setminus C^s(r)$, there exists $l \geq 1$ satisfying

$$(5.3) \quad d(f^l(x), \overline{O(r, f)}) \geq \varepsilon.$$

Now we show that f does not have the orbital inverse shadowing property. It is easy to construct a C^1 small perturbation g of f such that

$$(5.4) \quad \begin{aligned} g &= f \quad \text{on } M \setminus U', \quad g(U') = U, \quad \text{and} \\ g(N(\delta', r') \cap C^u(r')) \cap C^s(r) &= \emptyset. \end{aligned}$$

Let $\bar{\varepsilon} = \min\{\delta, \delta', \varepsilon/2, \varepsilon'/2\}$. Then the proof is completed by showing that there is no point $x \in M$ satisfying

$$d_H(\overline{O(x, g)}, \overline{O(r, f)}) \leq \bar{\varepsilon}.$$

Suppose that there exists a point $x \in M$ satisfying

$$d_H(\overline{O(x, g)}, \overline{O(r, f)}) \leq \bar{\varepsilon}.$$

Since U is wandering for g , we can see that the set $O(x, g) \cap N(\delta, r)$ contains only one point, say z ; and the set $O(x, g) \cap N(\delta', r')$ contains only one point, say z' . Then we have

$$g(z') = z, \quad O^+(z, f) = O^+(z, g) \quad \text{and} \quad O^-(z', f) = O^-(z', g).$$

The inequalities (5.2) and (5.3) imply that

$$z' \in N(\delta', r') \cap C^u(r') \quad \text{and} \quad z \in N(\delta, r) \cap C^s(r).$$

This contradicts to (5.4), and so completes the proof.

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