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Transitivity and the centre for maps of the circle

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Abstract. We study the dynamics of continuous maps of the circle with periodic points. We show that the centre is the closure of the periodic points and that the depth of the centre is at most two. We also characterize the property that every power is transitive in terms of transitivity of a single power and some periodic data.

1. *Introduction*

In this paper, we study the dynamics of continuous maps of the circle to itself. We prove analogues of results for maps of the interval due to A. N. Sarkovskii and to M. Barge and J. Martin.

In [**8**], Sarkovskii showed that for maps of the interval, the centre is the closure of the periodic points, and that the depth of the centre is at most two. The analogous result for maps of the circle is:

THEOREM A. *For continuous maps of the circle with periodic points, the centre is the closure of the periodic points, and the depth of the centre is at most two.*

For continuous maps of the circle without periodic points, the centre and its depth are easily determined from the description of such maps given by J. Auslander and Y. Katznelson [**1**], see § 3.

In [**2**], [**3**], Barge and Martin proved results which yield the equivalence of (1)–(4) in the following theorem.

THEOREM B (Barge & Martin). *Let $f: I \to I$ be a continuous map of a compat interval to itself. Then the following statements are equivalent.*

(1) *f is transitive and has a point of odd period greater than one.*
(2) *f^2 is transitive.*
(3) *f^n is transitive for every $n > 0$.*
(4) *f is topologically mixing.*

Furthermore, if f is piecewise monotone, then the following statement is equivalent to the rest.

(5) *For every interval $J \subseteq I$, there is an n such that $f^n(J) = I$.*

We complete the proof of theorem B by showing that (3) implies (5) if f is piecewise monotone, and we prove the following analogue of theorem B for maps of the circle.

THEOREM C. *Let $f: S \to S$ be a continuous map of the circle to itself. Then the following statements are equivalent.*

(1) *There is an m such that f^m is transitive and has a fixed point and a point of odd period greater than one.*

(2) *There is an m such that f^{2m} is transitive and f^m has a fixed point.*

(3) *f^n is transitive for every $n > 0$ and f has periodic points.*

(4) *f is topologically mixing.*

Furthermore, if f is piecewise monotone, then the following statement is equivalent to the rest.

(5) *For every interval $J \subseteq S$, there is an n such that $f^n(J) = S$.*

The authors thank Louis Block for stimulating remarks about the centre, which resulted in a much-improved § 3 of this paper.

2. *Preliminaries*

Throughout this paper $f: X \to X$ will denote a continuous map of a compact metric space to itself; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. (We reserve the lower-case letters i, j, k, m and n for non-negative integers.)

A set $E \subseteq X$ is *invariant* if $f(E) \subseteq E$, E^o and $\bar{E}$ denote the interior and closure of E, and the *orbit* of $x \in X$ is $\text{Orb}(x) = \{f^n(x) \mid n \geq 0\}$.

A point $x \in X$ is *periodic* if there is an $n > 0$ such that $f^n(x) = x$. The least such n is called the *period* of x. A point of period one is called a *fixed point*. x is *non-wandering* if for every neighbourhood U of x, there is an $n > 0$ such that $U \cap f^{-n}(U) \neq \emptyset$. We denote the periodic and non-wandering points of f by $P(f)$ and $\Omega(f)$.

The *centre* of f, denoted $C(f)$, is defined as follows. Let $\Omega_0 = X$, $\Omega_\alpha = \Omega(f | \Omega_{\alpha-1})$ for successor ordinals α, and $\Omega_\alpha = \bigcap_{\beta<\alpha} \Omega_\beta$ for limit ordinals α. There is a least (necessarily countable) ordinal δ such that $\Omega_\delta = \Omega_{\delta+1} = \ldots$. The centre of f is Ω_δ and δ is called the *depth of the centre*.

A map $f: X \to X$ is called *transitive* if any (and hence all) of the following equivalent conditions holds.

(1) For every pair U, V of non-empty open sets, there is an n such that $f^{-n}(U) \cap V \neq \emptyset$.

(2) The only closed invariant set K with $K^o \neq \emptyset$ is $K = X$.

(3) If $E^o \neq \emptyset$, then $\overline{\bigcup_{n\geq 0} f^n(E)} = X$.

(4) f is onto and there is a point with a dense orbit.

f is called *topologically mixing* if for every pair U, V of non-empty open sets, there is an N such that $f^{-n}(U) \cap V \neq \emptyset$ for every $n \geq N$.

Since we shall be concerned with transitivity for powers of a map, the following lemmas will be useful. They are probably folk-results. In any case, we omit the straightforward proofs.

LEMMA 2.1. *Let $n \geq 2$. If $f: X \to X$ is transitive but f^n is not, then there is a closed set $K \neq X$ with $K^o \neq \emptyset$ and a divisor $m > 1$ of n ($m = n$ is possible) such that*

(1) $f^m(K)=K$;
(2) $K\cup f(K)\cup\cdots\cup f^{m-1}(K)=X$;
(3) $[f^i(K)\cap f^j(K)]^o=\varnothing$ *for* $0\le i, j\le m-1$.

LEMMA 2.2. *If* f^m *and* f^n *are transitive, then so is* f^{mn}.

We will show in § 3 that for a transitive map of the circle with periodic points, the periodic points are dense. Thus for any such map f and for any $n>0$, f^n has dense periodic points, although f^n need not be transitive.

LEMMA 2.3. *If the periodic points of f are dense, then for every connected set E with* $E^0\neq\varnothing$, $\bigcup_{n\ge 0} f^n(E)$ *has finitely many components. These components have non-empty interior and are permuted by f.*

If X is either a compact interval I or the circle S, then by an *interval* $J\subseteq X$ we mean a non-degenerate, proper sub-interval of X.

Each continuous $f:S\to S$ can be lifted to (countably many) continuous maps $F:\mathbb{R}\to\mathbb{R}$ satisfying $f\circ\pi=\pi\circ F$, where $\pi:\mathbb{R}\to S=\mathbb{R}/\mathbb{Z}$ is the canonical projection. Whenever f has a lift F such that $F(I)\subseteq I$ for some compact interval $I\supseteq[0,1]$, this happens for maps of degree zero and for certain maps of degree ±1, we will abuse notation and write F and π in place of $F|I$ and $\pi|I$. Note that in this case, F is a map of the interval and f is a factor of F.

3. *The centre and its depth*: *theorem* A

A. N. Sarkovskii [**8**] showed that for maps of the interval, the centre is the closure of the periodic points, and that the depth of the centre is at most two. (For proofs in English, see [**6**] or [**9**].) Theorem A, proved in this section, is the analogous result for maps of the circle.

We say that an interval $J\subseteq S$ is *one-way* (with respect to $f:S\to S$) if whenever x, y, $f^m(x)$, $f^n(y)\in J$ and the circle is oriented so that if $[x,f^m(x)]\subseteq J$, then $[y,f^n(y)]\subseteq J$.

LEMMA 3.1. *If J is a one-way open interval, then* $J\cap\Omega(f|\Omega(f))=\varnothing$.

Proof. As in [**6**] or [**9**], one shows that if $x\in J\cap\Omega(f)$, then $f^n(x)\notin J$ for every $n>0$, and the result follows. □

LEMMA 3.2. *Let J be an interval which contains no periodic points. If* $\bigcup_{k\ge0} f^k(J)$ *is not the whole circle, then J is one-way.*

Proof. (*à la* Coppel [**6**]). Fix $z\notin\bigcup_{k\ge0} f^k(J)$. Suppose that x, $f^m(x)\in J$ and that the circle is oriented so that $[x,f^m(x)]\subseteq J$.

Claim: $f^{km}(x)\in[f^m(x), z)$ for every $k>0$.

If not, let k be the least positive integer for which the claim fails. Thus $k\ge2$ and $f^{(k-1)m}(x)\in[f^m(x), z)$. Since $z\notin f^{(k-1)m}[x,f^m(x)]$, it follows that $[x,f^m(x)]\cup f^{(k-1)m}[x,f^m(x)]$ is an interval and $f^{(k-1)m}$ maps x and $f^m(x)$ in opposite directions in this interval. Therefore $f^{(k-1)m}$ has a fixed point in $[x,f^m(x)]\subseteq J$. Since this is impossible, the claim is proved.

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Now suppose that $y, f^n(y) \in J$ and that $[y, f^n(y)]$ is not a subset of J. Then $[f^n(y), y] \subseteq J$ and the argument above shows that $f^{kn}(y) \in (z, f^n(y)]$ for every $k > 0$. It follows that $f^{mn}(x) \in [f^m(x), z)$, $f^{mn}(y) \in (z, f^n(y)]$, and $x \neq y$. Let K be the subinterval of J whose endpoints are x and y. Then $z \notin f^{mn}(K)$ and so $K \cup f^{mn}(K)$ is an interval. But f^{mn} moves the endpoints of K in opposite directions in this interval, and hence f^{mn} has a fixed point in K. This contradicts the assumption that there are no periodic points in J. □

LEMMA 3.3. *If f has non-zero degree and $f(x) \neq x$, then there are disjoint open intervals J and K such that $x \in J = f(K)$.*

We omit the straightforward proof.

THEOREM A. *For continuous maps of the circle with periodic points, the centre is the closure of the periodic points and the depth of the centre is at most two.*

Proof. If f has degree zero, let $F: I \to I$ be a lift of f to a map of a compact interval $I \supseteq [0, 1]$. The projection map $\pi: I \to S$ is finite-to-one, hence $\pi[P(F)] = P(f)$ and [**5**, proposition 3], $\pi[\Omega(F)] = \Omega(f)$. By Sarkovskii's result, $\Omega(F|\Omega(F)) = \overline{P(F)}$, hence $\Omega(f|\Omega(f)) = \overline{P(f)}$.

Suppose then that f has non-zero degree. It suffices to show that $\Omega(f|\Omega(f)) \subseteq \overline{P(f)}$. Suppose $x \notin \overline{P(f)}$. We may assume that for every open interval J containing x, $\bigcup_{k \geq 0} f^k(J)$ is the whole circle, for otherwise by lemmas 3.1 and 3.2, $x \notin \Omega(f|\Omega(f))$ and the proof is complete.

Let E be the orbit of a periodic point, and let n be the number of points in E. Since $f^i(x) \neq x$ for $1 \leq i \leq n$, there is an open interval K containing x such that $K \cap f^i(K) = \varnothing$ for $1 \leq i \leq n$. Applying lemma 3.3 to f^n, there are disjoint open intervals J_0 and J_n such that $x \in J_n = f^n(J_0)$. Without loss of generality, $J_n \subseteq K$ and there are no periodic points in $\overline{J_n}$. Let $J_i = f^i(J_0)$, $1 \leq i \leq n-1$. Then the collection $J_0, \ldots, J_n$ is pairwise disjoint. Let y be any point in a component of $S - (J_0 \cup \cdots \cup J_n)$ which doesn't meet E. There is an i such that $f^i(J_n) \cap E \neq \varnothing$ and an m such that $y \in f^m(J_n)$. Since f^i is onto, we can choose $m \geq i$. Thus $f^m(J_n)$ contains y and meets E, and hence J_n f^m-covers [**4**] at least one of the intervals $J_0, \ldots, J_n$. It follows that J_n f^k-covers itself for some $k \geq m$. But then $\overline{J_n}$ contains a periodic point. □

By embedding the appropriate maps of the interval in the circle, one may construct examples of maps of the circle with periodic points whose centres have depths zero, one, and two.

A map of the circle without periodic points is either topologically conjugate to an irrational rotation or is much like a Denjoy homeomorphism: there is a unique minimal (Cantor) set M such that for each component J of $S - M$, $f^k(J) \cap J = \varnothing$ for every $k > 0$ [**1**]. In the first case, the centre is the whole circle and its depth is zero, and in the second case, the centre is the minimal set M and its depth is one.

COROLLARY 3.4. *For transitive maps of the circle with periodic points, the periodic points are dense.*

This follows from theorem A and the fact that, in general, for transitive maps every point is non-wandering and hence the centre is the whole space.

4. *Transitivity for maps of the interval*: *theorem* B

In this section we complete the proof of theorem B and derive some consequences of it.

THEOREM B (Barge & Martin). *Let $f: I \to I$ be a continuous map of a compact interval to itself. Then the following statements are equivalent.*

(1) *f is transitive and has a point of odd period greater than one.*

(2) *f^2 is transitive.*

(3) *f^n is transitive for every $n > 0$.*

(4) *f is topologically mixing.*

Furthermore, if f is piecewise monotone, then the following statement is equivalent to the rest.

(5) *For every interval $J \subseteq I$, there is an n such that $f^n(J) = I$.*

We remark that (1)-(4) hold but (5) does not for the map in [**3**, example 5].

The implications (5)⇒(4)⇒(3)⇒(2) are immediate consequences of the definitions, (2)⇒(1) is theorem 13 of [**3**], (1)⇒(2)⇒(3) is lemma 2 of [**2**], and (3)⇒(4) is part of the proof of theorem 6 of [**3**].

To complete the proof of theorem B it suffices to show:

LEMMA 4.1. *Let $f: I \to I$ be piecewise monotone. If f^n is transitive for every $n > 0$, then for every interval $J \subseteq I$, there is an n such that $f^n(J) = I$.*

Proof. If there is an interval $K \subseteq I^o$ such that $f^2(K) = I$, then the conclusion of the lemma holds. For if $J \subseteq I$ is an interval, by [**3**, theorem 6] there is an n such that $f^n(J) \supseteq K$. Hence $f^{n+2}(J) = I$.

So we suppose that $f^2(K) \neq I$ for every interval $K \subseteq I^o$. Without loss of generality $I = [0, 1]$. Since f is onto, either $f^{-1}(0) \subseteq \{0, 1\}$ or $f^{-1}(1) \subseteq \{0, 1\}$. It follows that either $f^{-2}(0) = \{0\}$ or $f^{-2}(1) = \{1\}$. We assume the former.

Let t be the smallest turning point of f^2. Then f^2 has no fixed points in $(0, t)$ and $f^2(x) > x$ for every $x \in (0, t)$. (If not, then $f^2(x) < x$ for every $x \in (0, t)$, and hence $[0, t]$ is f^2-invariant.) But $f^2[t, 1] \subseteq [s, 1]$ for some $s > 0$. Thus $f^{2n}[t, 1] \subseteq [s, 1]$ for every $n > 0$, contradicting the assumption that f^2 is transitive. □

A standard compactness argument shows that the integer $n = n(J)$ in the lemma depends only on the length of J.

A piecewise monotone map $f: I \to I$ is called *strongly transitive* [7] if for every interval $J \subseteq I$, there is an n such that $\bigcup_{k=0}^{n} f^k(J) = I$.

LEMMA 4.2. *Every transitive, piecewise monotone map of the interval is strongly transitive.*

Proof. Let $f: I \to I$ be such a map.

If f^2 is transitive, then f is strongly transitive by theorem B.

If f^2 is not transitive, then by [**3**, lemma 2], $I = K \cup L$ where K and L are closed intervals with one point in common, $f(K) = L$ and $f(L) = K$, and $f^2|K$ and $f^2|L$ are transitive. It is easy to see that $(f^2|K)^2$ and $(f^2|L)^2$ are transitive. Now let $J \subseteq I$ be an interval. We may assume that $J \subseteq K$. By theorem B applied to $f^2|K$, there is an n such that $f^{2n}(J) = K$. Then $f^{2n}(J) \cup f^{2n+1}(J) = I$. □

COROLLARY 4.3. *Every transitive, piecewise monotone map f of the interval is topologically conjugate to a piecewise linear map whose linear pieces have slopes $\pm\beta$ (where $\log\beta$ is the topological entropy of f).*

Proof. Strongly transitive maps have this property [7, theorem 5]. □

5. *Transitivity for maps of the circle: theorem* C

THEOREM 5.1. *If f is a transitive map of the circle which has a fixed point, then f^n is transitive for every odd $n>0$.*

Proof. Suppose not and let $n\geq 3$ be the least odd integer such that f^n is not transitive. Let K and m be as in lemma 2.1. Then $m=n$, for otherwise f^m is not transitive.

Let J be a component of K with $J^o\neq\varnothing$. By corollary 3.4, the periodic points of f and hence of f^n are dense, and so by lemma 2.3, $J^*=\overline{\bigcup_{k\geq 0} f^{kn}(J)}$ has finitely many components; these components are intervals and they are permuted by f^n. Since $f^n(J^*)=J^*$, it follows that

$$J^*\cup f(J^*)\cup\cdots\cup f^{n-1}(J^*)=S,$$

and

$$[f^i(J^*)\cap f^j(J^*)]^o=\varnothing$$

whenever $0\leq i, j\leq n-1$.

Let p be a fixed point and let L be the component of J^* which contains p. Then $p\in L\cap f(L)\cap\cdots\cap f^{n-1}(L)$. Since $n\geq 3$, we must have $[f^i(L)\cap f^j(L)]^o\neq\varnothing$ for some i, j with $0\leq i, j\leq n-1$. But then $[f^i(J^*)\cap f^j(J^*)]^o\neq\varnothing$, which is not the case. □

We remark that the same proof shows that if f is a transitive map of the interval, then f^n is transitive for every odd $n>0$.

LEMMA 5.2 (cf [3, theorem 6]). *If f has periodic points and f^n is transitive for every $n>0$, then for every interval $J\subseteq S$, $\bigcup_{k\geq 0}[f^k(J)]^o$ misses at most one point. If such a point exists, then it is a fixed point.*

Proof. To show that $\bigcup[f^k(J)]^o$ misses at most one point, it suffices to show that for every $x\neq y$, the union contains either x or y.

We may assume that $x, y\notin J^o$, for otherwise there is nothing to prove. Let K be the open interval with endpoints x and y which is disjoint from J^o. By corollary 3.4, the periodic points are dense, so there is a periodic point $p\in J^o$. Let n be the period of p. Since f^n is transitive, there is a k such that $f^{-kn}(K)\cap J^o\neq\varnothing$. Thus $f^{kn}(J)$ is an interval which contains $p\in J^o$ and meets K. It follows that $[f^{kn}(J)]^o$ contains either x or y.

Now suppose that $\bigcup[f^k(J)]^o$ misses z. If z is not a fixed point, then there is an open interval K such that $z\in K$ and $f^{-1}(K)\cap K=\varnothing$. As in the proof of [3, theorem 6], there is an N such that $f^n(J)\supseteq S-K$ for every $n\geq N$. Since $S-K\supseteq f^{-1}(K)$, we have $f^{N+1}(J)$ contains both K and $S-K$. Thus $f^{N+1}(J)=S$, which is impossible. □

THEOREM C. *Let $f: S \to S$ be a continuous map of the circle to itself. Then the following statements are equivalent.*

(1) *There is an m such that f^m is transitive and has a fixed point and a point of odd period greater than one.*

(2) *There is an m such that f^{2m} is transitive and f^m has a fixed point.*

(3) *f^n is transitive for every $n > 0$ and f has periodic points.*

(4) *f is topologically mixing.*

Furthermore, if f is piecewise monotone, then the following statement is equivalent to the rest.

(5) *For every interval $J \subseteq S$, there is an n such that $f^n(J) = S$.*

Before proceeding with the proof we make two remarks. First, it is easy to construct maps of the circle for which the conditions of the theorem hold but which have no fixed points. Thus the unspecified m in (1) and (2) is necessary. Second, (1)-(4) hold but (5) does not for the map obtained by identifying the endpoints in the map of the interval in [**3**, example 5].

Proof. We show (1)⇒(2)⇒(3)⇒(4), (3)⇒(5) if f is piecewise monotone, and (4)⇒(1). (5)⇒(4) follows from the definition of topological mixing.

(1)⇒(2). It suffices to assume that $m = 1$ and show that f^2 is transitive.

Suppose that f^2 is not transitive. By lemma 2.1, there is a closed set $K \neq S$ such that $K^o \neq \varnothing, f^2(K) = K, K \cup f(K) = S$, and $[K \cap f(K)]^o = \varnothing$. Let L be a component of K with $L^o \neq \varnothing$ and let $L^* = \overline{\bigcup_{n \geq 0} f^{2n}(L)}$. By corollary 3.4 and lemma 2.3, L^* has finitely many components, each is a closed interval, and they are permuted by f^2. Thus $L^* \cup f(L^*)$ is the whole circle, and f cyclically permutes the intervals which are the components of L^* and $f(L^*)$ - otherwise f isn't transitive.

Suppose that L^* has k components. Then so does $f(L^*)$ and the components of L^* and $f(L^*)$ alternate around the circle. Let p be a point with odd period $n > 1$. Since $[L^* \cap f(L^*)]^o = \varnothing$ and $f^2(L^*) = L^*$, it follows that $p \in L^* \cap f(L^*)$. Since f cyclically permutes the components of L^* and $f(L^*)$, f^{2k} maps each of the two components containing p to itself, and hence $f^{2k}(p) = p$. Thus $2k$ is a multiple of n, and since n is odd, k is a multiple of n. On the other hand, f^{2n} maps these two components to themselves, so $2n$ is a multiple of $2k$, i.e. n is a multiple of k. Therefore $n = k$. But then the two points in $L^* \cap f(L^*)$ adjacent to p (one on each side of p) both have period $2k$. But this is impossible, since $L^* \cap f(L^*)$ is invariant and contains exactly $2k$ points.

(2)⇒(3). Let $g = f^m$. By theorem 5.1, g^n is transitive for every odd $n > 0$, and so by lemma 2.2, g^n is transitive for every $n > 0$. Thus f^n is transitive for every $n > 0$.

(3)⇒(4) and (3)⇒(5) if f is piecewise monotone. Suppose first that there is an interval J and an $n > 0$ such that $\bigcup_{k \geq 0} [f^{kn}(J)]^o$ is not the whole circle. Then by lemma 5.2, this union misses exactly one point, which is a fixed point of f^n. It follows that f^n has a lift $F: \mathbb{R} \to \mathbb{R}$ such that $F[0, 1] \subseteq [0, 1]$. The same conclusion holds (with $n = 1$) if there is no interval K such that $f(K) = S$. In either case, as in § 2 we write F in place of $F|[0, 1]$. For every $k > 0$, f^{kn} lifts to $F^k: [0, 1] \to [0, 1]$

and dense orbits lift to dense orbits. Therefore F^k is transitive for every $k>0$. Then by theorem B, F is topologically mixing, and hence so is f^n. But then f is topologically mixing as well. If, in addition, f is piecewise monotone, then so (by definition) is F, and again by theorem B, (5) holds for F, and hence also for f^n and f.

So we assume that for every interval J and for every $n>0$, $\bigcup_{k\geq 0}[f^{kn}(J)]^o$ is the whole circle and that there is an interval K such that $f(K)=S$. We show that for every interval J, there is an n such that $f^n(J)=S$. Replacing J by a smaller interval, we may assume that either $\bar{J}\cap\bar{K}=\emptyset$ or $J\subseteq K$.

Suppose that $\bar{J}\cap\bar{K}=\emptyset$. Choose periodic points p and q, one in each component of $S-(\bar{J}\cup\bar{K})$. There is an n such that $f^n(J)$ contains both p and q. Hence $f^n(J)$ contains either J or K. If $f^n(J)\supseteq J$, then $\{[f^{kn}(J)]^o\}$ is an increasing sequence of open sets whose union is the whole circle. By compactness, $f^{kn}(J)=S$ for some k. If $f^n(J)\supseteq K$, then $f^{n+1}(J)=S$.

Suppose that $J\subseteq K$. Choose distinct periodic points p and q in $S-\bar{K}$. Again there is an n such that $f^n(J)$ contains both p and q. Thus $f^n(J)$ contains either K or J', the closed interval with endpoints p and q which is disjoint from $\bar{K}$. As above, there is an m such that $f^m(J')=S$. If $f^n(J)\supseteq K$, then $f^{n+1}(J)=S$, and if $f^n(J)\supseteq J'$, then $f^{n+m}(J)=S$.

(4)$\Rightarrow$(1). It suffices to show that f has a non-periodic, non-wandering point with a finite orbit. For then by [**5**, theorem B+], f has points with periods $n<m$ where m/n is not a power of 2, and hence there is a k such that f^{kn} has a fixed point and a point of odd period greater than one.

A transitive map of the circle without periodic points is topologically conjugate to an irrational rotation [**1**] and hence is not topologically mixing. Thus f has periodic points. By corollary 3.4, the periodic points are dense and so every point is non-wandering. Let p be a point with period greater than one (if all the periodic points were fixed, then f would be the identity map) and let J be an interval. We may assume that $J\cap \mathrm{Orb}\,(p)=\emptyset$. Since $\bigcup_{k\geq 0}[f^k(J)]^o$ misses at most one point, which must be a fixed point of f, this union must contain p. Thus J contains the desired non-periodic, non-wandering point with a finite orbit. □

REFERENCES

[1] J. Auslander & Y. Katznelson. Continuous maps of the circle without periodic points. *Israel J. Math.* **32** (1979), 375–381.

[2] M. Barge & J. Martin. Chaos, periodicity and snakelike continua. Preprint, 1984.

[3] M. Barge & J. Martin. Dense orbits on the interval. Preprint, 1984.

[4] L. Block. Periodic orbits of continuous mappings of the circle. *Trans. Amer. Math. Soc.* **260** (1980), 555–562.

[5] L. Block, E. Coven, I. Mulvey & Z. Nitecki. Homoclinic and nonwandering points for maps of the circle. *Ergod. Th. & Dynam. Sys.* **3** (1983), 521–532.

[6] A. Coppel. Continuous maps of an interval. Xeroxed notes, 1984.

[7] W. Parry. Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.* **122** (1966), 368–378.

[8] A. N. Sarkovskiĭ. Fixed points and the center of a continuous mapping of the line into itself. (Ukrainian, Russian and English summaries.) *Dopovidi Akad. Nauk. Ukrain.* RSR **1964**, 865–868.

[9] J.-C. Xiong. $\Omega(f|\Omega(f))=\overline{P(f)}$ for every continuous map f of the interval. *Kexue Tongbao* (English Ed.) **28** (1983), 21–23.

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