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EVALUATION OF CERTAIN EXOTIC $_3F_2(1)$-SERIES

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Abstract. A class of exotic $_3F_2(1)$-series is examined by integral representations, which enables the authors to present relatively easier proofs for a few remarkable formulae. By means of the linearization method, these $_3F_2(1)$-series are further extended with two integer parameters. A general summation theorem is explicitly established for these extended series, and several sample summation identities are highlighted as consequences.

§1. Introduction and motivation

For an indeterminate x and an integer $n \in \mathbb{Z}$, the shifted factorial (also called Pochhammer symbol) is defined as $(x)_0 \equiv 1$ and

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n>0,\\ \dfrac{1}{(x-1)(x-2)\cdots(x+n)}, & n<0,\end{cases}$$

whose multiparameter form is abbreviated compactly to

$$\begin{bmatrix} \alpha,\beta,\ldots,\gamma \\ A,B,\ldots,C \end{bmatrix}_n = \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}.$$

The shifted factorial can also be expressed as a Γ-function quotient

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad \text{where} \quad \Gamma(x) = \int_0^\infty \tau^{x-1}e^{-\tau}d\tau.$$

There are several useful properties of the Γ-function (see [15, §17]). One of them is the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Following [3], the classical hypergeometric series reads as

$${}_{1+p}F_p\left[\begin{matrix} a_0,\, a_1,\, \ldots,\, a_p \\ b_1,\, \ldots,\, b_p\end{matrix}\middle| z\right] = \sum_{k=0}^\infty \frac{(a_0)_k(a_1)_k\cdots(a_p)_k}{k!(b_1)_k\cdots(b_p)_k} z^k.$$

This series is always convergent for $|z|<1$. However, for $z=1$, the series converges only when the real part of the sum of the numerator parameters is less than that of the denominator parameters.

There exist numerous summation identities of hypergeometric series in the literature (see, e.g., [7], [8], [10]–[13]). Recently, Asakura *et al.* [1] examined the following exotic $_3F_2$-series:

$$\mathcal{F}(a,b,x) = {}_3F_2\left[\begin{matrix} a,\ b,\ x \\ a+b, 1+x\end{matrix}\middle| 1\right], \quad \text{where} \quad a,b,x \in \mathbb{Q}.$$

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Assuming further that

$$a,b,x,\ x-a,x-b,x-a-b \notin \mathbb{Z}$$

and

$$\{\lambda x\}+\{\lambda(x-a-b)\}=\{\lambda(x-a)\}+\{\lambda(x-b)\} \text{ for any } \lambda \in \hat{\mathbb{Z}}^{\times},$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$, they showed that

$$B(a,b)\times \mathcal{F}(a,b,x) \in \bar{\mathbb{Q}}+\bar{\mathbb{Q}}\log\bar{\mathbb{Q}}^{\times}.$$

Here, $\bar{\mathbb{Q}}+\bar{\mathbb{Q}}\log\bar{\mathbb{Q}}^{\times}$ denotes the $\bar{\mathbb{Q}}$-linear subspace spanned by $1, 2\pi\mathbf{i}$ and $\log\alpha$ for all $\alpha \in \bar{\mathbb{Q}}^{\times}$. This result characterizes the $\mathcal{F}(a,b,x)$ values in quality, but not in quantity, that is, it does not contribute to find explicit formulae for the $\mathcal{F}(a,b,x)$ values.

Some of the $\mathcal{F}(a,b,x)$ values were subsequently determined by Asakura and Yabu [2], who listed without proofs the $\mathcal{F}(\frac{1}{6},\frac{5}{6},x)$-values for all the irreducible proper rational numbers x with denominators in $\{2,3,4,6\}$. In particular, they provide a lengthy proof of five pages and half to the following elegant formula:

$$\mathcal{F}(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2}) = {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{2}\\ 1,\frac{3}{2}\end{matrix}\middle|1\right] = \frac{3\sqrt{3}}{2\pi}\ln(2+\sqrt{3}).$$

Chen [5] examined further the series $\mathcal{F}(\frac{1}{6},\frac{5}{6},x+n)$ with $n\in\mathbb{Z}$ and succeeded in expressing this series explicitly as a linear combination of π^{-1} and $\mathcal{F}(\frac{1}{6},\frac{5}{6},x)$ with rational coefficients.

The objective of this paper is twofold. First, we derive an integral representation for the $\mathcal{F}(\alpha,1-\alpha,\frac{1}{2})$-series, which enables us to provide relatively easy proofs for the closed formulae of $\mathcal{F}(\alpha,1-\alpha,\frac{1}{2})$ when α is one of the irreducible proper rational numbers with denominators belonging to $\{2,3,4,5,6,8,10,12\}$.

Second, we extend Chen's series by considering the following ${}_3F_2$-series:

$${}_3F_2\left[\begin{matrix}\alpha,1-\alpha, & x+m\\ 1, & 1+x+n\end{matrix}\middle|1\right] \quad \text{for} \quad m,n\in\mathbb{Z} \quad \text{subject to} \quad m\le n.$$

By means of the linearization method (cf. [6], [9], [14]), we show that this series is always evaluable as a linear function of $\mathcal{F}(\alpha,1-\alpha,x)$ with the coefficients being two sums of finite terms. In particular, when $x=\frac{1}{2}$, several explicit formulae for the above ${}_3F_2$-series are presented as consequences.

§2. Integral representation of $\mathcal{F}(\alpha,1-\alpha,\frac{1}{2})$

Recalling the Beta function (see [15, §16])

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \text{where} \quad a,b>0,$$

we have the equalities

$$\begin{aligned}B(\alpha+k,1-\alpha+k) &= \frac{(\alpha)_k(1-\alpha)_k}{(2k)!}B(\alpha,1-\alpha)\\ &= \frac{\pi}{\sin(\pi\alpha)}\times\frac{(\alpha)_k(1-\alpha)_k}{(2k)!}.\end{aligned}$$

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Then we can rewrite the $\mathcal{F}(\alpha,1-\alpha,\frac{1}{2})$-series as

$$\begin{aligned}\mathcal{F}(\alpha,1-\alpha,\tfrac{1}{2}) &= \sum_{k=0}^{\infty}\frac{(\alpha)_k(1-\alpha)_k}{(2k+1)\,(k!)^2}\\ &= \frac{\sin(\pi\alpha)}{\pi}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{B(\alpha+k,1-\alpha+k)}{2k+1}\\ &= \frac{\sin(\pi\alpha)}{\pi}\sum_{k=0}^{\infty}\binom{2k}{k}\int_0^1\frac{x^{k-1+\alpha}(1-x)^{k-\alpha}}{2k+1}dx\\ &= \frac{\sin(\pi\alpha)}{\pi}\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k(1-x)^k}{2k+1}dx,\end{aligned}$$

where the interchanging between summation and integral is justified by Lebesgue's monotone convergence theorem.

Taking into account the Maclaurin series

$$\arcsin y = \sum_{k=0}^{\infty}\binom{2k}{k}\frac{y^{2k+1}}{4^k(2k+1)},$$

we can evaluate, in closed forms,

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k(1-x)^k}{2k+1} = \frac{\arcsin\left(2\sqrt{x(1-x)}\right)}{2\sqrt{x(1-x)}}.$$

By substitution, we get the following integral expression

$$\mathcal{F}(\alpha,1-\alpha,\tfrac{1}{2}) = \frac{\sin(\pi\alpha)}{2\pi}\int_0^1\frac{\arcsin\left(2\sqrt{x(1-x)}\right)}{x^{\frac{3}{2}-\alpha}(1-x)^{\frac{1}{2}+\alpha}}dx. \tag{1}$$

According to $[0,1]=[0,\frac{1}{2}]\cup[\frac{1}{2},1]$, splitting the above integral into two and then making changes of variables $x=\sin^2\theta$ and $x=\cos^2\theta$, respectively, we can state the resulting expression as

$$\mathcal{F}(\alpha,1-\alpha,\tfrac{1}{2}) = \frac{2\sin(\pi\alpha)}{\pi}\Big\{I(\alpha)+I(1-\alpha)\Big\},$$
$$\text{where}\quad I(\alpha)=\int_0^{\frac{\pi}{4}}\frac{\theta}{\sin^{2\alpha}\theta\cos^{2-2\alpha}\theta}d\theta.$$

Making further change of variables $\theta=\arctan y$, we deduce the next expression:

$$I(\alpha)=\int_0^1\frac{\arctan y}{y^{2\alpha}}dy.$$

When $\alpha=\frac{1}{2}$, we come across the Catalan constant G (cf. [16]):

$$I(\tfrac{1}{2})=\int_0^1\frac{\arctan y}{y}dy=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 y^{2n}dy=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G.$$

From this, we recover the following remarkable evaluation formula due to Ramanujan (cf. [4, (29.3)]):

$$\mathcal{F}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = \frac{2\sin\frac{\pi}{2}}{\pi}(G+G) = \frac{4}{\pi}G.$$

By means of the integration by parts, we can further reformulate

$$\begin{aligned} I(\alpha)+I(1-\alpha) &= \int_0^1 \left\{\frac{\arctan y}{y^{2\alpha}} + \frac{\arctan y}{y^{2-2\alpha}}\right\} dy \\ &= \arctan y \times \left\{\frac{y^{1-2\alpha}}{1-2\alpha} - \frac{y^{2\alpha-1}}{1-2\alpha}\right\}_0^1 \\ &\quad + \frac{1}{1-2\alpha}\int_0^1 \frac{y^{2\alpha-1}-y^{1-2\alpha}}{1+y^2} dy. \end{aligned}$$

Keeping in mind that $0<\alpha<1$ with $\alpha \neq \frac{1}{2}$, we can check that the expression in the middle line vanishes. By making the substitution $y \to \sqrt{y}$ in the last integral, we get a slightly reduced one

$$I(\alpha)+I(1-\alpha) = \frac{1}{2-4\alpha}\int_0^1 \frac{y^{\alpha-1}-y^{-\alpha}}{1+y} dy.$$

Therefore, we finally arrive at an integral expression as in the theorem below.

THEOREM 1. *Let α be a real number with $0<\alpha<1$ and $\alpha \neq \frac{1}{2}$. The following formula holds:*

$$\mathcal{F}(\alpha,1-\alpha,\tfrac{1}{2}) = \frac{\sin(\pi\alpha)}{\pi(1-2\alpha)}\int_0^1 \frac{y^{\alpha-1}-y^{-\alpha}}{1+y} dy.$$

When α is a rational number, the above integral can be transformed into that about rational functions. By assigning α to small rational numbers, we can compute, by means of this theorem, the exact values of the corresponding $\mathcal{F}(\alpha,1-\alpha,\frac{1}{2})$-series. The first values are highlighted in the following proposition.

PROPOSITION 2. (Evaluation formulae for series $\mathcal{F}(\alpha,1-\alpha,\frac{1}{2})$)

$$\begin{aligned} &\text{(a)}\ \mathcal{F}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = \frac{4}{\pi}G, \\ &\text{(b)}\ \mathcal{F}(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2}) = \frac{3\sqrt{3}}{\pi}\ln 2, \\ &\text{(c)}\ \mathcal{F}(\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2}) = \frac{4}{\pi}\ln(1+\sqrt{2}), \\ &\text{(d)}\ \mathcal{F}(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2}) = \frac{3\sqrt{3}}{2\pi}\ln(2+\sqrt{3}). \end{aligned}$$

Among these four values, the first one (a) is already determined.

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Proof of (b). By making the substitution $y \to x^3$, we can rewrite

$$\mathcal{F}(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2}) = \frac{3\sin\frac{\pi}{3}}{\pi}\int_0^1 \frac{y^{-\frac{2}{3}}-y^{-\frac{1}{3}}}{1+y}dy = \frac{9\sqrt{3}}{2\pi}\int_0^1 \frac{1-x}{1+x^3}dx.$$

According to the partial fraction decomposition

$$\frac{1-x}{1+x^3} = \frac{2}{3(1+x)} + \frac{1-2x}{3(1-x+x^2)},$$

the integral can be evaluated as follows:

$$\begin{aligned}\mathcal{F}(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2}) &= \frac{3\sqrt{3}}{2\pi}\int_0^1 \left\{\frac{2}{1+x} + \frac{1-2x}{1-x+x^2}\right\}dx \\ &= \frac{3\sqrt{3}}{2\pi}\Big(2\ln 2 + 0\Big) = \frac{3\sqrt{3}}{\pi}\ln 2.\end{aligned}$$

This confirms the value given in (b). ∎

Proof of (c). By making the substitution $y \to x^4$, we can rewrite

$$\mathcal{F}(\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2}) = \frac{2\sin\frac{\pi}{4}}{\pi}\int_0^1 \frac{y^{-\frac{3}{4}}-y^{-\frac{1}{4}}}{1+y}dy = \frac{4\sqrt{2}}{\pi}\int_0^1 \frac{1-x^2}{1+x^4}dx.$$

According to the partial fraction decomposition

$$\frac{1-x^2}{1+x^4} = \frac{1}{2\sqrt{2}}\left\{\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right\},$$

the integral can be evaluated as follows:

$$\begin{aligned}\mathcal{F}(\tfrac{1}{4},\tfrac{3}{4},\tfrac{1}{2}) &= \frac{2}{\pi}\int_0^1 \left\{\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right\}dx \\ &= \frac{2}{\pi}\Big\{\ln(2+\sqrt{2}) - \ln(2-\sqrt{2})\Big\} = \frac{4}{\pi}\ln(1+\sqrt{2}).\end{aligned}$$

This confirms the value given in (c). ∎

Proof of (d). By making the substitution $y \to x^6$, we can rewrite

$$\mathcal{F}(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2}) = \frac{3\sin\frac{\pi}{6}}{2\pi}\int_0^1 \frac{y^{-\frac{5}{6}}-y^{-\frac{1}{6}}}{1+y}dy = \frac{9}{2\pi}\int_0^1 \frac{1-x^4}{1+x^6}dx.$$

According to the partial fraction decomposition

$$\frac{1-x^4}{1+x^6} = \frac{\sqrt{3}}{6}\left\{\frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1} - \frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1}\right\},$$

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the integral can be evaluated as follows:

$$\begin{aligned}\mathcal{F}(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2}) &= \frac{3\sqrt{3}}{4\pi}\int_0^1 \left\{\frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}-\frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1}\right\}dx\\ &= \frac{3\sqrt{3}}{4\pi}\left\{\ln(2+\sqrt{3})-\ln(2-\sqrt{3})\right\} = \frac{3\sqrt{3}}{2\pi}\ln(2+\sqrt{3}).\end{aligned}$$

This confirms the value given in (d). ∎

By carrying out the same procedure, we can further evaluate, in closed forms, the following series by expressing their values in terms of radicals:

$$\mathcal{F}(\tfrac{1}{5},\tfrac{4}{5},\tfrac{1}{2}) = \frac{5\sqrt{5-\sqrt{5}}}{6\pi\sqrt{2}}\left\{\sqrt{5}\ln\frac{\sqrt{5}+1}{\sqrt{5}-1}+2\ln 2\right\},$$

$$\mathcal{F}(\tfrac{2}{5},\tfrac{3}{5},\tfrac{1}{2}) = \frac{5\sqrt{5+\sqrt{5}}}{2\pi\sqrt{2}}\left\{\sqrt{5}\ln\frac{\sqrt{5}+1}{\sqrt{5}-1}-2\ln 2\right\};$$

$$\mathcal{F}(\tfrac{1}{8},\tfrac{7}{8},\tfrac{1}{2}) = \frac{2\sqrt{2}}{3\pi}\ln\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}+\frac{2(2-\sqrt{2})}{3\pi}\ln\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}},$$

$$\mathcal{F}(\tfrac{3}{8},\tfrac{5}{8},\tfrac{1}{2}) = \frac{2\sqrt{2}}{\pi}\ln\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}-\frac{2(2+\sqrt{2})}{\pi}\ln\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}};$$

$$\begin{aligned}\mathcal{F}(\tfrac{1}{10},\tfrac{9}{10},\tfrac{1}{2}) &= \frac{5\sqrt{10-2\sqrt{5}}}{16\pi}\ln\frac{2\sqrt{5-\sqrt{5}}+\sqrt{10}}{2\sqrt{5-\sqrt{5}}-\sqrt{10}}\\ &\quad+\frac{5\sqrt{5-2\sqrt{5}}}{8\pi}\ln\frac{2\sqrt{5+\sqrt{5}}+\sqrt{10}}{2\sqrt{5+\sqrt{5}}-\sqrt{10}},\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\tfrac{3}{10},\tfrac{7}{10},\tfrac{1}{2}) &= \frac{5\sqrt{10+2\sqrt{5}}}{8\pi}\ln\frac{2\sqrt{5-\sqrt{5}}+\sqrt{10}}{2\sqrt{5-\sqrt{5}}-\sqrt{10}}\\ &\quad-\frac{5\sqrt{5+2\sqrt{5}}}{4\pi}\ln\frac{2\sqrt{5+\sqrt{5}}+\sqrt{10}}{2\sqrt{5+\sqrt{5}}-\sqrt{10}};\end{aligned}$$

$$\mathcal{F}(\tfrac{1}{12},\tfrac{11}{12},\tfrac{1}{2}) = \frac{24\ln(1+\sqrt{2})}{5\pi(1+\sqrt{3})}+\frac{3\sqrt{3}}{5\pi(1+\sqrt{3})}\ln\frac{5+2\sqrt{6}}{5-2\sqrt{6}},$$

$$\mathcal{F}(\tfrac{5}{12},\tfrac{7}{12},\tfrac{1}{2}) = \frac{24\ln(1+\sqrt{2})}{\pi(1-\sqrt{3})}-\frac{3\sqrt{3}}{\pi(1-\sqrt{3})}\ln\frac{5+2\sqrt{6}}{5-2\sqrt{6}}.$$

§3. Extension by integer parameters m and n

In this section, we examine the series

$${}_3F_2\left[\begin{matrix}\alpha,1-\alpha,x+m\\1,1+x+n\end{matrix}\middle|1\right] \quad\text{for}\quad m,n\in\mathbb{Z}\quad\text{with}\quad m\le n.$$

By means of the linearization method (cf. [6], [9], [14]), a surprising formula (see Theorem 5) is shown, which expresses the above series on the left as a linear function of $\mathcal{F}(\alpha,1-\alpha,x)$ with the coefficients being two finite sums.

We begin with the following recurrence relation.

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LEMMA 3. (Recurrence relation)

$$\mathcal{F}(\alpha,1-\alpha,x)=\frac{(x+\alpha)(1+x-\alpha)}{x(1+x)}\mathcal{F}(\alpha,1-\alpha,x+1)-\frac{\sin(\pi\alpha)}{\pi x}.$$

Proof. As done by Chen [5], by putting the initial term aside, we can write

$$\begin{aligned}\mathcal{F}(\alpha,1-\alpha,x)&=1+\sum_{k=1}^{\infty}\frac{(\alpha)_k(1-\alpha)_k(x)_k}{(k!)^2(1+x)_k}\qquad\boxed{k\to k+1}\\&=1+\sum_{k=0}^{\infty}\frac{(\alpha)_k(1-\alpha)_k}{(k!)^2}\left\{\frac{(\alpha+k)(1-\alpha+k)x}{(k+1)^2(1+x+k)}\right\}.\end{aligned}$$

According to the partial fractions

$$\frac{(\alpha+k)(1-\alpha+k)x}{(k+1)^2(1+x+k)}=\frac{\alpha-\alpha^2+x+x^2}{(1+x+k)x}+\frac{\alpha-\alpha^2}{(k+1)^2}-\frac{\alpha-\alpha^2+x}{(k+1)x},$$

we can evaluate, by making use of the Gauss summation theorem [3, §1.3]

$$ {}_2F_1\left[\begin{matrix}a,b\\c\end{matrix}\middle|1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad\text{where}\quad\Re(c-a-b)>0,\tag{2}$$

the three corresponding sums

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(\alpha)_k(1-\alpha)_k}{(k!)^2}\frac{\alpha-\alpha^2+x+x^2}{(1+x+k)x}&=\frac{(x+\alpha)(1+x-\alpha)}{x(1+x)}\\&\quad\times{}_3F_2\left[\begin{matrix}\alpha,1-\alpha,1+x\\1,\ 2+x\end{matrix}\middle|1\right],\\\sum_{k=0}^{\infty}\frac{(\alpha)_k(1-\alpha)_k}{(k!)^2}\frac{\alpha-\alpha^2}{(k+1)^2}&=-1+{}_2F_1\left[\begin{matrix}\alpha-1,-\alpha\\1\end{matrix}\middle|1\right]\\&=-1+\frac{\Gamma(1)\ \Gamma(2)}{\Gamma(1+\alpha)\Gamma(2-\alpha)},\\\sum_{k=0}^{\infty}\frac{(\alpha)_k(1-\alpha)_k}{(k!)^2}\frac{\alpha-\alpha^2+x}{(k+1)x}&=\frac{\alpha-\alpha^2+x}{x}{}_2F_1\left[\begin{matrix}\alpha,1-\alpha\\2\end{matrix}\middle|1\right]\\&=\frac{\alpha-\alpha^2+x}{x}\frac{\Gamma(1)\ \Gamma(2)}{\Gamma(1+\alpha)\Gamma(2-\alpha)}.\end{aligned}$$

Putting them together and making some simplifications, we find that

$$\begin{aligned}\mathcal{F}(\alpha,1-\alpha,x)&=\frac{(x+\alpha)(1+x-\alpha)}{x(1+x)}{}_3F_2\left[\begin{matrix}\alpha,1-\alpha,1+x\\1,\ 2+x\end{matrix}\middle|1\right]-\frac{(\alpha-\alpha^2)\ \Gamma(1)\ \Gamma(2)}{x\ \Gamma(1+\alpha)\Gamma(2-\alpha)}\\&=\frac{(x+\alpha)(1+x-\alpha)}{x(1+x)}\mathcal{F}(\alpha,1-\alpha,1+x)-\frac{\sin(\pi\alpha)}{\pi x}.\end{aligned}$$

This proves the recurrence relation in Lemma 3. ∎

Iterating n times the equation in Lemma 3 gives rise to the following general one:

$$\begin{aligned}\mathcal{F}(\alpha,1-\alpha,x)=\mathcal{F}(\alpha,1-\alpha,x+n)&\begin{bmatrix}x+\alpha,1+x-\alpha\\x,\ 1+x\end{bmatrix}_n\\&-\frac{\sin(\pi\alpha)}{\pi x}\sum_{k=0}^{n-1}\begin{bmatrix}x+\alpha,1+x-\alpha\\1+x,\ 1+x\end{bmatrix}_k.\end{aligned}\tag{3}$$

PROPOSITION 4. $(n\in\mathbb{N}_0)$

$$\begin{aligned}\text{(a) }\mathcal{F}(\alpha,1-\alpha,x+n)=&\begin{bmatrix}x,\ 1+x\\x+\alpha,1+x-\alpha\end{bmatrix}_n\Bigg\{\mathcal{F}(\alpha,1-\alpha,x)\\&+\frac{\sin(\pi\alpha)}{\pi x}\sum_{k=0}^{n-1}\begin{bmatrix}x+\alpha,1+x-\alpha\\1+x,\ 1+x\end{bmatrix}_k\Bigg\},\\\text{(b) }\mathcal{F}(\alpha,1-\alpha,x-n)=&\begin{bmatrix}\alpha-x,1-\alpha-x\\-x,\ 1-x\end{bmatrix}_n\Bigg\{\mathcal{F}(\alpha,1-\alpha,x)\\&-\frac{\sin(\pi\alpha)}{\pi x}\sum_{k=1}^{n}\begin{bmatrix}-x,\ -x\\\alpha-x,1-\alpha-x\end{bmatrix}_k\Bigg\}.\end{aligned}$$

It should be pointed out that when $\alpha=\frac{1}{6}$, the corresponding transformation formulae in Proposition 4 were previously obtained by Chen [5].

Proof. The first formula (a) is obtained by inverting (3).

To prove the second one (b), making the replacement $x\to x-n$ in (3), we have the following two equalities:

$$\begin{aligned}\frac{(x+\alpha-n)_n(1+x-\alpha-n)_n}{(x-n)_n(1+x-n)_n}&=\begin{bmatrix}\alpha-x,1-\alpha-x\\1-x,\ -x\end{bmatrix}_n,\\\frac{(x+\alpha-n)_k(1+x-\alpha-n)_k}{(x-n)_{k+1}(1+x-n)_k}&=\frac{1}{x}\begin{bmatrix}\alpha-x,1-\alpha-x\\1-x,\ -x\end{bmatrix}_n\\&\quad\times\begin{bmatrix}-x,\ -x\\\alpha-x,1-\alpha-x\end{bmatrix}_{n-k}.\end{aligned}$$

Then the resulting identity under the replacement $k\to n-k$ becomes (b). This completes the proof of Proposition 4. ∎

Observing further that

$$\begin{aligned}\begin{bmatrix}\alpha-x,1-\alpha-x\\-x,\ 1-x\end{bmatrix}_{-n}&=\begin{bmatrix}x,\ 1+x\\x+\alpha,1+x-\alpha\end{bmatrix}_n,\\\begin{bmatrix}-x,\ -x\\\alpha-x,1-\alpha-x\end{bmatrix}_{-k}&=\begin{bmatrix}x+\alpha,1+x-\alpha\\1+x,\ 1+x\end{bmatrix}_k,\end{aligned}$$

we can unify, for $m\in\mathbb{Z}$, the two formulae in Proposition 4 to a single one

$$\mathcal{F}(\alpha,1-\alpha,x+m)=\left[\begin{matrix}x,\ 1+x\\ x+\alpha,1+x-\alpha\end{matrix}\right]_m\left\{\mathcal{F}(\alpha,1-\alpha,x)+\operatorname{sgn}(m)\frac{\sin(\pi\alpha)}{\pi x}\sum_{k\in\sigma(m)}\left[\begin{matrix}x+\alpha,1+x-\alpha\\ 1+x,\ 1+x\end{matrix}\right]_k\right\},\tag{4}$$

where $\operatorname{sgn}(k)$ and $\sigma(m)$ are given, respectively, by

$$\operatorname{sgn}(k)=\begin{cases}1, & k>0,\\ 0, & k=0,\\ -1, & k<0,\end{cases}\quad\text{and}\quad \sigma(m)=[0,m)\cup[m,0)=\begin{cases}[0,m), & m>0,\\ [m,0), & m<0.\end{cases}$$

Now, we are ready to prove the following general theorem.

THEOREM 5. $(m,n\in\mathbb{Z}$ with $m\le n)$

$$_3F_2\left[\begin{matrix}\alpha,1-\alpha,x+m\\ 1,1+x+n\end{matrix}\middle|1\right]=\sum_{k=m}^{n}\binom{-x-m}{k-m}\binom{x+n}{n-k}\left[\begin{matrix}x,\ 1+x\\ x+\alpha,1+x-\alpha\end{matrix}\right]_k \times\left\{\mathcal{F}(\alpha,1-\alpha,x)+\operatorname{sgn}(k)\frac{\sin(\pi\alpha)}{\pi x}\sum_{j\in\sigma(k)}\left[\begin{matrix}x+\alpha,1+x-\alpha\\ 1+x,\ 1+x\end{matrix}\right]_j\right\}.$$

The importance of the above theorem lies in the fact that for any pair $m,n\in\mathbb{Z}$ with $m\le n$, the $_3F_2$-series on the left results always in a linear function of $\mathcal{F}(\alpha,1-\alpha,x)$ with the coefficients being two sums of finite terms. Therefore, for any specific integers m,n and an irreducible rational number x considered in the last section, we can explicitly compute, by means of this theorem, the corresponding $_3F_2$-series in closed forms.

Proof. Recall the Chu–Vandermonde convolution formula

$$1=\sum_{k=m}^{n}\binom{-x-m-i}{k-m}\binom{x+n+i}{n-k}.$$

By putting the above relation inside the $_3F_2$-series and then applying the equations

$$\binom{-x-m-i}{k-m}=\binom{-x-m}{k-m}\frac{(x+k)_i}{(x+m)_i},$$
$$\binom{x+n+i}{n-k}=\binom{x+n}{n-k}\frac{(1+x+n)_i}{(1+x+k)_i},$$

we can manipulate the double sum as follows:

$$\begin{aligned}_3F_2\left[\begin{matrix}\alpha,1-\alpha,x+m\\ 1,1+x+n\end{matrix}\middle|1\right]&=\sum_{i=0}^{\infty}\left[\begin{matrix}\alpha,1-\alpha,x+m\\ 1,1,1+x+n\end{matrix}\right]_i\sum_{k=m}^{n}\binom{-x-m-i}{k-m}\binom{x+n+i}{n-k}\\ &=\sum_{k=m}^{n}\binom{-x-m}{k-m}\binom{x+n}{n-k}\sum_{i=0}^{\infty}\left[\begin{matrix}\alpha,1-\alpha,x+k\\ 1,1,1+x+k\end{matrix}\right]_i.\end{aligned}$$

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The resulting equality can be stated as

$$ {}_3F_2\left[\begin{matrix}\alpha,1-\alpha,x+m\\1,1+x+n\end{matrix}\Big|1\right]=\sum_{k=m}^{n}\binom{-x-m}{k-m}\binom{x+n}{n-k}\mathcal{F}(\alpha,1-\alpha,x+k). \tag{5}$$

Now, replacing the rightmost series $\mathcal{F}(\alpha,1-\alpha,x+k)$ by (4), we arrive at the formula displayed in Theorem 5. □

By specifying $x=\frac{1}{2}$ and $\alpha\in\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\right\}$ in Theorem 5, we exhibit below several concrete infinite series identities in conjunction with Proposition 2.

Corollary 6. *Twenty formulae for* $\boldsymbol{{}_3F_2}={}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}+m\\1,\frac{3}{2}+n\end{matrix}\right]$ *with* $\mathcal{A}=\frac{4G}{\pi}$*:*

m	n	$\boldsymbol{{}_3F_2}$	m	n	$\boldsymbol{{}_3F_2}$
0	0	$\mathcal{A}$	-3	-3	$\frac{178}{45\pi}$
0	1	$\frac{9\mathcal{A}}{8}-\frac{3}{4\pi}$	-3	-2	$\frac{12}{5\pi}$
0	2	$\frac{615\mathcal{A}}{512}-\frac{285}{256\pi}$	-3	-1	$\frac{16}{15\pi}$
0	3	$\frac{5{,}145\mathcal{A}}{4{,}096}-\frac{24{,}955}{18{,}432\pi}$	-2	-2	$\frac{10}{3\pi}$
1	1	$\frac{3\mathcal{A}}{4}+\frac{3}{2\pi}$	-2	-1	$\frac{4}{3\pi}$
1	2	$\frac{105\mathcal{A}}{128}+\frac{45}{64\pi}$	-2	0	$\frac{3\mathcal{A}}{8}+\frac{13}{12\pi}$
1	3	$\frac{1{,}785\mathcal{A}}{2{,}048}+\frac{1{,}015}{3{,}072\pi}$	-2	1	$\frac{33\mathcal{A}}{64}+\frac{79}{96\pi}$
2	2	$\frac{45\mathcal{A}}{64}+\frac{65}{32\pi}$	-1	-1	$\frac{2}{\pi}$
2	3	$\frac{385\mathcal{A}}{512}+\frac{2765}{2{,}304\pi}$	-1	0	$\frac{\mathcal{A}}{2}+\frac{1}{\pi}$
3	3	$\frac{175\mathcal{A}}{256}+\frac{2{,}723}{1{,}152\pi}$	-1	1	$\frac{21\mathcal{A}}{32}+\frac{9}{16\pi}$

Corollary 7. *Twenty formulae for* $\boldsymbol{{}_3F_2}={}_3F_2\left[\begin{matrix}\frac{1}{3},\frac{2}{3},\frac{1}{2}+m\\1,\frac{3}{2}+n\end{matrix}\right]$ *with* $\mathcal{B}=\frac{3\sqrt{3}}{\pi}\ln 2$*:*

m	n	$\boldsymbol{{}_3F_2}$	m	n	$\boldsymbol{{}_3F_2}$
0	0	$\mathcal{B}$	-3	-3	$\frac{1{,}001\mathcal{B}}{6{,}561}+\frac{7{,}021}{1{,}215\sqrt{3}\pi}$
0	1	$\frac{39\mathcal{B}}{35}-\frac{27\sqrt{3}}{70\pi}$	-3	-2	$\frac{287\mathcal{B}}{2{,}187}+\frac{2{,}879}{810\sqrt{3}\pi}$
0	2	$\frac{1{,}185\mathcal{B}}{1{,}001}-\frac{2{,}295\sqrt{3}}{4{,}004\pi}$	-3	-1	$\frac{187\mathcal{B}}{2{,}187}+\frac{2{,}723}{1{,}620\sqrt{3}\pi}$
0	3	$\frac{56{,}993\mathcal{B}}{46{,}189}-\frac{128{,}853\sqrt{3}}{184{,}756\pi}$	-2	-2	$\frac{35\mathcal{B}}{243}+\frac{44}{9\sqrt{3}\pi}$
1	1	$\frac{27\mathcal{B}}{35}+\frac{27\sqrt{3}}{35\pi}$	-2	-1	$\frac{23\mathcal{B}}{243}+\frac{37}{18\sqrt{3}\pi}$
1	2	$\frac{837\mathcal{B}}{1{,}001}+\frac{729\sqrt{3}}{2{,}002\pi}$	-2	0	$\frac{107\mathcal{B}}{243}+\frac{59}{36\sqrt{3}\pi}$
1	3	$\frac{40{,}797\mathcal{B}}{46{,}189}+\frac{31{,}833\sqrt{3}}{184{,}756\pi}$	-1	-1	$\frac{\mathcal{B}}{9}+\frac{\sqrt{3}}{\pi}$
2	2	$\frac{729\mathcal{B}}{1{,}001}+\frac{1{,}044\sqrt{3}}{1{,}001\pi}$	-1	0	$\frac{5\mathcal{B}}{9}+\frac{\sqrt{3}}{2\pi}$
2	3	$\frac{35{,}721\mathcal{B}}{46{,}189}+\frac{57{,}267\sqrt{3}}{92{,}378\pi}$	-1	1	$\frac{73\mathcal{B}}{105}+\frac{39\sqrt{3}}{140\pi}$
3	3	$\frac{32{,}805\mathcal{B}}{46{,}189}+\frac{55{,}989\sqrt{3}}{46{,}189\pi}$	-1	2	$\frac{6{,}997\mathcal{B}}{9{,}009}+\frac{547\sqrt{3}}{4{,}004\pi}$

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COROLLARY 8. *Twenty formulae for* $\boldsymbol{{}_3F_2} = {}_3F_2\left[\begin{matrix}\frac14,\frac34,\frac12+m\\1,\frac32+n\end{matrix}\right]$ *with* $\mathcal{C} = \frac{4}{\pi}\ln(1+\sqrt{2})$:

m	n	$\boldsymbol{{}_3F_2}$	m	n	$\boldsymbol{{}_3F_2}$
0	0	$\mathcal{C}$	-3	-3	$\frac{21\mathcal{C}}{64} + \frac{149}{40\sqrt{2}\pi}$
0	1	$\frac{11\mathcal{C}}{10} - \frac{2\sqrt{2}}{5\pi}$	-3	-2	$\frac{37\mathcal{C}}{128} + \frac{559}{240\sqrt{2}\pi}$
0	2	$\frac{65\mathcal{C}}{56} - \frac{25}{21\sqrt{2}\pi}$	-3	-1	$\frac{103\mathcal{C}}{512} + \frac{1{,}141}{960\sqrt{2}\pi}$
0	3	$\frac{8{,}267\mathcal{C}}{6{,}864} - \frac{287}{198\sqrt{2}\pi}$	-2	-2	$\frac{5\mathcal{C}}{16} + \frac{19}{6\sqrt{2}\pi}$
1	1	$\frac{4\mathcal{C}}{5} + \frac{4\sqrt{2}}{5\pi}$	-2	-1	$\frac{7\mathcal{C}}{32} + \frac{17}{12\sqrt{2}\pi}$
1	2	$\frac{6\mathcal{C}}{7} + \frac{8\sqrt{2}}{21\pi}$	-2	0	$\frac{67\mathcal{C}}{128} + \frac{53}{48\sqrt{2}\pi}$
1	3	$\frac{257\mathcal{C}}{286} + \frac{2\sqrt{2}}{11\pi}$	-2	1	$\frac{811\mathcal{C}}{1{,}280} + \frac{397}{480\sqrt{2}\pi}$
2	2	$\frac{16\mathcal{C}}{21} + \frac{68\sqrt{2}}{63\pi}$	-1	-1	$\frac{\mathcal{C}}{4} + \frac{\sqrt{2}}{\pi}$
2	3	$\frac{344\mathcal{C}}{429} + \frac{64\sqrt{2}}{99\pi}$	-1	0	$\frac{5\mathcal{C}}{8} + \frac{1}{\sqrt{2}\pi}$
3	3	$\frac{320\mathcal{C}}{429} + \frac{124\sqrt{2}}{99\pi}$	-1	1	$\frac{119\mathcal{C}}{160} + \frac{11}{20\sqrt{2}\pi}$

COROLLARY 9. *Twenty formulae for* $\boldsymbol{{}_3F_2} = {}_3F_2\left[\begin{matrix}\frac16,\frac56,\frac12+m\\1,\frac32+n\end{matrix}\right]$ *with* $\mathcal{D} = \frac{3\sqrt{3}}{2\pi}\ln(2+\sqrt{3})$:

m	n	$\boldsymbol{{}_3F_2}$	m	n	$\boldsymbol{{}_3F_2}$
0	0	$\mathcal{D}$	-3	-3	$\frac{3{,}584\mathcal{D}}{6{,}561} + \frac{6{,}469}{3{,}645\pi}$
0	1	$\frac{69\mathcal{D}}{64} - \frac{27}{64\pi}$	-3	-2	$\frac{1{,}088\mathcal{D}}{2{,}187} + \frac{1{,}378}{1{,}215\pi}$
0	2	$\frac{8{,}067\mathcal{D}}{7{,}168} - \frac{4{,}509}{7{,}168\pi}$	-3	-1	$\frac{1{,}661\mathcal{D}}{4{,}374} + \frac{1{,}561}{2{,}430\pi}$
0	3	$\frac{151{,}963\mathcal{D}}{131{,}072} - \frac{100{,}485}{131{,}072\pi}$	-2	-2	$\frac{128\mathcal{D}}{243} + \frac{41}{27\pi}$
1	1	$\frac{27\mathcal{D}}{32} + \frac{27}{32\pi}$	-2	-1	$\frac{98\mathcal{D}}{243} + \frac{20}{27\pi}$
1	2	$\frac{1{,}593\mathcal{D}}{1{,}792} + \frac{729}{1{,}792\pi}$	-2	0	$\frac{1{,}249\mathcal{D}}{1{,}944} + \frac{121}{216\pi}$
1	3	$\frac{60{,}399\mathcal{D}}{65{,}536} + \frac{12{,}879}{65{,}536\pi}$	-2	1	$\frac{90{,}431\mathcal{D}}{124{,}416} + \frac{5{,}735}{13{,}824\pi}$
2	2	$\frac{729\mathcal{D}}{896} + \frac{1{,}017}{896\pi}$	-1	-1	$\frac{4\mathcal{D}}{9} + \frac{1}{\pi}$
2	3	$\frac{13{,}851\mathcal{D}}{16{,}384} + \frac{11{,}259}{16{,}384\pi}$	-1	0	$\frac{13\mathcal{D}}{18} + \frac{1}{2\pi}$
3	3	$\frac{6{,}561\mathcal{D}}{8{,}192} + \frac{53{,}829}{40{,}960\pi}$	-1	1	$\frac{623\mathcal{D}}{768} + \frac{69}{256\pi}$

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