BULL. AUSTRAL. MATH. SOC. 10F35
VOL. 33 (1986), 59-65.

TRANSCENDENCE MEASURES BY MAHLER'S TRANSCENDENCE METHOD

PAUL-GEORG BECKER-LANDECK

Let $f(z)$ be an analytic function in the unit circle satisfying the functional equation $f(z) = a(z)\, f(z^\rho) + b(z)$, where ρ is a natural number and $a(z)$, $b(z)$ are polynomials. If α is an algebraic number, we give a transcendence measure for $f(\alpha)$. This improves earlier results of Galochkin and Miller.

1. Introduction and statement of the Theorem

Let $T : \mathbb{C} \longrightarrow \mathbb{C}$ be the transformation defined by $Tz = z^\rho$, where $\rho \in \mathbb{N}$, $\rho \neq 1$. Let $f(z)$ be a transcendental function which is holomorphic in the unit circle and which satisfies the functional equation

$$(1) \qquad f(z) = a(z)\, f(Tz) + b(z) ,$$

where $a(z)$ and $b(z)$ are polynomials with algebraic coefficients and $f(0)$ is algebraic.

For these functions and for a wider class of functions satisfying functional equations similar to (1), K. Mahler [5] has shown the following result:

Received 19 April, 1985.

https://doi.org/10.1017/S0004972700002860 Published online by Cambridge University Press

If α is an algebraic number, $0 < |\alpha| < 1$, and if $a(T^k\alpha) \neq 0$ for any $k \in \mathbb{N}_0$, then $F(\alpha)$ is transcendental.

The paper intends to prove a quantitative version of Mahler's result.

THEOREM. *Let* $f(z)$ *and* α *be as above. There exists a constant* $C > 0$, *such that for all polynomials* $Q(X) \in \mathbb{Z}[X]$ *with degree* d *and height* H *the following estimate holds*

$$|Q(f(\alpha))| > \exp(-Cd(\log H + d^2 \log d)) .$$

In the special case of the functions considered here the Theorem is an improvement of earlier results of A.I. Galochkin [3] and W. Miller [6]. Galochkin has shown

$$|Q(f(\alpha))| > \exp(-Cd \log H)$$

under the additional hypothesis $H > H_0(d)$, whereas Miller proved

$$|Q(f(\alpha))| > \exp(-Cd^2(\log H + d^2)) .$$

The transcendence type (see [8], p.100) of $f(\alpha)$ which can be bounded by 4 using Miller's result is now shown to be smaller than $3 + \varepsilon$ for arbitrary $\varepsilon > 0$.

2. Notation and preliminary results

The height $H(P)$, $H(\alpha)$ and the degree $\deg P$, $\deg \alpha$ of polynomials P and algebraic numbers α have their usual meaning. If α is an algebraic number, then $\overline{|\alpha|}$, the house of α, denotes the maximum over the absolute values of all conjugates of α. We define the length $\Lambda(P)$ of a polynomial P with algebraic coefficients to be the sum of the houses of the coefficients.

Let K be the algebraic number field containing the coefficients of $a(z)$ and $b(z)$ and the number $f(0)$. I_K denotes the ring of algebraic integers in K. The positive constants $C_1, C_2, \ldots$ are explicitly computable and will depend only on $a(z)$, $b(z)$, $f(z)$, ρ and α.

Without loss of generality we can assume in the proof: (i) $f(0) = 0$ and (ii) the coefficients of $b(z)$ are algebraic integers. Substituting $f(z)$ by $\lambda f(z) + \mu$ with appropriate algebraic

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numbers λ and μ we can show this easily. The Theorem is proved in an equivalent form as a statement concerning approximations $|\xi - f(\alpha)|$ for algebraic numbers ξ (see [4], p.61).

In the proof, ideas of Galochkin and Miller are combined with an improvement in the estimates for the houses and denominators of the power series coefficients of $f(z)$.

3. Proof of the Theorem

LEMMA 1. *For* $j \in \mathbb{N}_0$, $h \in \mathbb{N}$, *let us denote by* $f_{j,h}$ *the power series coefficients of* z^h *in the expansion for* $f(z)^j$. *There is a natural number* D_1 *with* $D_1^{j[\log h]+1} f_{j,h} \in I_K$ *for all* $j \in \mathbb{N}_0$, $h \in \mathbb{N}$. *Furthermore the following estimate holds*

$$\overline{|f_{j,h}|} \le C_1 \exp(C_2 j \log h) \ .$$

Proof. Iterating the functional equation $(k-1)$ times we write

$$f(z) = a^{(k)}(z)\, f(T^k z) + b^{(k)}(z) \ , \tag{2}$$

where

$$a^{(i)}(z) = a(z)\ldots a(T^{i-1}z)$$

and

$$b^{(k)}(z) = \sum_{i=0}^{k-1} a^{(i)}(z)\, b(T^i z) \ .$$

All $f_{1,h}$ with $h < \rho^k$ are determined by the second part of the right hand side in (2). Hence, we get

$$\overline{|f_{1,h}|} \le \sum_{i=0}^{k-1} \Lambda(b(T^i z)) \prod_{j=0}^{i-1} \Lambda(a(T^j z))$$

$$\le C_3^k \le C_1 \exp(C_2 \log h) \ .$$

The last inequality follows by choosing k with $\rho^{k-1} \le h < \rho^k$. From the recursion formula

$$f_{j,h} = \sum_{\ell=1}^{h-1} f_{j-1,\ell}\, f_{1,h-\ell} \ , \tag{3}$$

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the estimate of the lemma can now be deduced easily. Using (2) and (3) the second part of the lemma follows by a similar argument.

For a power series $g(z) = \sum_{\ell=0}^{\infty} \beta_\ell z^\ell$, we denote by $\operatorname{ord} g(z)$ the first index ℓ with $\beta_\ell \neq 0$.

LEMMA 2. *Let* m $(\geq C_4)$ *and* n $(\geq C_5 m)$ *be natural numbers. Then exists a polynomial* $P(w,z) \in \mathbb{Z}[w,z]$ *with the following properties:*

(a) $1 \leq \deg_w P \leq m$, $\deg_z P \leq n$;

(b) $H(P) \leq \exp(C_6 m \log nm)$;

(c) $\operatorname{ord} P(f(z),z) \geq C_7 nm$;

(d) $\operatorname{ord} P(f(z),z) \leq C_8 nm$.

Proof. Using Siegel's Lemma (see [7], p.141) we can construct a polynomial satisfying (a), (b) and (c). The estimate (d) is a consequence of Lemma 4 in [3]. The hypothesis $n \geq C(m)$ required there can be specified to $n \geq m$, if we restrict Galochkin's proof to the functions considered here.

We introduce the auxiliary parameter k and obtain the following lemma, which describes the growth of $\log|P(f(T^k\alpha), T^k\alpha)|$, $k \longrightarrow \infty$, for a polynomial P constructed according to Lemma 2.

LEMMA 3. *Let* m,n *and* k *be natural numbers satisfying* $m \geq C_4$, $n \geq C_5 m$ *and* $\rho^k \geq C_9 m \log nm$. *Then*

$$\exp(-C_{10} nm\rho^k) < |P(f(T^k\alpha), T^k\alpha)| < \exp(-C_{11} nm\rho^k) .$$

Proof. Let β_ℓ denote the coefficients of the power series expansion of $P(f(z),z)$ at 0. As a consequence of Lemma 2 we obtain

$$\log \overline{|\beta_\ell|} \leq C_{12} m \log \ell , \tag{4}$$

$$D_2^{m[\log \ell]} \beta_\ell \in I_K \tag{5}$$

with an appropriate $D_2 \in \mathbb{N}$. Assuming $|z| \leq \frac{1}{2}$ we easily deduce

(6) $$\left|\sum_{\ell=\lambda+1}^{\infty} \beta_\ell z^\ell\right| \le |z|^{\lambda+1} \exp(C_{13} m \log \lambda) ,$$

where $\lambda = \operatorname{ord} P(f(z),z)$. Liouville estimates combined with (4) and (5) yield

(7) $$\log |\beta_\lambda| \ge - C_{14} m \log \lambda .$$

Combining (6) and (7) we get

(8) $$(\tfrac{1}{2}) \ |\beta_\lambda z^\lambda| < |P(f(z),z)| < (\tfrac{3}{2}) \ |\beta_\lambda z^\lambda| ,$$

if $|z| \le \exp(-C_{15} m \log nm)$ is assumed. Supposing $z = T^k \alpha$, $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$, we see that this assumption is equivalent to $\rho^k \ge C_9 m \log nm$. Lemma 3 follows now directly from (8).

Let ξ be an algebraic number of degree d and height H . Guided by the functional equation (2) we define ξ_k by

$$\xi_k = (\xi - b^{(k)}(\alpha))/a^{(k)}(\alpha)$$

and similar to Lemma 4 from [6] the following lemma can be proved.

LEMMA 4. *Let* m , n *and* k *be natural numbers satisfying* $m, k \ge C_{16}$ *and* $n \ge C_{17} m$. *There is a positive number* C_{18} *with the following property. If*

$$|\xi - f(\alpha)| \le \exp(-C_{18} nm\rho^k)$$

then

$$|P(\xi_k, T^k\alpha) - P(f(T^k\alpha), T^k\alpha)| < \exp(-C_{10} nm\rho^k)$$

with the constant C_{10} *of Lemma* 3.

The next lemma yields a lower bound for $|P(\xi_k, T^k\alpha)|$.

LEMMA 5. *Let* m , n *and* k *be natural numbers satisfying* $m \ge C_{19}$, $n \ge C_{20}$ *and* $\rho^k \ge C_{21} \log nm$. *If* $P(\xi_k, T^k\alpha) \ne 0$, *then*

$$|P(\xi_k, T^k\alpha)| > H^{-C_{22} m} \exp(-C_{23} dn\rho^k) .$$

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Proof. For each non-negative integer k, we define $P_k(w,z) = a^{(k)}(z)^m P((w-b^{(k)}(z))/a^{(k)}(z), T^k z) \in K[w,z]$. From the definition of ξ_k it is obvious that $P(\xi_k, T^k\alpha) = P_k(\xi,\alpha)\, a^{(k)}(\alpha)^{-m}$. By straightforward calculation we get $\deg_w P \le m$, $\deg_z P \le C_{24}(m+n)\rho^k$, $\Lambda(P_k) \le \Lambda(P)\, C_{25}$ and with a number D_3 chosen appropriately by $D_3^{mk} P_k(w,z) \in I_K[w,z]$. Liouville estimates (see [3], Lemma 5) applied to $P_k(\xi,\alpha)$ and $a^{(k)}(\alpha)$ lead to the inequality of the lemma.

Proof of the Theorem. With constants large enough to satisfy the hypotheses of all lemmas we assume

(9) $$m \ge C_{26},\ n \ge C_{27}m,\ \rho^k \ge C_{28} m \log mn .$$

Using Lemma 3 and the triangle inequality we get

(10) $$\exp(-C_{11}nm\rho^k) > |P(f(T^k\alpha), T^k\alpha)| \ge$$
$$|P(\xi_k, T^k\alpha)| - |P(\xi_k, T^k\alpha) - P(f(T^k\alpha), T^k\alpha)| .$$

Under the additional assumption $|\xi - f(\alpha)| \le \exp(-C_{18}nm\rho^k)$ we infer by combining (10), Lemma 3, Lemma 4 and Lemma 5:

(11) $$\exp(-C_{11}nm\rho^k) > H^{-C_{22}m} \exp(-C_{23}nd\rho^k) - \exp(-C_{10}nm\rho^k) .$$

With the choice

(12) $$m \ge 2C_{23}C_{11}^{-1}d$$

it is an immediate consequence of (11) that $H > \exp(C_{29}n\rho^k)$ which is false, whenever

$$n\rho^k \ge C_{30} \log H .$$

Hence, the contrary of the assumption in Lemma 3 is true, and we have

(14) $$|\xi - f(\alpha)| > \exp(-C_{18}nm\rho^k) ,$$

if the parameters m, n and k are chosen with respect to the conditions (9), (12), and (13). From (14), the Theorem can be derived in the usual way.

4. Remarks

In the case $a(z) = \eta$ with η a root of unity the estimate of the theorem can be sharpened to

$$|Q(f(\alpha))| > \exp(-Cd(\log H + d^2 \log\log d)) .$$

The proof is quite similar to the above one, and therefore it is omitted. We note that we can also give a quantitative version of some of Mahler's results on algebraic independence (see [2]). Both that and the theorem proved here are parts of the author's thesis [1].

The author is grateful to P. Bundschuh for introducing him to this problem and for giving much advice.

References

[1] P.-G. Becker-Landeck, *Quantitative Resultate im Zusammenhang mit der Mahlerschen Transzendenzmethode* (Dissertation, Universität zu Köln, 1984).

[2] P.-G. Becker-Landeck, "Maße für algebraische Unabhängigkeit", (to appear).

[3] A.I. Galochkin, "Transcendence measure of values of functions satisfying certain functional equation", *Math. Notes* 27 (1980), 83-88.

[4] S. Lang, *Introduction to transcendental numbers*, (Addison-Wesley, Reading, 1966).

[5] K. Mahler, "Arithmetische Eigenschaften der Lösungan einer Klasse von Funktionalgleichungen", *Math. Ann.* 101 (1929), 342-366.

[6] W. Miller, "Transcendence measures by a method of Mahler", *J. Austral. Math. Soc.* Ser. A 32 (1982), 68-78.

[7] T. Schneider, *Einführung in die transzendenten Zahlen* (Springer, Berlin, 1957).

[8] M. Waldschmidt, *Nombres transcendants*, (Lecture notes in mathematics 402, Springer, Berlin, 1974).

Remigiusstrasse 37,
D-5000 KÖLN 41,
West Germany.