6
Lattice techniques

6.1 Introduction

As we mentioned in the previous chapter, the definition of Yang–Mills theories in the continuum in terms of loops requires a regularization and the resulting eigenvalue equations are, in the non-Abelian case, quite involved. Lattice techniques appear to be a natural way to deal with both these difficulties. First of all since on a lattice there is a minimum length (the lattice spacing), the theory is naturally regularized. An important point is that this is a gauge invariant regularization technique. Secondly, formulating a theory on a lattice reduces an infinite-dimensional problem to a finite-dimensional one. It is set naturally to be analyzed using a computer.

Apart from these technical advantages, the reader may find interest in this chapter from another viewpoint. In terms of lattices one can show explicitly in simple models many of the physical behaviors of Wilson loops that we could only introduce heuristically in previous chapters.

Lattice gauge theories were first explored in 1971 by Wegner [104]. He considered a usual Ising model with up and down spins but with a local symmetry. He associated a spin to each link in the lattice and considered an action that was invariant under a spin-flip of all the spins associated with links emanating from a vertex. He noted that this model could undergo phase transitions, but contrary to what happens with usual Ising models, his model did not magnetize. The absence of the magnetization posed him with the problem of distinguishing the phases of the theory. That lead him to introduce correlation functions associated with loops ("loop correlation functions") and to find laws of area and perimeter very much in the same spirit as the ones introduced in the previous chapter.

In a similar fashion, usual gauge theories can be introduced in the lattice, associating to each link an element of the corresponding gauge group.

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Many lattice formulations of a certain theory may be written down. This is totally analogous to trying to discretize a differential equation in the sense that many discretized versions of a single equation in the continuum may exist.

The first application of lattice techniques to Yang–Mills theories is due to Wilson [48], who showed how to quantize a gauge theory on a lattice using path integral techniques. Making a Wick rotation to a Euclidean spacetime, he showed that the computation of the Green functions of the field theory essentially coincides with the computation of the correlation functions of a Euclidean four-dimensional statistical mechanics system. Wilson also noticed that the lattice theory admits a strong coupling regime in which there are no free quarks, i.e., confinement appears explicitly. The strong coupling expansion is not completely satisfactory since it does not preserve Lorentz invariance. Kogut and Susskind [49] were the first to introduce a Hamiltonian formulation for lattice gauge theories. In this case space is discretized but time is retained as a continuous variable. They studied the $SU(2)$ theory, which then becomes a quantum mechanical problem, and they studied the strong coupling expansion, which becomes the usual time independent perturbation theory.

Exploiting the connection pointed out above between gauge theories and four-dimensional statistical mechanical problems has allowed the introduction of Monte Carlo techniques for the covariant description of lattice field theories. These computational techniques were developed in the 1950s and a widely used practical implementation is due to Metropolis *et al.* [105]. In the context of lattice gauge theories these techniques were first applied by Wilson [106] and further developed by Creutz [107]. The application of these methods has allowed a concrete prediction of the mass spectra of the physical excitations of the theory and has been implemented on supercomputers yielding values of elementary excitations within 10% error of experimental measurements.

The main limitation of lattice approaches is that the number of degrees of freedom increases very rapidly with lattice size. The situation is worse in higher dimensions and when the theory is coupled to fermions. Progress in lattice approaches to gauge theories is therefore more dependent on the development of new analytical techniques and the identification of the relevant degrees of freedom than on the development of faster computers. We have argued in previous sections that loops are natural objects for describing gauge theories in a gauge invariant fashion. This raises hopes that the loop representation could be a useful tool for addressing some of the difficulties that arise in lattice formulations. The lattice context is very useful for putting in a concrete and rigorous setting many of the formal results discussed in the previous chapters and for gaining an intuitive feel for the loop representation. Loop representations on the lattice have

been developed for the $Z(2)$ model [108] and more recently some progress has been made for non-Abelian gauge theories. Concrete calculations in $2+1$ dimensions have been performed for $SU(2)$ [109, 110] and $SU(3)$ [95].

One is usually interested in the continuum limit of a lattice gauge theory. This involves shrinking the separation between lattice points to zero and increasing the number of lattice points to infinity in such a way that distances are conserved. The theory on the lattice involves interactions between the variables in different lattice sites. These interactions give rise to correlations. If the correlations are short range with respect to sites, when taking the continuum limit the correlations vanish for non-vanishing lengths. Therefore, in order to have a non-trivial continuum limit, a lattice model needs to allow a regime (at least for some value of the coupling constants) such that the system becomes scale-free and long range correlations appear. These regimes correspond to second class phase transitions in the statistical mechanics sense, and it is in this regime that the continuum limit is usually taken.

The organization of this chapter is as follows. In the following section we discuss as a toy model the $Z(2)$ model on the lattice. We analyze it in the covariant and Hamiltonian versions in terms of the usual variables and then study the loop representation. In the following section we repeat the analysis for $SU(2)$ and in the last section we discuss the inclusion of fermions in an open-path representation.

6.2 Lattice gauge theories: the $Z(2)$ example

Wegner [104] introduced the Ising lattice gauge theories in 1971. He was interested in building a model similar to the Ising model but which did not exhibit spontaneous magnetization and which had a non-trivial phase structure. He wanted to study how to characterize the phases of a model without local order parameters.

The treatment of this section will follow closely the presentation due to Kogut [74], to which we refer the reader for further details.

6.2.1 Covariant lattice theory

Consider a cubic lattice in a three-dimensional Euclidean spacetime. We label the lattice sites by a triplet of integers n and the unit vectors on the lattice which we characterize by unit vectors along the lattice directions. The lattice is oriented and the unit vectors will have a $+$ or $-$ sign in front according to their orientations. Notice that each link corresponds to two possible arrangements of (n, μ), since $(n, \mu) = (n + \mu, -\mu)$. At each link

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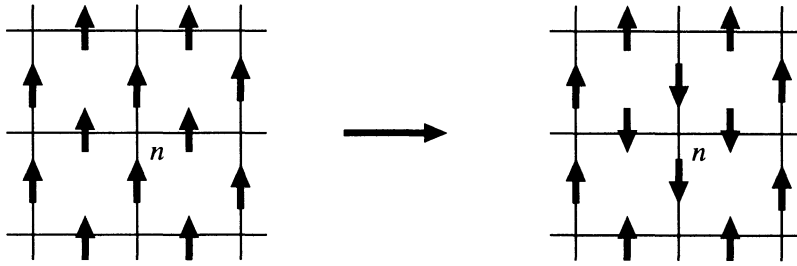

Fig. 6.1. The local gauge transformations of the Wegner $Z(2)$ model.

we associate an Ising spin $\sigma = \pm 1$ and therefore each configuration of the system is associated with an assignment of a spin orientation to each link of the lattice.

Consider now a transformation (which we will call "a local gauge transformation") such that it flips all the spins connected with one site in the lattice. An example is shown in figure 6.1.

We now consider the following action for the model

$$S(\sigma) = -\beta \sum_{n,\mu,\nu} \sigma(n,\mu)\sigma(n+\mu,\nu)\sigma(n+\mu+\nu,-\mu)\sigma(n+\mu,-\nu), \quad (6.1)$$

given by the sum of the products of all spins around each elementary plaquette of the lattice. β is the coupling constant of the model. This action is invariant under the gauge transformations introduced above. This can be readily checked noticing that a gauge transformation at n simultaneously changes the sign of $\sigma(n,\mu)$ and $\sigma(n+\mu,-\nu)$. Notice that any product of spins around any loop on the lattice will be invariant. We readily see how for this simple model the ideas of loop and holonomy play an important role.

If one attempts to define an order parameter for this model in the same spirit as the one defined for the Ising model — the magnetization — one finds that the statistical mean value of such an order parameter identically vanishes. This is a particular case of the result due to Elitzur [111] that states that taking the statistical mean value of a local gauge dependent quantity averages it over the gauge orbits. For a compact Lie group (or a discrete group like $Z(2)$) this means that the mean value vanishes.

Wegner proposed the idea of considering as an order parameter for the model the gauge invariant quantity

$$W_\gamma = \prod_{l\in\gamma} \sigma(l), \quad (6.2)$$

which represents the product of all the spins situated at the links l that compose the closed loop γ on the lattice. This idea appears natural

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in view of what we discussed in the previous chapter, but it should be remembered that historically it appeared before Wilson's proposal. It was the first time that a "Wilson loop" was proposed as an order parameter for a gauge theory.

The statistical mean value of the operator W_γ is given by

$$< W_\gamma >= \frac{1}{Z} \sum_{\sigma_l} W_\gamma \exp(-S(\sigma)), \tag{6.3}$$

where $Z = \sum_{\sigma_l} \exp(-S(\sigma))$ is the partition function. The summations on σ_l above mean summing over all possible spin configurations on the lattice (i.e., assignments of spin values to the links). Notice that in this context one can reinterpret β as the inverse temperature in a statistical model. Wegner proved that at small values of β (high temperatures), $< W_\gamma > \sim \exp(-\text{area}(\gamma))$. At low temperatures — large values of the coupling constant — it decreases as $\sim \exp(-\text{length}(\gamma))$ and therefore the expectation value of this operator allows us to distinguish the high and low temperature phases.

Let us discuss the proof of the area behavior via a high temperature expansion. We start by considering the identity

$$\begin{aligned} \exp(-S(\sigma)) &= \exp(\beta \sigma\, \sigma\, \sigma\, \sigma) \\ &= \cosh(\beta) + \sigma\, \sigma\, \sigma\, \sigma \sinh(\beta) \qquad (6.4) \\ &= (1 + \sigma\, \sigma\, \sigma\, \sigma \tanh(\beta)) \cosh(\beta), \qquad (6.5) \end{aligned}$$

valid for $\sigma \in Z(2)$. Therefore

$$< W_\gamma >= \frac{\sum_{\sigma_l} \prod_\square (1 + \sigma\, \sigma\, \sigma\, \sigma \tanh(\beta)) \prod_{l \in \gamma} \sigma(l)}{\sum_{\sigma_l} \prod_\square (1 + \sigma\, \sigma\, \sigma\, \sigma \tanh(\beta))}, \tag{6.6}$$

where symbolically the product of four sigmas represents a product along a plaquette like the one considered in equation (6.3). The product over $\square$ means over all plaquettes in the lattice.

In order to evaluate the above expression one should recall that,

$$\sum_{\sigma_l} \sigma = 0, \qquad \sum_{\sigma_l} \sigma^2 = 2. \tag{6.7}$$

Therefore the only contributions that survive in the numerator are those in which each link is traversed at least twice, in particular the links of the loop γ. This means that the interior of the loop has to be filled by the plaquettes in the product. This ensures that each link in the loop and all links in the internal plaquettes are traversed twice. Notice that in three dimensions this could be accomplished by many configurations of plaquettes, not just planar ones. Similarly in a compact lattice the same effect could be achieved in the exterior of the loop. We will see immediately that all those possibilities are suppressed and the minimal

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area surrounded by the loop gives the dominant term. In order to see this, recall that $\tanh(\beta) \ll 1$ and therefore the leading contribution to the numerator will be of order $(\tanh\beta)^N$ where N is the minimal number of plaquettes that fill the loop γ and, therefore, to leading order

$$< W_\gamma >= (\tanh(\beta))^N + \cdots = \exp(N \ln(\tanh(\beta))) + \cdots. \tag{6.8}$$

Since the minimal area inside the loop is given by N times the area of the elementary plaquette we get the area law

$$< W_\gamma >= \exp(-f(\beta)\text{area}), \tag{6.9}$$

where the leading term in the expansion of $f(\beta)$ is $-\ln(\tanh(\beta))$.

A similar perturbative expansion for the behavior at low temperature gives a dependence proportional to the length of the loop. We refer the reader to reference [74] for details.

6.2.2 The transfer matrix method

A lattice Hamiltonian version of a quantum gauge theory can be introduced in two different ways. One would be simply to consider the theory in a Hamiltonian fashion in the continuum and to propose a discretization on a lattice. There is another method, which is commonly used, in which one discretizes the covariant theory on a Euclidean spacetime lattice and then takes the continuum limit in the time direction to end with a Hamiltonian formulation. This procedure is called the transfer matrix formalism and was introduced by Schulz *et al.* [113]. Why would one proceed in this way? It turns out that for several theories it is more immediate to write a discretized version of the covariant theory and it exhibits in a clearer fashion the symmetries of the theory. For statistical models that do not come from a discretization of a continuum theory (like the example we are considering), the transfer matrix method is the only way to construct a Hamiltonian theory from the covariant one. The Hamiltonian version of a statistical theory can only agree with the covariant version at critical points since it is a partial continuum limit (in the time direction) of the latter.

To illustrate the transfer matrix method, let us consider it for a simple mechanical system, a particle in a potential. We follow the treatment due to Creutz [112]. The transfer matrix method is based on the close analogy between the Euclidean path integral formulation of quantum mechanics and statistical mechanics. The idea is the following. One starts with a theory, the covariant lattice version of it giving a statistical mechanical system with a discretized time. One then writes the partition function in terms of a product of elements of a certain matrix. This matrix is then reinterpreted, in the limit in which the discrete time intervals go to

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zero, as the matrix elements of the evolution operator of a Hamiltonian quantum theory. The vacuum energy of this theory may be identified with the free energy of the statistical mechanical system. The propagator can be identified with the correlation functions and the mass gap with the inverse of the correlation length.

We will now prove for the simplified case of a particle in a potential the relation between the partition function and the energy of a quantum Hamiltonian. Let us consider the Lagrangian of a particle in a potential $V(x)$

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2 + V(x), \tag{6.10}$$

where the + sign is due to the Euclidianization. Then the path integral is given by

$$Z = \int Dx(t)\, \exp(-S), \tag{6.11}$$

where the integral is over all possible trajectories $x(t)$ from an initial configuration at t_0 to a configuration at t_N. We will perform this integral in a lattice in which space is continuous but time is discrete, divided into intervals spaced by $a = (t_N - t_0)/N$. The discretized version of the action is

$$S = a\sum_i \left[\frac{1}{2}m\left(\frac{x_{i+1} - x_i}{a}\right)^2 + V(x_i)\right]. \tag{6.12}$$

The functional integral is now precisely defined as a multiple integral,

$$Z = \int \prod_1^{N-1} dx_i\, \exp(-S). \tag{6.13}$$

Notice that if one considers periodic boundary conditions in time and sums for all $x(t_0) = x(t_N)$, Z becomes the partition function of a statistical mechanic system.

We will now see that evaluating this partition function is equivalent to solving a quantum mechanical Hamiltonian system. To see this, let us write the partition function

$$Z = \int \prod_{i=1}^{N} dx_i\, T(x_{i+1}, x_i), \tag{6.14}$$

where $T(x, x')$ are the elements of the transfer matrix, given by,

$$T(x, x') = \exp\left(-\frac{m}{2a}(x - x')^2 - \frac{a}{2}(V(x') + V(x))\right). \tag{6.15}$$

Consider now a Hilbert space $\Psi(x)$ with the usual inner product, in

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which we define the position and translation operators,

$$\hat{x}\Psi(x) = x\Psi(x), \tag{6.16}$$

$$\exp(ib\hat{p})\Psi(x) = \Psi(x+b). \tag{6.17}$$

We wish to identify the elements of the transfer matrix as the matrix element of an operator in the position representation,

$$T(x,x') =< x|\hat{T}|x' > . \tag{6.18}$$

It can be seen that the operator,

$$\hat{T} = \int db \exp\left(-\frac{aV(\hat{x})}{2}\right) \exp\left(-\frac{b^2 m}{2a} - ib\hat{p}\right) \exp\left(-\frac{aV(\hat{x})}{2}\right) \tag{6.19}$$

$$= \sqrt{2\pi a/m} \exp\left(-\frac{aV(\hat{x})}{2}\right) \exp\left(-\frac{a\hat{p}^2}{2m}\right) \exp\left(-\frac{aV(\hat{x})}{2}\right) \tag{6.20}$$

gives the desired result. In the limit in which the lattice spacing is small, one can rewrite the operator as

$$\hat{T} = \sqrt{\frac{2\pi a}{m}} \exp(-a\hat{H} + \mathcal{O}(a^2)), \tag{6.21}$$

and one recognizes the usual Hamiltonian operator,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{6.22}$$

In terms of the $\hat{T}$ operator, the partition function can be written as

$$Z = \mathrm{Tr}(\hat{T}^N) = \mathrm{Tr}(\exp(-\hat{H}(t_N - t_0))). \tag{6.23}$$

It is immediate to establish the relation between the partition function and the vacuum energy of the associated Hamiltonian formulation. Suppose we are in a basis that diagonalizes $\hat{H}$ with eigenvalues E_j. Then,

$$Z = \sum_j \exp(-E_j(t_N - t_0)), \tag{6.24}$$

and in the limit in which the time interval is large, the dominant term is given by $\exp(-E_0 t)$.

6.2.3 Hamiltonian lattice theory

Let us now apply the transfer matrix method to the $Z(2)$ theory. For this purpose we consider an asymmetric lattice with time spacing τ and spatial spacing a. Let β_τ be the coupling constant of the plaquettes that contain time-like links $\Box_t$ and β the constant associated with the purely

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spatial plaquettes $\square_s$. The action can be split into two types of terms

$$S = -\beta_\tau \sum_{\square_t} \sigma\,\sigma\,\sigma\,\sigma - \beta \sum_{\square_s} \sigma\,\sigma\,\sigma\,\sigma. \tag{6.25}$$

We now fix the gauge in such a way that the spins associated with all the temporal links take the value $+1$ (this is called the "temporal gauge" and is analogous to $A_0 = 0$ in a gauge theory). The contribution of the plaquettes that include time-like links can then be rewritten as

$$\beta_\tau \sum_{n,\mu} \sigma(n,\mu)\sigma(n+\tau,\mu), \tag{6.26}$$

where the sum is over all the spatial links, i.e., μ is a space-like unit vector of the lattice. This can be immediately rewritten — apart from an irrelevant additive constant — as

$$S = \tfrac{1}{2}\beta_\tau \sum_{n,\mu} (\sigma(n+\tau,\mu) - \sigma(n,\mu))^2 - \beta \sum_{\square_s} \sigma\,\sigma\,\sigma\,\sigma. \tag{6.27}$$

If we now denote by σ^i the spatial spins associated with the spatial surface $t = t_i$, the partition function can be rewritten

$$Z = \sum_{\substack{\text{spatial spin} \\ \text{configurations}}} \prod_i \exp\left(\frac{\beta_\tau}{2} \sum_{l_s} (\sigma^{i+1}(l) - \sigma^i(l))^2 + \beta \sum_{\square_s} \sigma\,\sigma\,\sigma\,\sigma \right). \tag{6.28}$$

We can write the partition function as

$$Z = \sum_k \prod_i T(k_{i+1}, k_i), \tag{6.29}$$

where k_i is a spatial configuration of spins and T is the transfer matrix. If the lattice has Q spatial links at $t = t_i$ there will be 2^Q different configurations and therefore the transfer matrix will be of dimension $2^Q \times 2^Q$. Following the procedure outlined in the previous subsection, we can introduce a Hilbert space of functions that depend on the spin configurations on a given spatial surface $\Psi(k_i) = \Psi(\sigma_1^i, \ldots, \sigma_Q^i)$. The inner product is given by $< \psi|\phi >= \sum_k \psi^*(k)\phi(k)$. The product of two given configurations is $< k|k' >= \delta_{\sigma_1,\sigma_1'} \cdots \delta_{\sigma_Q,\sigma_Q'}$.

Let us now write the diagonal matrix elements of the transfer matrix among the configurations at time t_i and the configurations at time t_{i+1}. We denote the diagonal elements as "zero flip", meaning that all the spins are unchanged,

$$T(0\,\text{flips}) \equiv T(k_i, k_i), = \exp\left(\beta \sum_{\square_s} \sigma\,\sigma\,\sigma\,\sigma \right), \tag{6.30}$$

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where we see that all the terms of the first summation of the action (6.29) vanish since the configurations are the same.

Let us consider the contribution when the configuration at t_i differs by one spin from that at t_{i+1}, which we denote as "one flip",

$$T(1\,\text{flip}) = \exp\left(-2\beta\tau + \beta\sum_{\square_s}\sigma\,\sigma\,\sigma\,\sigma\right). \tag{6.31}$$

We see that only one term contributes to the first summation in the action, the one corresponding to the lattice site where the spin has been flipped. In general,

$$T(n\,\text{flips}) = \exp\left(-2n\beta\tau + \beta\sum_{\square_s}\sigma\,\sigma\,\sigma\,\sigma\right). \tag{6.32}$$

We would now like to adjust the parameters β and τ such that in the limit $t_{i+1}-t_i \to 0$, the transfer matrix becomes $\exp(\tau H) \sim 1-\tau H$ with H the Hamiltonian. This immediately leads us to the following conclusions (in the limit),

$$\beta \sim \tau, \tag{6.33}$$

$$\exp(-2\beta\tau) \sim \tau, \tag{6.34}$$

which leads us to identify $\beta = \lambda\exp(-2\beta\tau)$, where λ is a constant. Therefore from the expression of the elements of the transition matrix we can infer the elements of the quantum Hamiltonian,

$$H(0\,\text{flips}) \equiv H(k_i, k_i) = \left(\lambda\sum_{\square_s}\sigma\,\sigma\,\sigma\,\sigma\right), \tag{6.35}$$

whereas

$$H(1\,\text{flips}) = 1 + \mathcal{O}(t^2). \tag{6.36}$$

Let us give a representation for the action of the Hamiltonian operator. We represent an upward pointing spin by the two-dimensional vector $(1,0)$ and the downward pointing spin by $(0,1)$. The operator that produces a spin-flip is the Pauli matrix $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, whereas the diagonal operator in this basis is $\sigma_3 = \text{diag}(1,-1)$. Therefore the Hamiltonian can be written as

$$H = -\sum_{l_s}\sigma_1 - \lambda\sum_{\square_s}\sigma_3\,\sigma_3\,\sigma_3\,\sigma_3. \tag{6.37}$$

The operator that materializes the gauge transformations in this model can be written as

$$\Omega(n) = \prod_{l_n}\sigma_1, \tag{6.38}$$

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where l_n are the spatial links that emanate from the lattice site n. It can be readily checked that this operator commutes with the Hamiltonian.

We see that via the transfer matrix method we have obtained from a $(d+1)$-dimensional covariant theory a quantum Hamiltonian theory in d dimensions. We will now show that one can build a loop representation for such a theory.

6.2.4 Loop representation

In the same spirit as we discussed in the section of Yang–Mills theory, we introduce a set of operators that are gauge invariant and that are based on loops,

$$T^0(\gamma) = \prod_{l\in\gamma} \sigma_3^{(l)}, \tag{6.39}$$

where γ is a loop on the lattice. We only need to consider loops without repeated links, since $(\sigma_3)^2 = \mathrm{diag}(1,1)$, and therefore the repeated links do not contribute to the product. Also note that $T^0(\gamma) = T^0(\gamma^{-1})$. These identities reflect the fact that the group in question is $Z(2)$ and could be viewed as "Mandelstam identities" for this simple case.

The second invariant operator is

$$T^1(l) = \sigma_1, \tag{6.40}$$

where we see that the operator T^1 depends on a particular link. Notice the similarity with the construction for Maxwell theory, where we could have taken as operators the Wilson loop and the electric field. There we decided to multiply the electric field by the holonomy to keep the similarity with the non-Abelian case. Here we decide to make the variable T^1 loop independent. The commutator of the two operators is

$$[T^1(l), T^0(\gamma)] = 2T^1(l)T^0(\gamma) \sum_{l'\in\gamma} \delta_{l\,l'}. \tag{6.41}$$

Defining $X_l(\gamma)$ as $\sum_{l'\in\gamma} \delta_{l\,l'}$ (a lattice analogue of the first rank multitensor $X^{ax}(\gamma)$ in the continuum) we get the T algebra

$$[T^0(\gamma), T^0(\eta)] = 0, \tag{6.42}$$

$$[T^1(l), T^1(l)] = 0, \tag{6.43}$$

$$[T^1(l), T^0(\eta)] = 2X_l(\gamma)T^1(l)T^0(\gamma). \tag{6.44}$$

Notice that one could define in analogy with Maxwell theory an "electric field operator" $E(l)$,

$$E(l) = 1 - \tfrac{1}{2}T^1(l). \tag{6.45}$$

As in the case of Maxwell theory, where we defined a special group of loops to reflect the symmetries of the theory, one can define an Abelian

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group of loops that reflects the symmetries of $Z(2)$. In order to do this, one starts from the Abelian group of loops defined in section 4.1 and identifies the squares of loops with the identity. The elements of this group therefore satisfy

$$\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1, \tag{6.46}$$

$$\gamma \circ \gamma = 1, \tag{6.47}$$

$$\gamma_1 = \gamma_1 \circ (\bar{\gamma}_1)^2 = \bar{\gamma}_1. \tag{6.48}$$

We now give a representation of the T algebra in terms of a space of wavefunctions dependent on loops $\Psi(\gamma)$

$$T^0(\eta)\Psi(\gamma) = \Psi(\gamma \circ \eta), \tag{6.49}$$

$$T^1(l)\Psi(\gamma) = (1 - 2X_l(\gamma))\Psi(\gamma). \tag{6.50}$$

The electric field has the usual action for an Abelian theory,

$$E(l)\Psi(\gamma) = X_l(\gamma)\Psi(\gamma). \tag{6.51}$$

The Hamiltonian in this representation can be written in terms of the elementary operators,

$$\hat{H}\Psi(\gamma) = -\lambda \sum_{\square_s} T^0(\square_s) - \sum_{l_s} T^1(l), \tag{6.52}$$

and is called the Wegner Hamiltonian. To make contact with usual gauge theories it is common to define a Hamiltonian which differs from Wegner's by a constant, $\hat{H} = \lambda^{-1}(H_{\text{Wegner}} + \Lambda)$ where Λ is the product of the number of sites in the lattice and the dimension of the space. The modified Hamiltonian then reads

$$\hat{H}\Psi(\gamma) = \mu \sum_{l} E(l) - \sum_{\square_s} \sigma\,\sigma\,\sigma\,\sigma, \tag{6.53}$$

where $\mu = -2/\lambda$ and we see the appearance of an "electric" and a "magnetic" piece. This is clarified by studying the action on a state,

$$\hat{H}\Psi(\gamma) = -\sum_{\square_s} \Psi(\square_s \circ \gamma) + \mu l(\gamma)\Psi(\gamma), \tag{6.54}$$

where $l(\gamma)$ is the length of the loop. Usually one would expect a length squared in the term arising from the electric part. However, for the $Z(2)$ case loops are only traversed once.

This Hamiltonian can be put on a computer and used to study the vacuum energy, observables and mass spectra of the theory [114] and the results can be compared with those obtained with other methods. There is a great wealth of knowledge about this model because — in three dimensions — it is the dual of the Ising model. Duality in this context means that one can associate to the lattice a dual lattice in which

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to each cube one associates a lattice site and to each plaquette a link and rewrite the action of the Wegner model as the action of an Ising model on the dual lattice [74]. This allows us to import all the knowledge about the Ising model to the Wegner model.

Let us now sketch one of the approximation techniques that appear naturally in the loop representation on the lattice. It is called the collective variables technique.

Given the known behavior at long lengths of the Wilson loop, the length and area of a loop appear as natural variables with which to study the theory. The length and area of loops are two examples of possible collective variables that characterize loops in the asymptotic region of large loop lengths. One could refine this picture by introducing other variables that give additional information about the loop, such as information about corners [114].

Let us therefore propose a description of the model in which we consider wavefunctions that are a function of the length of the loop, the area of the loop and a variable c that codes the information about the number and kinds of corners of the loop $\Psi(l, A, c)$.

The Hamiltonian in terms of these variables can be constructed from the action of the Hamiltonian in the loop representation (6.52). The action of the electric part is trivial, it just multiplies the wavefunction and the value of the length. The magnetic part adds a plaquette. The effect of this will depend on where the Hamiltonian is acting. If the plaquette is completely exterior to the loop, the area is increased by one unit, the length is increased by four units and four corners are added. If the plaquette is completely inside the loop, the area decreases by one unit, and the length and number of corners increase by four units. In the case where the plaquette shares a link with the loop and there are no corners adjacent the length increases by two units, the area by (plus or minus) one and the number of corners by four units. All actions are weighted by a factor that states how many plaquettes with the action of interest are possible. The action of the Hamiltonian is therefore coded in a finite difference equation involving the three variables of the problem and the total number of plaquettes in the lattice.

One can search for solutions of the finite difference equation minimizing the energy per plaquette. Since one is aware of possible exponential behaviors with length and area — as we argued in the previous chapter — one can propose a solution $\Psi(l, A, c) = Y^A X^l Z^c$ with X, Y, Z constants. The typical behaviors of the energy and the Y constant as a function of the coupling constant μ is shown in figure 6.2. Notice that between $\mu = 0$ and a certain critical value (weak coupling regime) the constant $Y = 1$ and therefore the wavefunction does not depend on the area. For $\mu > \mu_0$ the variable $Y < 1$ and therefore the wavefunction decreases exponen-

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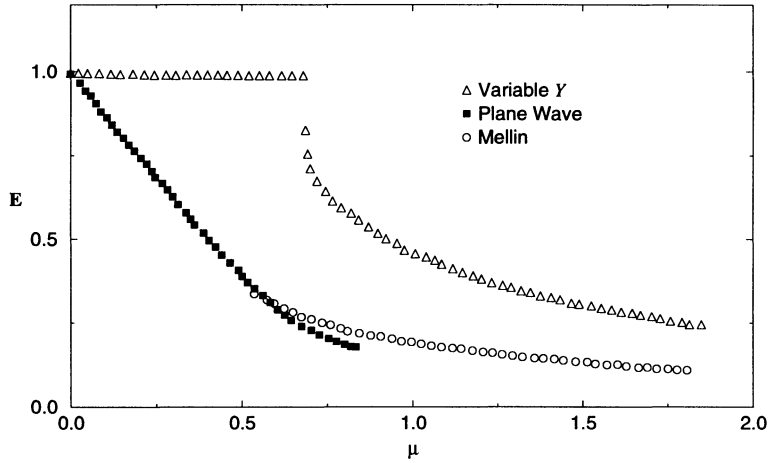

Fig. 6.2. Dependence of the energy per plaquette and the variable Y as a function of the coupling constant μ in the Wegner model.

tially with the area. For the energy we show two curves, the one in the weak coupling regime was obtained as discussed in the text, the one in the strong coupling regime was obtained by a Mellin transform method [108, 114]. Notice that there is a discontinuity suggesting a second order phase transition for the value $\mu = 0.68$ which agrees to within 2% of the value observed with other methods.

So we see that the use of the loop representation allows a very natural and intuitive action for the Hamiltonian constraint. Collective variables representing qualitative properties of the loop allow an understanding of the phase structure and the observables of the theory with small computing power. Unlike statistical methods, such as Monte Carlo simulations, they produce analytic results. The results can be plotted with a computer as we chose to do, but they are available in analytic form.

These initial results should act as an encouragement for further work on the use of collective variables on the lattice and the development of other approximation methods in the loop representation.

6.3 The $SU(2)$ theory

We will now discuss a more realistic lattice gauge theory, related to a continuum theory. Many of the techniques developed here are also applicable to other cases of direct physical interest, such as QED and QCD.

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Let us start by setting up a lattice version of the connection representation. We assign to each link in the lattice an element of the $SU(2)$ gauge group $\mathbf{U}(l)$. This element is the parallel transport operator along the link. Notice the parallel with Wegner's model. Each link variable $\mathbf{U}(l)$ is not gauge invariant, the gauge transformations act at each site. We will be able to construct gauge invariant quantities by taking the product of link variables along closed contours. Notice that the variables $\mathbf{U}(l)$ are just the natural counterparts of the holonomies along open paths that we discussed in chapter 1. A point to notice is that in the continuum descriptions of gauge theories one usually takes variables defined on the algebra of the gauge group whereas the fundamental variables on the lattice take values in the group.

6.3.1 Hamiltonian lattice formulation

A field configuration is determined by an assignment of $SU(2)$ matrices to each link on the lattice. The assignment depends on an orientation of the lattice: if the link l has associated with it the matrix $\mathbf{U}(l)$, the reversed link has $\mathbf{U}(l)^{-1}$. We denote by $\mathbf{A}(l)$ the element of the algebra associated with $\mathbf{U}(l) = \exp(iag\mathbf{A}(l))$ where a is the lattice spacing and g is the coupling constant of the theory. We want to introduce a variable canonically conjugate to U that in the limit of zero lattice spacing plays the role of the electric field and that in general has the same transformation properties under gauge transformations that the electric field has in the continuum case. We introduce the variable $\mathbf{E}(l)$ which takes values in the $SU(2)$ algebra and which has Poisson brackets,

$$\{\mathbf{U}_A^B(l), \mathbf{U}_C^D(l')\} = 0, \tag{6.55}$$

$$\{\mathbf{E}_j(l), \mathbf{U}_A^B(l')\} = -i\delta_{l,l'}(X_j)_A^C \mathbf{U}_C^B(l), \tag{6.56}$$

$$\{\mathbf{E}_j(l), \mathbf{E}_k(l')\} = \sqrt{2}\epsilon_{jkm}\mathbf{E}_m(l)\delta_{l,l'}, \tag{6.57}$$

where as usual the indices j refer to components in a basis of generators of the algebra, $(\mathbf{X}_j)_A^B$, l and l' are in the positive orientation and $\delta_{l,l'} = 1$ if $l = l'$ and zero otherwise.

The continuum limit of these variables (when the lattice spacing a goes to zero) is defined as

$$\lim_{a\to 0} \mathbf{A}(l) = \lim_{a\to 0} \mathbf{A}(n, \mu_b) = \mathbf{A}_b(n), \tag{6.58}$$

$$\lim_{a\to 0} \mathbf{U}(l) = \lim_{a\to 0} (\mathbf{1} + ia\mathbf{A}(l)) = \mathbf{1}, \tag{6.59}$$

$$\lim_{a\to 0} \mathbf{E}_j(l) = \lim_{a\to 0} \mathbf{E}_j(n, \mu_b) = \lim_{a\to 0} a^2 \mathbf{E}_j^b(n), \tag{6.60}$$

where $\mathbf{A}(n, \mu_b)$ is the value of the field at the lattice position n where the link l starts and in the direction of the vector μ_b along the link l (the μs

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were a triad of vectors along the lattice directions and b is a triad index).

With this definition we can check that the Poisson brackets correspond to the usual canonical brackets of Yang–Mills theory,

$$\lim_{a\to 0}\frac{1}{a^3}\{\mathbf{E}_j(n,\mu_b),\mathbf{U}(n',\mu_c)\} = -i\lim_{a\to 0}\frac{1}{a^3}\delta_{n,n'}\delta_{bc}\mathbf{X}^j, \tag{6.61}$$

and in the limit,

$$\{\mathbf{E}_b^j(x),\mathbf{A}_c(x')\} = -\delta(x-x')\delta_{b,c}\mathbf{X}^i, \tag{6.62}$$

where x and x' are the coordinate positions of the lattice sites n and n' respectively. Following a similar calculation one can show that the Poisson bracket of the $\mathbf{E}$ variables leads to the usual vanishing Poisson bracket of electric fields.

We now write the Gauss law and the Hamiltonian for the classical theory. The Gauss law is

$$\mathcal{G}_j(n) = \sum_{l_n}\mathbf{E}_j(l_n) = 0, \tag{6.63}$$

where l_n are all the links emanating from the site n. In the limit of vanishing lattice spacing it can be checked that this equation gives rise to the usual divergence of E. To perform this limit the reader should be aware of the commutation relation of $\mathbf{E}_j(\bar{l})$ where $\bar{l}$ is the link l with a reversed orientation. This Poisson bracket is derived from the ones introduced above and recalling that $\mathbf{U}(\bar{l}) = \mathbf{U}^{-1} = \mathbf{U}^\dagger(l)$ where $\dagger$ is the conjugate transpose matrix. The result is

$$\{\mathbf{E}_j(l),\mathbf{U}_A^B(l')\} = -i\delta_{l,l'}\mathbf{U}_A^C(l)(X_j)_C^B. \tag{6.64}$$

It is possible to show that the Gauss law generates infinitesimal canonical transformations associated with gauge transformations on the lattice,

$$\mathbf{U}(n,\mu)\to\mathbf{V}(n)\mathbf{U}(n,\mu)\mathbf{V}^\dagger(n+\mu), \tag{6.65}$$

$$\mathbf{E}_j(n,\mu)\to\mathbf{V}(n)\mathbf{E}_j(n,\mu)\mathbf{V}^\dagger(n). \tag{6.66}$$

The Hamiltonian is

$$\mathcal{H} = \frac{g^2}{2}\sum_{l>0}\mathbf{E}_j(l)\mathbf{E}_j(l) - \tfrac{1}{g^2}\sum_{\square}\mathrm{Tr}(\mathbf{U}(\square)), \tag{6.67}$$

where $l>0$ means all the positive oriented links of the lattice and $\mathbf{U}(\square)$ is the product of the four U variables associated with the links of the elementary plaquettes over which the sum runs. We have set $a=1$ as is customary in the lattice. If not, the electric part would be altered by a factor $1/a$ and the magnetic part by a factor a. In the continuum limit, the electric piece immediately reproduces the continuum electric field squared. The magnetic part is more complicated. In order to recover the usual magnetic part one should subtract a negative constant proportional to

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the number of plaquettes. The Hamiltonian that we consider is therefore bounded from below but is not positive-definite.

To construct a quantum theory in the **U** representation we introduce wavefunctions $\Psi(\mathbf{U})$ where **U** denotes a configuration of the system that assigns to each element on the lattice an $SU(2)$ matrix. The variables **U** and **E** become multiplicative and purely derivative operators in this space. This space is endowed with a natural inner product

$$< \Psi|\Phi >= \int \prod_{l>0} d\mathbf{U}_l \Psi^*(\mathbf{U})\Psi(\mathbf{U}). \tag{6.68}$$

The measure is, for every lattice link, the Haar measure associated with $SU(2)$ [98]. In this space the operators associated with **U** and **E** have the following action:

$$\hat{U}(l)^B_A \Psi(U) = \mathbf{U}(l)^B_A \Psi(U), \tag{6.69}$$

$$\hat{E}_j(l)\Psi(U) = -(X_j)^B_A \mathbf{U}(l)^C_B \frac{\partial}{\mathbf{U}(l)^C_A}\Psi(\mathbf{U}), \tag{6.70}$$

and the quantum expressions for the Hamiltonian and Gauss law can be constructed straightforwardly.

6.3.2 Loop representation in the lattice

Following the same steps that we used when discussing the Wegner model, it is immediate to introduce gauge invariant variables for the $SU(2)$ theory on the lattice. Loop representations on the lattice have been considered by several authors [115, 110, 109]. In this section we will follow closely the treatment of reference [110].

Let us consider an algebra of classical gauge invariant quantities on the lattice defined by*:

$$T^0(\gamma) = \tfrac{1}{2}\mathrm{Tr}\left[\prod_{l\in\gamma} \mathbf{U}(l)\right] \equiv \tfrac{1}{2}\mathrm{Tr}\left[\mathbf{U}(\gamma)\right], \tag{6.71}$$

$$T^1_l(\gamma) = \tfrac{1}{2}\mathrm{Tr}\left[\mathbf{U}(\gamma^n_n)\mathbf{E}(l)\right], \tag{6.72}$$

where as usual $l = (n, \mu)$ and $\mathbf{U}(\gamma^n_n)$ denotes the product of $\mathbf{U}(l)$ with l links in γ starting at n and ending at n. The Poisson algebra is

$$\{T^0(\gamma), T^0(\eta)\} = 0, \tag{6.73}$$

* Our conventions in this chapter for the T variables differ from those in the rest of the book by a factor $\frac{1}{2}$. We do this in order to facilitate the comparison with the particle physics literature which usually includes that factor in the definition of the Wilson loops.

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$$\{T^0(\gamma), T^1_l(\eta)\} = \frac{i}{2} g \sum_{l' \in \gamma} \delta_{l,l'} \left[T^0(\gamma^n_n \circ \eta^{n'}_{n'}) - T^0(\gamma) T^0(\eta) \right], \tag{6.74}$$

$$\{T^1_l(\gamma), T^1_{l'}(\eta)\} = \frac{i}{2} g \sum_{l'' \in \gamma} \delta_{l,l''} \left[T^1_{l'}(\eta^{n''}_{n'} \circ \gamma^n_n \circ \eta^{n'}_{n''}) - T^0(\gamma) T^1_{l'}(\eta) \right] - \frac{i}{2} g \sum_{l'' \in \eta} \delta_{l',l''} \left[T^1_l(\gamma^{n''}_n \circ \eta^{n'}_{n'} \circ \gamma^n_{n''}) - T^0(\eta) T^1_l(\gamma) \right], \tag{6.75}$$

where $\delta_{l,l'} = \pm 1$ if l and l' are the same link, the sign depending on the orientation of links, and being zero otherwise.

We now proceed to quantize the theory. We need to realize the algebra of classical quantities that we just introduced on a space of functions of loops on the lattice. As we argued in chapter 3, for the case of $SU(2)$ it is sufficient to consider wavefunctions of a single loop. This was due to the fact that the Mandelstam identities (for $SU(2)$) allow us to express any product of Wilson loops as a linear combination of Wilson loops. In this chapter however, we will not take advantage of this fact and we will consider a representation in terms of wavefunctions of multiloops ("clusters" in lattice notation).

The reason for this is that the use of multiloops will lead us naturally to calculational techniques that are more economical and efficient from the point of view of the lattice. In short we will trade aesthetics (having a single loop) for calculational efficiency. For instance, we may consider the action of the magnetic term of the Yang–Mills Hamiltonian in the loop representation. In terms of a representation based on multiloops its action is to add a plaquette to the loops in the argument of the wavefunction. In terms of functions of a single loop, one obtains reroutings at intersections (as is typical of the use of the Mandelstam identities).

As we discussed in chapter 5, in order to represent the Hamiltonian of Yang–Mills theory, it is not enough to consider the "small" T algebra formed by T^0 and T^1. As in the discussion of $SU(N)$ in the continuum, we will not study the representation of the "large" T algebra but just of the T^0 and T^1 supplemented with the Hamiltonian constraint. As we argued in the continuum, one can always find a representation of the large T algebra such that the representation we introduce for the T^0, T^1 and Hamiltonian is reproduced simply by considering the expression of the Ts in terms of E and U and choosing a factor ordering (E to the right in this particular case). The action of the operators on the space of wavefunctions of multiloops is

$$\hat{T}^0(\eta)\Psi(\gamma_1, \ldots, \gamma_n) = \Psi(\eta, \gamma_1, \ldots, \gamma_n), \tag{6.76}$$

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$$\hat{T}_l^1(\eta)\Psi(\gamma_1,\ldots,\gamma_n) = \frac{g}{2}\sum_{k=1}^{n}\sum_{l'\in\gamma_k}\delta_{l,l'} \times\Big(\Psi(\gamma_1,\ldots,\eta_n^n\circ(\gamma_k)_{n'}^{n'},\ldots,\gamma_n)-\Psi(\eta,\gamma_1,\ldots,\gamma_n)\Big) \quad (6.77)$$

To realize the electric part of the Hamiltonian we proceed in the same way as in chapter 5, recalling the commutation relation of the electric part of the Hamiltonian,

$$\hat{\mathcal{E}} = \frac{g^2}{2}\sum_{l>0}\hat{E}_j(l)\hat{E}_j(l), \quad (6.78)$$

with the Wilson loop (which can be derived in the U representation),

$$[\hat{\mathcal{E}},\hat{T}^0(\gamma)] = g^2\sum_{l,l'\in\gamma}\delta_{l,l'}[\hat{T}^0(\gamma_n^{n'})\hat{T}^0(\gamma_{n'}^n)-\tfrac{1}{4}T^0(\gamma)]-g\sum_{l\in\gamma}\hat{T}_l^1(\gamma). \quad (6.79)$$

From here we can read off the realization of the electric part of the Hamiltonian on a wavefunction, which consists of four distinct contributions $\mathcal{E}=\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3+\mathcal{E}_4$, given by

$$\hat{\mathcal{E}}_1\Psi(\gamma_1,\ldots,\gamma_n) = g^2\sum_{j=1}^{n}L(\gamma_j)\Psi(\gamma_1,\ldots,\gamma_n), \quad (6.80)$$

$$\hat{\mathcal{E}}_2\Psi(\gamma_1,\ldots,\gamma_n) = -\frac{g^2}{4}\sum_{i,j=1}^{n}\Lambda(\gamma_i,\gamma_j)\Psi(\gamma_1,\ldots,\gamma_n), \quad (6.81)$$

where $L(\gamma)$ is the number of links in the loop γ, $\Lambda(\gamma,\eta)=\sum_{l\in\gamma}\sum_{l\in\eta}\delta_{l,l'}$ (sometimes called the "quadratic length" in the case $\gamma=\eta$),

$$\hat{\mathcal{E}}_3\Psi(\gamma_1,\ldots,\gamma_n) = \frac{g^2}{2}\sum_{k<j=1}^{n}\sum_{l\in\gamma_k}\sum_{l'\in\gamma_j}\delta_{l,l'} \times\Psi(\gamma_1,\ldots,\gamma_{i-1},(\gamma_k)_n^n\circ(\gamma_j)_{n'}^{n'},\ldots,\gamma_{j-1},\gamma_{j+1},\ldots\gamma_n), \quad (6.82)$$

$$\hat{\mathcal{E}}_4\Psi(\gamma_1,\ldots,\gamma_n) = g^2\sum_{j=1}^{n}\sum_{l,l'\in\gamma_j}\delta_{l,l'} \times\Psi(\gamma_1,\ldots,\gamma_{j-1},(\gamma_j)_n^{n'},(\gamma_j)_{n'}^n,\gamma_{i+1},\ldots,\gamma_n). \quad (6.83)$$

The magnetic part of the Hamiltonian,

$$\hat{\mathcal{B}} = -\frac{1}{g^2}\sum_{\square}\mathrm{Tr}(\hat{U}(\square)), \quad (6.84)$$

has also a very simple action,

$$\hat{\mathcal{B}}\Psi(\gamma_1,\ldots,\gamma_n) = -\frac{1}{g^2}\sum_{\square}\Psi(\square,\gamma_1,\ldots,\gamma_n), \quad (6.85)$$

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and the loop could have been inserted in any entry. The order of loops in the multiloop is irrelevant.

We therefore see that the Hamiltonian in the lattice has a beautifully simple geometric action. The term $\mathcal{E}_1$ measures the length of the loop, the term $\mathcal{E}_2$ measures "overlapping" and quadratic effects in the length. The term $\mathcal{E}_3$ fuses two loops if they have an intersecting point and $\mathcal{E}_4$ takes any loop with a self-intersection and produces a fission. The magnetic terms simply add a loop. For the case $SU(N)$ the action of the Hamiltonian is exactly the same (apart from factors dependent on the dimensionality of the group) [110].

We will now discuss methods for treating the Hamiltonian on the lattice.

6.3.3 Approximate loop techniques

The cluster approximation techniques we are to describe are based on the combination of strong coupling expansions and collective variables. Let us therefore start with a brief discussion of the strong coupling expansion in terms of loops. If one takes the limit $g \to \infty$, the magnetic term in the Hamiltonian of Yang–Mills theory drops out and the Hamiltonian eigenvalue problem can be solved exactly. Remembering that all the terms in the electric part of the Hamiltonian are proportional to loop lengths, it is immediate to realize that the vacuum is a ket with zero loops $|0>$. The energy of the vacuum vanishes. The first excited state is given by a plaquette excitation $|\Box>$. The second involves at most two plaquettes and so on. In this approximation the magnetic term is considered a perturbation of the electric term. The effect of the magnetic term is to add a plaquette. Therefore, in the perturbative expression of the vacuum in the strong coupling regime, terms involving many plaquettes are suppressed by a power of $1/g^4$.

A cluster is a set of loops in a finite region of space. The quantum states we will consider will be based on sets of clusters, which we assume to be far apart from each other.

Examples of clusters are:

- a single plaquette,
- two plaquettes nearby,
- a rectangle,
- a plaquette traversed twice.

The idea of this approximation is based on the action of the Hamiltonian we described above. Since the clusters are assumed to be far away

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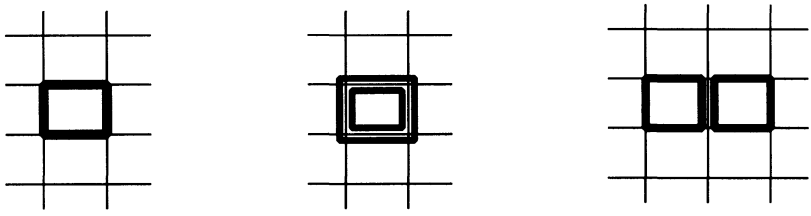

Fig. 6.3. The three clusters considered in the example

from each other, the Hamiltonian never connects them. The approximation is based on truncating the basis of all possible states (all possible clusters) and considering a finite number of clusters and the action of the Hamiltonian in the truncated basis.

Let us consider a concrete example of the action of the Hamiltonian of $SU(2)$ theory in $2+1$ dimensions on a particular set of clusters. We truncate the basis of clusters to only three types of cluster. This is clearly a toy model only and later we will give a procedure for constructing approximations to any desired order.

Cluster type 1 is a plaquette. Cluster type 2 is a plaquette traversed twice and type 3 consists of two plaquettes with a common link in a plane, as shown in figure 6.3. The three elements of the cluster basis considered are

$$\text{type 1}: \quad T^0(\square)|0>, \tag{6.86}$$

$$\text{type 2}: \quad T^0(\square)T^0(\square)|0>, \tag{6.87}$$

$$\text{type 3}: \quad T^0(\square)T^0(\square')|0>, \tag{6.88}$$

where $\square'$ has a common link with $\square$. The states spanned by this basis are denoted by

$$|n_1, n_2, n_3>, \tag{6.89}$$

where n_i indicates the number of clusters of type i present. The lattice position of the clusters is immaterial (as long as the clusters are far apart) since the action of the Hamiltonian is local and sums over all clusters.

Let us study the action of the Hamiltonian. For convenience we rescale it by a factor $g^2/2$. The magnetic term adds a plaquette. Its action can be written as

$$\begin{aligned} g^2\hat{\mathcal{B}}|n_1, n_2, n_3> &= n_1|n_1-1, n_2+1, n_3> \\ &+4n_1|n_1-1, n_2, n_3+1> +5n_2|n_1, n_2-1, n_3> \\ &+8|n_1, n_2, n_3-1> +(P-5n_1-5n_2-8n_3)|n_1+1, n_2, n_3>, \end{aligned} \tag{6.90}$$

where P is the number of plaquettes in the lattice. The first term corresponds to the addition of a plaquette on top of one of the single plaquettes

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and therefore producing a cluster of type 2. The second term introduces a plaquette adjacent to one of the clusters of type 1, forming a cluster of type 3. The third and fourth terms destroy clusters of type 2 and type 3 respectively and produce a state rigorously out of the basis of clusters considered. The last term corresponds to the addition of a plaquette without contact with any of the existing clusters. We see that the action of the magnetic part of the Hamiltonian does not leave the basis invariant. There are two possibilities: either one ignores this fact and the calculation becomes valid only in the strong coupling limit, or one tries to encode the missing information in an extra set of variables (collective variables).

The electric part of the Hamiltonian gives a diagonal contribution that can be written as

$$\hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_2 | n_1, n_2, n_3 > = \frac{g^4}{2}(3n_1 + 4n_2 + \tfrac{13}{2})|n_1, n_2, n_3 > . \tag{6.91}$$

The remaining terms are the fission and fusion terms. The fission terms do not contribute because we are not considering loops with self-intersections at this order of approximation. The fusion terms give a non-diagonal contribution,

$$\begin{aligned}\hat{\mathcal{E}}_3|n_1, n_2, n_3 > = \frac{g^4}{2}(&4n_2|n_1, n_2, n_3 > -2n_2|n_1, n_2 - 1, n_3 > \\ &-\tfrac{1}{2}n_3|n_1, n_2, n_3 - 1 >).\end{aligned} \tag{6.92}$$

The first and second term originate in the action of the fusion terms on a cluster of type 2; this leads to a $\hat{T}^0(\square \circ \square)$ which can be rearranged using the Mandelstam identities into a linear combination of a cluster of type 2 and the vacuum. The last term comes from the fusion of the two plaquettes present in the clusters of type 3 and leads to a rectangle, which does not appear at the present order.

We are now in a position to cast the problem of finding the vacuum and the excited states of $SU(2)$ Yang–Mills theory in terms of a finite difference equation. We start from

$$< \Psi | H | n_1, n_2, n_3 > = E < \Psi | n_1, n_2, n_3 >, \tag{6.93}$$

and get

$$\begin{aligned}&n_1[5\Psi(n_1 + 1, n_2, n_3) - \Psi(n_1 - 1, n_2 + 1, n_3) \\ &-4\Psi(n_1 - 1, n_2, n_3 + 1) + \tfrac{3}{2}g^4\Psi(n_1, n_2, n_3)] \\ &n_2[5\Psi(n_1 + 1, n_2, n_3) - 5\Psi(n_1, n_2 - 1, n_3) \\ &+4g^4\Psi(n_1, n_2, n_3) - g^4\Psi(n_1, n_2 - 1, n_3)] \\ &n_3[8\Psi(n_1 + 1, n_2, n_3) - 8\Psi(n_1, n_2, n_3 - 1) \\ &+\tfrac{13}{4}g^4\Psi(n_1, n_2, n_3) - \tfrac{1}{4}g^4\Psi(n_1, n_2, n_3 - 1)] = \\ &P[\Psi(n_1 + 1, n_2, n_3) + \epsilon\Psi(n_1, n_2, n_3)],\end{aligned} \tag{6.94}$$

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where P is the number of plaquettes and $\epsilon = E/P$ is the energy per plaquette.

We now propose a power-law solution for the vacuum (as discussed above),

$$\Psi_0(n_1, n_2, n_3) = x_1^{n_1} x_2^{n_2} x_3^{n_3}, \tag{6.95}$$

with x_i constants, that leads to a system of non-linear equations,

$$(5x_1 + \tfrac{3}{2}g^4)x_1 - x_2 - 4x_3 = 0, \tag{6.96}$$

$$(5x_1 + 4g^4)x_2 - g^4 - 5 = 0, \tag{6.97}$$

$$(8x_1 + \tfrac{13}{4}g^4)x_3 - \frac{g^4}{4} - 8 = 0, \tag{6.98}$$

and the relation $\epsilon_0 = -x_1$. Note that in spite of the fact that one started from a strong-coupling approximation, if one takes $g = 0$, then $x_1 = x_2 = x_3 = 1$, which corresponds to the exact solution of the system when $g = 0$. This last fact can be better seen in the U representation. In such a representation the magnetic part is just a multiplication by $-T^0(\Box)$, which has a minimum at -1, and this implies that T^0 is 1 and therefore it corresponds to a configuration in which each link has associated the element $U = 1$ (up to gauge). This implies that the vacuum in the loop representation is $\Psi(\gamma) = 1$ for any γ in the weak coupling limit, which is the result we found above.

The excited states are found by trial of ansätze of the form

$$(1 + a_1 n_1 + a_2 n_2 + a_3 n_3) x_1^{n_1} x_2^{n_2} x_3^{n_3}, \tag{6.99}$$

which resemble the kind of polynomial construction we performed for the excited states of Maxwell theory.

We now return to the general discussion of the cluster approximation technique. A generic state will be characterized by a list of clusters, $|n_1, \ldots, n_k, \ldots >$. We now propose a recursive ordering among clusters. This idea of order is associated with the different orders in the strong coupling approximation. The zeroth order will be the vanishing loop, the first order is a single plaquette. The nth order is obtained from the $(n-1)$th one through the action of the Hamiltonian on the basis of $(n-1)$th clusters. We restrict the action of the Hamiltonian to the addition of a plaquette immediately adjacent to the existing one. The combined action of the electric and magnetic terms will give rise to loops of large area and disconnected (but close). It is simple to see that through this procedure one can obtain any loop on the lattice. Therefore, there exists a natural approximation scheme that consists in considering clusters up to a certain order. In particular, the previous example is an approximation of order 2. This proposal for the construction of a basis of clusters has the drawback that one may consider clusters that are equivalent under the Mandelstam

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identities, that in the conventions used in this section read (in the loop representation)

$$2\Psi(\ldots,\gamma_i,\gamma_j,\ldots) = \Psi(\ldots,\gamma_i \circ \gamma_j,\ldots) + \Psi(\ldots,\gamma_i \circ \bar{\gamma}_j,\ldots). \quad (6.100)$$

An interesting point is that if one is only interested in the energy spectrum of the Hamiltonian, one can ignore completely the Mandelstam identities. The only consequence (apart from the obvious one of working on a larger set of states) is that the level of degeneracy of each energy eigenstate is increased (since one is considering as independent states that are not). But the point is that the energy levels are unchanged. This statement is an exact one as long as one does not truncate the basis. With a truncation one should be careful because clusters apparently of higher order could be Mandelstam rearranged to a lower order. An explicit calculation for the $SU(2)$ case in $2+1$ dimensions taking into account the complete set of Mandelstam identities [109] shows no appreciable difference in the energy levels from those in which the Mandelstam identities were taken into account only partially [110], as we shall discuss later.

Although it is difficult to exhaust entirely the content of the Mandelstam identities to reduce the set of states, it is possible to use them partially in a simple way to curb the number of states considerably. The Mandelstam identity for two adjacent loops sharing a common link with the same orientation implies that the state can be completely rewritten in terms of loops in which the link appears to be traversed only once. This means that one can automatically eliminate from the set of states those which include links traversed twice or more in the same direction.

A technique that considerably improves the performance of the cluster approximation is the use of collective variables. The idea is to supplement the information one has about the clusters with the use of certain variables, similar to the ones we discussed for the Wegner model. An example of a collective variable is the length of a loop L. The value of a collective variable Q for a certain configuration of clusters $|n_1,\ldots,n_k>$ is given by $Q = \sum_{i=1}^{k} n_i Q_i$ where the coefficients Q_i are the values of the collective variable for the cluster i (for instance, it could be the length of the cluster i). One usually normalizes the variables such that $Q_1 = 1$. The wavefunctions $\Psi(n_1,\ldots,n_k)$ can be reexpressed as $\Psi'(Q,n_2,\ldots,n_k)$, and therefore to search for the vacuum one chooses the ansatz $x_1^Q y_2^{n_2}\ldots y_k^{n_k}$ with $x_1,y_2,\ldots,y_k$ are constants. In spite of the fact that this description may seem quite similar to the one presented before, the use of the collective variable Q allows us to take into account when the action of the Hamiltonian goes out of the space of clusters considered. Note that in searching for the vacuum one solves a system of non-linear equations. It can be seen that the equations associated with the collective variables are non-linear and that all the others are linear.

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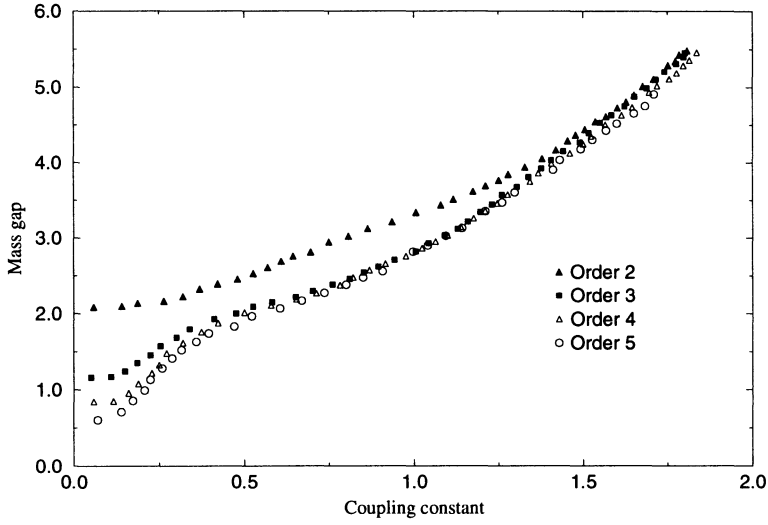

Fig. 6.4. Energy of the first excited state minus the vacuum energy (mass gap) as a function of the coupling constant. The calculation was performed using clusters up to order five and one collective variable.

The use of collective variables allows the approximation to remain meaningful to a certain extent in the weak coupling regime, as can be seen in figure 6.4. This figure corresponds to the use of a single collective variable, appropriately chosen to fit the analytic behavior of the energy in the weak coupling regime. The collective variable considered is $Q = ((N+1)L - \Lambda)/4N$, where N is the order of the approximation, L the length and Λ the quadratic length.

In the figure one can see the transition between strong and weak coupling regimes around $g^2 = 1$. If one does not use collective variables, the approximation breaks down around $g^2 = 2$ ([109] figure 8). In the weak coupling regime the mass gap should go to zero as $g \to 0$ linearly in g^2 (as can be seen in perturbation theory [116]). We see that there are signs of convergence to the expected behavior for the higher cluster orders. The slope for the best approximation (fifth order) is 4.03 [110], whereas strong coupling calculations predict a value of 4.4 ± 0.5[117, 118] and the Monte Carlo result is 4.06 ± 0.6 [119].

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6.4 Inclusion of fermions

Yang–Mills theories arise in nature as the theories of the vector bosons that mediate the interactions of fermionic matter fields. An example would be the photons that allow particles with electric charge to interact. Another example could be the interaction of colored particles, such as quarks, via the exchange of $SU(3)$ gluons. In all these examples the particles that interact via the Yang–Mills fields are charged fermions. Therefore to study interactions of gauge theories with matter we need to incorporate fermions in the discussion. We have already mentioned that fermions give rise to open paths in the language of loops both in chapter 1 and in the discussion of the ideas about confinement. We are also going to present a discussion of the interaction of fermions with gravity in chapter 9. In this section we do not intend to develop in great detail the discussion of gauge theories interacting with fermions in the loop representation. We simply want to introduce very briefly some techniques that have been developed to deal with fermions interacting with gauge fields. These techniques are formulated on the lattice and that is the rationale for including the discussion of fermions in this chapter.

The action of a gauge field interacting with a charged fermion is given by

$$S = S_{\text{Free}} + \bar{\psi}^I(\delta_I^J \gamma^\mu(\partial_\mu - ig(A_\mu)_I^J))\psi_J, \tag{6.101}$$

where ψ_I is a group-valued four-component Dirac spinor. We have omitted the Dirac indices, as is usually done. γ_μ are the four Dirac matrices and $\bar{\psi}^I = (\psi^I)^\dagger\gamma^0$ where $(\psi^I)^\dagger$ is the complex conjugate of the transpose of ψ_J viewed as a four-component vector. The indices I, J are those of a representation of the gauge group.

If one constructs the Hamiltonian theory of this action one finds that the canonical variables are ψ_I and its canonically conjugate momentum is $(\psi^I)^\dagger$. From here one can quantize and arrive at a "connection representation" in which wavefunctions are labeled by the Yang–Mills connection and the spinor field ψ_I, $\Psi[A, \psi]$.

One would like to find an analogue of the loop representation. In order to do this, one wants to introduce a transform in which one expands the wavefunctions of the connection representation in terms of a basis of gauge invariant quantities. The natural quantities that arise in this context are holonomies along open paths with fermions at their ends,

$$W(\pi_x^y) = (\psi^I)^\dagger H(\pi_x^y)_I^J \psi_J, \tag{6.102}$$

where we have used the letter W to stress the analogy with the Wilson loop. Notice that W has two Dirac indices which we omit, since we do not

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assume a contraction between $(\psi^I)^\dagger$ and ψ_J on the Dirac indices, which would be too restrictive.

Notice that we immediately face a difficulty in the sense that the above quantity depends on both the configuration variable and its conjugate momentum. We therefore cannot use it to expand the wavefunctions in the connection representation in terms of it. There do not exist, in general, natural invariants associated with a single open path that are functions only of the configuration variables. For example, in the Abelian case (QED) the only gauge invariant quantity has the form (6.102).

For particular gauge groups there are different alternatives for tackling this problem. For instance, one can make contractions (if the gauge group is special, like in $SU(N)$) with the Levi-Civita symbol in the gauge group and obtain gauge invariant quantities that only depend on the configuration variables. For example, for $SU(2)$ one can construct an object depending on a single path

$$W(\pi_x^y) = \psi_I(x)\epsilon^{IJ}H(\pi_x^y)_J^K\psi_K(y) \tag{6.103}$$

and we will see in chapter 10 a detailed discussion of the resulting representation in the case of fermions interacting with gravity.

Unfortunately for $SU(N)$ with $N > 2$ the Levi–Civita symbol has more than two indices and one is forced to consider more than a single path in order to construct an invariant. For instance, for $SU(3)$,

$$W(\pi_x^y, \eta_x^z, \gamma_x^w) = \epsilon_{LMN}H(\pi_x^y)_L^I H(\eta_x^z)_M^J H(\gamma_x^y)_N^K \psi_I(y)\psi_J(z)\psi_K(w). \tag{6.104}$$

The above object corresponds naturally to physical excitations of the theory in the confining phase. It represents a baryon constructed as three quarks at the ends of three gluon lines that join at the point x.

Evidently, constructing a representation in terms of the above objects is more complicated than we expected. It also leads to completely different representations, even at the most basic kinematical level, for the different gauge groups. Notice also that the above construction does not work for the simplest case, that of a $U(1)$ gauge theory.

A possibility for solving this problem, which has not been explored, would be to decompose the Dirac spinors in their up and down components and construct a representation with wavefunctions that are functionals of the connection, $(\psi^I_{\rm up})^\dagger$ and $\psi_{I\,\rm down}$. One can then construct gauge invariants that only depend on the configuration variables that are based on a single open path.

All this has led to the use of a different approach for the inclusion of fermions in the loop representation of gauge theories, inspired by the last observation about decomposing the Dirac spinor into its different components [121, 122]. The resulting procedure makes use of the staggered

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fermion technique introduced by Susskind in the context of lattice gauge theories [120]. This technique arose as a solution of the "fermion doubling" problem that is present in lattice gauge theories.

We will only present a discussion of this technique in a simplified context, that of a free theory in one spatial dimension. In that case, Dirac spinors have only two components (ψ_1, ψ_2). The spinors can be group-valued, but since we are not considering interactions this does not play any role, so we drop the group index in the subsequent discussion. The Dirac matrices in one dimension are given by $\gamma^0 = \sigma_3$ and $\alpha = \gamma^0\gamma^1 = \sigma_1$, where σ_i are the Pauli matrices and the (massless) Dirac equation is

$$\frac{\partial \psi}{\partial t} = -\alpha \frac{\partial \psi}{\partial x}. \tag{6.105}$$

If one now considers a one-dimensional lattice of spacing a the discretized equation is

$$\dot{\psi}(n) = -\frac{i\alpha}{2a}(\psi(n+1) - \psi(n-1)). \tag{6.106}$$

The solution of the continuum equation is given by plane waves of the form $\exp(ik_1 x - k_0 t)$, which lead to the eigenvalue problem $k_0\psi = k\alpha\psi$. The solution of the eigenvalue problem leads to a dispersion relation $k_0 = \pm k_1$. The discrete equation, on the other hand, has solutions of the form $\exp(ik_1 na - k_0 t)$. In this case k_1 takes a discrete set of values $|k_1| = \pi m/Na$ where N is the number of lattice sites and $m \leq N$. This corresponds to a Brillouin zone of $|k| \leq \pi/a$. The resulting eigenvalue problem is

$$k_0\psi = \alpha \frac{\sin(ka)}{a}\psi, \tag{6.107}$$

which leads to a dispersion relation $k_0 = \pm \sin(k_1 a)/a$.

There are two values of $k_1 a$ that lead to a continuum limit, $k_1 a = 0$ and $k_1 a = \pm\pi$. For a given value of k_0 close to zero, there are two values of k_1 allowed by the dispersion relation, each close to the two values of $k_1 a$ that lead to continuum limits. In one case the corresponding k_1 is positive and in the other negative. This is the root of the fermion doubling problem in the lattice: in the continuum limit one gets two fermions moving in opposite directions.

The staggered fermion technique consists in putting the different components of the Dirac spinor in different lattice positions. For the one-dimensional case we are considering this amounts to putting the two components in alternating positions in the lattice. In $3 + 1$ dimensions it is considerably more complicated, since one has to double each dimension of the lattice and therefore there is an eight-fold increase in the components.

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The end result is that the up components lie at the even sites and the down components at the odd sites. See reference [120] for more details.

After staggering the lattice position of the two components of the Dirac fermion, the discretized Dirac equation reads in components,

$$\dot{\psi}_1(n) = \frac{1}{2a}(\psi_2(n+1) - \psi_2(n-1)), \tag{6.108}$$

$$\dot{\psi}_2(n) = \frac{1}{2a}(\psi_1(n+1) - \psi_1(n-1)). \tag{6.109}$$

We now introduce a field $\phi(n)$ defined by

$$\phi(n) = \begin{cases} \psi_1(n) \text{ for } n \text{ even}, \\ \psi_2(n) \text{ for } n \text{ odd}, \end{cases} \tag{6.110}$$

in terms of which we can rewrite the Dirac equation as

$$\dot{\phi}(n) = \frac{1}{2a}(\phi(n+1) - \phi(n-1)). \tag{6.111}$$

We therefore see that the resulting equation (6.111) is equivalent to the original Dirac equation but with a double lattice spacing. This translates in terms of the momentum space into a reduction of the Brillouin zone to half its original size, i.e, $|k| \leq \pi/(2a)$. This excises from the dispersion relation the second continuum limit point.

With this idea in hand, we are now in a position to return to the main argument which was to define gauge invariant quantities depending only on configuration variables to introduce a geometric formulation for Yang–Mills theories interacting with fermions. To do that one considers as configuration variables the Dirac fields $\psi(y_{\text{odd}})$ at the odd sites and their conjugate momenta $\psi(x_{\text{even}})^\dagger$ at the even sites. One introduces the following quantities:

$$W(\pi_x^y) = \psi(x_{\text{even}})^\dagger H(\pi_x^y)\psi(y_{\text{odd}}), \tag{6.112}$$

in terms of which one can define a transform to a representation purely in terms of paths. In this representation one can now realize the action of physical excitations, such as the baryonic excitation (6.104). New Mandelstam identities arise relating baryonic excitations and open-path mesonic excitations. We will not present the details here, the reader is referred to reference [121].

6.5 Conclusions

We have seen several examples of the formulation of gauge theories in the lattice in terms of loop representations. It was shown that practical calculations of excitation energies and observables are feasible in the language

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of loops. The main advantage is that the formulation is gauge invariant and the action of the operators admits very simple geometric formulations in the lattice. In the case of the inclusion of fermions through the use of open paths one does not need to introduce Grassmann variables, which leads to computational economies. The main drawback is that the basis of loops grows very rapidly with the lattice size and since the description is Hamiltonian one does not have at hand statistical methods, like the Monte Carlo techniques, to deal efficiently with a large number of degrees of freedom. The use of cluster techniques, as we have seen, allows us with relative simplicity to obtain a complete approximate description of the phase diagram of theories, although it is not a systematic approximation procedure.