A Type of Alternant

By F. W. Ponting

(*Received 4th May*, 1950. *Read 3rd November*, 1950.)

1. *Introduction.*

We define

$$\alpha_j^{(k)} = (\alpha_j + \beta_1)(\alpha_j + \beta_2) \ldots (\alpha_j + \beta_k),$$

where $\alpha_p \neq \alpha_q$ when $p \neq q$. If $N = \Sigma \lambda_i$, then the partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of N with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ is denoted by (λ) and we set

$$l_j = \lambda_j + n - j.$$

All partitions will be in descending order and the usual notation for repeated parts will be used.

The determinant with $f(s, t)$ in row s and column t will be denoted by $|f(s, t)|$. The use of s and t implies that the determinant is of order n. For other orders σ and τ will be used.

We consider the function

$$\{\alpha;\ (\lambda);\ \beta\}$$

defined by

$$|\alpha_s^{(l_t)}| = \{\alpha;\ (\lambda);\ \beta\}\,|\alpha_s^{n-t}|.$$

If every $\beta_i = 0$, then we have the S-function $\{\alpha;\ (\lambda)\}$ defined by

$$|\alpha_s^{l_t}| = \{\alpha;\ (\lambda)\}\,|\alpha_s^{n-t}|$$

[**1**, chap. VI].

When $0 < v \leqslant u$, we define $b(u, v)$ to be the v-th elementary symmetric function of $\beta_1, \beta_2, \ldots, \beta_u$. We set $b(0, 0) = 1 = b(u, 0)$ and $b(u, v) = 0$ if $v < 0$ or $u < v$. We take $H(u, v)$ as the v-th complete homogeneous symmetric function of $\beta_1, \beta_2, \ldots, \beta_u$ when $0 < v$, and $H(0, 0) = H(u, 0) = 1$, $H(u, v) = 0$ if $v < 0$.

In this note we prove the following theorems:

Theorem 1. *If* $b\{(l), (r)\} = |b(l_s, l_s - l_t + r_t)|$, *then*

$$\{\alpha;\ (\lambda);\ \beta\} = \Sigma \{\alpha;\ (\lambda_1 - r_1, \lambda_2 - r_2, \ldots, \lambda_n - r_n)\}\, b\{(l), (r)\},$$

where the summation is taken over all non-negative r_i *such that*

$$\lambda_1 - r_1 \geqslant \lambda_2 - r_2 \geqslant \ldots \geqslant \lambda_n - r_n \geqslant 0.$$

https://doi.org/10.1017/S0013091500013997 Published online by Cambridge University Press

THEOREM 2. *If* $\lambda_1 \leqslant n$, (μ) *and* $(\mu - r')$ *are partitions conjugate to* (λ) *and* $(\lambda - r)$ *respectively and* $m_s = \mu_s + n - s$, *then*

$$b\{(l), (r)\} = |H(2n - m_s, m_s - m_t + r_t')| = H\{(m), (r')\}, \text{ say.}$$

THEOREM 3. *The function* $b\{(l), (r)\}$ *may be expanded as a polynomial in* $\beta_1, \beta_2, \ldots, \beta_{l_1}$ *with positive integral coefficients.*

Since Theorem 2 is solely concerned with the β_i, we can choose $n \geqslant \lambda_1$ by adding a sufficient number of zero parts to (λ).

Hirsch [**2**] considered the case of $(\lambda) = (1^{n-k}, 0^k)$ and his result may be put in the form[1]

$$\{\alpha;\ (1^{n-k}, 0^k);\ \beta\} = \sum_{r=0}^{n-k} \{\alpha;\ (1^{n-k-r})\} H(k+1, r). \tag{1}$$

We may obtain the dual result

$$\{\alpha;\ (n-k, 0^{n-1});\ \beta\} = \sum_{r=0}^{n-k} \{\alpha;\ (n-k-r)\} b(2n-k-1, r) \tag{2}$$

by subtracting appropriate multiples of the columns of

$$|\alpha_s^{(l_t)}|, \quad (l_1 = 2n-k-1,\ l_2 = n-2,\ l_3 = n-3,\ \ldots,\ l_n = 0),$$

from the preceding columns.

Using Theorem 1, we find that in the expansion of (1) we have a term with

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{n-k} = 1, \quad \lambda_{n-k+1} = \lambda_{n-k+2} = \ldots = \lambda_n = 0,$$

$$r_1 = r_2 = \ldots = r_t = 0, \quad r_{t+1} = r_{t+2} = \ldots = r_{n-k} = 1, \quad r_{n-k+1} = \ldots = r_n = 0;$$

so that $\qquad (\lambda) = (1^{n-k}), \quad (\lambda - r) = (1^t)$

and in Theorem 2

$$(\mu) = (n-k), \quad (\mu - r') = (t).$$

Thus the coefficient of $\{\alpha;\ (1^t)\}$ is

$$b\{(l), (r)\} = |H(2n - m_s, m_s - m_t + r_t')|$$

where

$$m_1 = 2n-k-1, \quad m_2 = n-2, \quad m_3 = n-3, \quad \ldots, \quad m_n = 0,$$

$$r_1' = n-k-t, \quad r_2' = r_3' = \ldots = r_n' = 0.$$

The first column of $|H(2n - m_s, m_s - m_t + r_t')|$ now has

$$H(k+1, n-k-t)$$

in the first row and zero below, since $m_s - m_1 + r_1' < 0$ for $s > 1$. The other columns will have unity on the principal diagonal position and zero below. Hence we have (1).

[1] We omit zero parts when there is no danger of ambiguity.

https://doi.org/10.1017/S0013091500013997 Published online by Cambridge University Press

In (2), we find that we have a term with

$$\lambda_1 = n-k, \quad r_1 = r, \quad \lambda_j = r_j = 0 \quad (j>2),$$

and the coefficient of $\{\alpha;\ (1^{n-k-r})\}$ is $b(2n-k-1,\ r)$ from Theorem 1.

As a further example we consider

$$\begin{vmatrix} \alpha_1^{(4)} & \alpha_1^{(2)} & 1 \\ \alpha_2^{(4)} & \alpha_2^{(2)} & 1 \\ \alpha_3^{(4)} & \alpha_3^{(2)} & 1 \end{vmatrix} = \begin{vmatrix} \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^2 & \alpha_3 & 1 \end{vmatrix} \quad \{\alpha;\ (2,1,0);\ \beta\}.$$

We denote $b(u, v)$ by u, v, $\{\alpha;\ \lambda\}$ by $\{\lambda\}$, $-t$ by $\bar{t}$, and find that

$$\{\alpha;\ (2,1,0);\ \beta\} = \{2,1\}\begin{vmatrix} 4,0 & 4,2 & 4,4 \\ 2,\bar{2} & 2,0 & 2,2 \\ 0,\bar{4} & 0,\bar{2} & 0,0 \end{vmatrix} + \{2\}\begin{vmatrix} 4,0 & 4,3 & 4,4 \\ 2,\bar{2} & 2,1 & 2,2 \\ 0,\bar{4} & 0,\bar{1} & 0,0 \end{vmatrix}$$

$$+\{1^2\}\begin{vmatrix} 4,1 & 4,2 & 4,4 \\ 2,\bar{1} & 2,0 & 2,2 \\ 0,\bar{3} & 0,\bar{2} & 0,0 \end{vmatrix} + \{1\}\begin{vmatrix} 4,1 & 4,3 & 4,4 \\ 2,\bar{1} & 2,1 & 2,2 \\ 0,\bar{3} & 0,\bar{1} & 0,0 \end{vmatrix} + \begin{vmatrix} 4,2 & 4,3 & 4,4 \\ 2,0 & 2,1 & 2,2 \\ 0,\bar{2} & 0,\bar{1} & 0,0 \end{vmatrix}$$

$$= \{2,\ 1\} + \{2\}b(2,\ 1) + \{1^2\}b(4,\ 1) + \{1\}b(4,\ 1)b(2,\ 1) + \begin{vmatrix} b(4,\ 2) & b(4,\ 3) \\ b(2,\ 0) & b(2,\ 1) \end{vmatrix}.$$

As an example of Theorem 3, we consider

$$\begin{vmatrix} 4,\ 2 & 4,\ 3 & 4,\ 4 \\ 2,\ 0 & 2,\ 1 & 2,\ 2 \\ 1,\ \bar{1} & 1,\ 0 & 1,\ 1 \end{vmatrix} = b\{(4,\ 2,\ 1);\ (2,\ 1,\ 1)\}.$$

This is the term independent of the α_i in the expansion of $\{\alpha;\ (2, 1^2);\ \beta\}$ and it does not factorise into determinants of the same type but lower order. The term independent of the α_i in the expansion of $\{\alpha;\ (1^3);\ \beta\}$ also has this property and it is $h(1,\ 3)$. These two terms are the first of order 3 which have the property.

Now

$$b\{(4,\ 2,\ 1),\ (2,\ 1,\ 1)\}$$

$$= \begin{vmatrix} \beta_1\beta_2+(\beta_1+\beta_2)(\beta_3+\beta_4)+\beta_3\beta_4 & \beta_1\beta_2(\beta_3+\beta_4)+(\beta_1+\beta_2)\beta_3\beta_4 & \beta_1\beta_2\beta_3\beta_4 \\ 1 & \beta_1+\beta_2 & \beta_1\beta_2 \\ 0 & 1 & \beta_1 \end{vmatrix}$$

$$= (\beta_3+\beta_4)\begin{vmatrix} \beta_1+\beta_2 & \beta_1\beta_2 & 0 \\ 1 & \beta_1+\beta_2 & \beta_1\beta_2 \\ 0 & 1 & \beta_1 \end{vmatrix} + \begin{vmatrix} \beta_1\beta_2 & 0 & 0 \\ 1 & \beta_1+\beta_2 & \beta_1\beta_2 \\ 0 & 1 & \beta_1 \end{vmatrix}$$

$$= (\beta_3+\beta_4)\beta_1^3 + \beta_1^3\beta_2.$$

https://doi.org/10.1017/S0013091500013997 Published online by Cambridge University Press

Since this note was first submitted, Foulkes [8] has given a different method of obtaining (1), (2) and Theorem 1.

2. *Expansion of* $\{\alpha;\ (\lambda);\ \beta\}$.

We have, when $k \leqslant l_1$,

$$\alpha_j^{(k)} = \sum_{i=k-l_1}^{k} b(k, i)\, \alpha_j^{k-i},$$

so that $|\alpha_s^{(l_t)}|$ is the determinant of the product of the matrices

$$A = [\alpha_s^{l_1-\tau}] \quad \text{and} \quad B = [b(l_t,\ l_t - l_1 + \sigma)],$$

where $\sigma, \tau = 0, 1, 2, \ldots, l_1;\ s, t = 1, 2, \ldots, n$.

It is well-known [*cf.* **3**, 86] that the determinant $|AB|$ is the sum of the $\binom{l_1+1}{n}$ products of pairs of corresponding n-th order determinants which can be formed from A and B', the transpose of B, each determinant occurring once only.

The determinant in B' corresponding to $|\alpha_s^{l_t - r_t}|$ is

$$|b(l_s,\ l_s - l_t + r_t)|. \tag{3}$$

We may select, and account for all n-th order determinants from A, by demanding that

$$l_1 - r_1 > l_2 - r_2 > \ldots > l_n - r_n, \quad \textit{i.e.} \quad \lambda_1 - r_1 \geqslant \lambda_2 - r_2 \geqslant \ldots \geqslant \lambda_n - r_n.$$

Moreover, $r_i \leqslant \lambda_i$ since $l_i - r_i \geqslant n - i$, the least possible exponent for column i. Hence the coefficient of $\{\alpha;\ (\lambda_1 - r_1, \lambda_2 - r_2, \ldots, \lambda_n - r_n)\}$ in the expansion of $\{\alpha;\ (\lambda);\ \beta\}$ is

$$|b(l_s,\ l_s - l_t + r_t)|.$$

We have $l_h \leqslant l_g$ and $l_g - l_j + r_j \leqslant r_g$ for $j \leqslant g \leqslant h$. Hence, if $r_g < 0$, then $b(l_h,\ l_h - l_j + r_j) = 0$. In this case (3) vanishes, having zero elements in the first g terms of the last $n-g+1$ rows. This completes the proof of Theorem 1.

If $r_{g-1} < l_{g-1} - l_g$, *i.e.* $r_{g-1} \leqslant \lambda_{g-1} - \lambda_g$, we can show similarly that (3) factorises into two lower order determinants of the same kind as (3). We note that the graph [**1**, 67] of the partition $(\lambda_1 - r_1, \lambda_2 - r_2, \ldots, \lambda_n - r_n)$ must be regular for a non-zero term. However, if we construct this graph by removing the last r_i nodes from row i of the graph of (λ) for $i = 1, 2, \ldots, n$ in succession, and if we have a regular graph at any, except the last, stage, then (3) will factorise.

https://doi.org/10.1017/S0013091500013997 Published online by Cambridge University Press

3. *Duality.*

Lemma 1. *The p-th order matrices* $H_1 = [H(p-\tau+1, \tau-\sigma)]$ *and* $B_1 = [(-1)^{\tau-\sigma} b(p-\sigma, \tau-\sigma)]$ *are reciprocal and hence adjoint.*

Proof. If $H = [H(p, \tau-\sigma)]$, $B = [(-1)^{\tau-\sigma} b(p, \tau-\sigma)]$, then $HB = I$, the unit matrix [*cf.* **3**, 115]. We set Q_{ur} as the p-th order square matrix with 1 on the principal diagonal, $-\beta_u$ in row $r-1$ of column r, and zero elsewhere,

$$Q_u = Q_{u,p}\, Q_{u,p-1} \cdots Q_{u,p-u+2} \quad \text{and} \quad Q = Q_p\, Q_{p-1} \cdots Q_2.$$

Now
$$b(i+1, j+1) = b(i, j+1) + \beta_{i+1} b(i, j),$$
$$H(i+1, j+1) = H(i, j+1) + \beta_{i+1} H(i+1, j).$$

Then since $H(r, 0) = b(r, 0) = 1 = b(0, 0)$, we have $HQ = H_1$ and $Q^{-1} B = B_1$. Hence $H_1 B_1 = I$, and since the determinant of H_1 is 1, then H_1 and B_1 are adjoint.

Lemma 2. *If* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *and* $(\mu_1, \mu_2, \ldots, \mu_n)$ *are conjugate partitions, then* $n+s-\lambda_s$ *and* $n+1+\mu_s-s$ $(s = 1, 2, \ldots, n)$, *form a permutation of* $1, 2, \ldots, 2n$.

This is merely a re-statement of Aitken's rule [**5**], that

$$(\lambda_n, \lambda_{n-1}+1, \ldots, \lambda_1+n-1) \quad \text{and} \quad (\mu_n, \mu_{n-1}+1, \ldots, \mu_1+n-1)$$

form bicomplementary sets in relation to the set $0, 1, 2, \ldots, 2n-1$.

Proof of Theorem 2. This is based on a similar proof in [**5**]. If $r_1+r_2+\ldots+r_n = r$, then $|(-1)^{l_s-l_t+r_t} b(l_s, l_s-l_t+r_t)|$ is the minor of B_1 (for $p = 2n$) formed by rows $n+s-\lambda_s$ and columns $n+t-(\lambda_t-r_t)$, which by Jacobi's theorem and Lemma 2, is equal to $(-1)^r$ times the minor formed from the transpose of H_1 by rows $n+1+\mu_s-s$ and columns $n+1+\mu_t-r_t'-t$. Hence we have

$$\begin{aligned} |b(l_s, l_s-l_t+r_t)| &= (-1)^r |(-1)^{l_s-l_t+r_t} b(l_s, l_s-l_t+r_t)| \\ &= |H\{2n+1-(n+1+\mu_s-s),\ n+1+\mu_s-s-(n+1+\mu_t-r_t'-t)\}| \\ &= |H(2n-m_s,\ m_s-m_t+r_t')|. \end{aligned}$$

This completes the proof of Theorem 2.

We note that from Lemma 1 we may deduce modified Wronski recurrence formulae; for $1 \leqslant r \leqslant p-1$, we have

$$H(p, 0)\, b(p-1, r) - H(p-1, 1)\, b(p-2, r-1) + H(p-2, 2)\, b(p-3, r-2) - + \ldots = 0,$$

$$b(p-1, 0)\, H(p-r, r) - b(p-1, 1)\, H(p-r, r-1) + b(p-1, 2)\, H(p-r, r-2) - + \ldots = 0.$$

https://doi.org/10.1017/S0013091500013997 Published online by Cambridge University Press

If we replace $b(u, r)$ by $H(2n-u, r)$ in $b\{(l), (r)\}$, we obtain

$$|H(2n-l_s, l_s-l_t+r_t)|$$

which is equal to $$|b(m_s, m_s-m_t+r'_t)|,$$

thus illustrating the duality.

As in Section 2, we find that $H\{(m), (r')\}$ will factorise if

$$r'_{g-1} < m_{g-1}-m_g, \quad i.e. \quad r'_{g-1} \leqslant \mu_{g-1}-\mu_g.$$

Hence if we remove r_i' nodes from column i of the graph of (λ) in succession, and if, at any stage except the last, we have a regular graph, then $H\{(m), (r')\}$ will factorise into two determinants of the same type but of lower order.

4. *Proof of Theorem* 3.

We assume that

$$r_{g-1} \geqslant l_{g-1}-l_g \quad (g = 2, 3, \ldots, n),$$

so that $b\{(l), (r)\}$ does not factorise.

When $0 < q \leqslant l_{u-1}-l_u$, we define

$$(l_{u-1}, l_u \between q)$$

to be the q-th elementary symmetric function of

$$\beta_{l_u+1}, \beta_{l_u+2}, \ldots, \beta_{l_{u-1}}.$$

We set $(l_{u-1}, l_u \between 0) = 1$ and $(l_{u-1}, l_u \between r) = 0$ if $r < 0$ or $r > l_{u-1}-l_u$.

We denote the n-th order determinant

$$\begin{vmatrix} b(l_u, l_u-l_t+r_t+v_{s'}) \\ \cdots\cdots\cdots\cdots \\ b(l_{s''}, l_{s''}-l_t+r_t) \end{vmatrix} \quad (s' = 1, 2, \ldots, u-1;\ s'' = u, u+1, \ldots, n),$$

by $$[v_1, v_2, \ldots, v_{u-1}].$$

Now

$$b(l_u, l_u-l_t+r_t) = \sum_{p=0}^{l_u-l_{u+1}} (l_u, l_{u+1} \between l_u-l_{u+1}-p)\, b(l_{u+1}, l_{u+1}-l_t+r_t+p). \quad (4)$$

Hence $$b\{(l), (r)\} = \sum_{p_{11}=1}^{l_1-l_2} (l_1, l_2 \between l_1-l_2-p_{11})\, [p_{11}],$$

the term for $p_{11} = 0$ vanishing.

Similarly,

$$[p_{11}] = \sum_{p_{12}=0}^{l_2-l_3} \sum_{p_{22}=1}^{l_2-l_3} (l_2, l_3 \between l_2-l_3-p_{12})(l_2, l_3 \between l_2-l_3-p_{22})\, [p_{11}+p_{12}, p_{22}]$$

$$= \Sigma \begin{vmatrix} (l_2, l_3 \between l_2-l_3-p_{12}) & (l_2, l_3 \between l_2-l_3-(p_{22}-p_{11})) \\ (l_2, l_3 \between l_2-l_3-p_{11}-p_{12}) & (l_2, l_3 \between l_2-l_3-p_{22}) \end{vmatrix} [p_{11}+p_{12}, p_{22}], \quad (5)$$

https://doi.org/10.1017/S0013091500013997 Published online by Cambridge University Press

where the summation is taken over

$$0 \leqslant p_{12} \leqslant l_2 - l_3, \quad 1 \leqslant p_{22} \leqslant l_2 - l_3,$$

subject to

$$p_{11} + p_{12} > p_{22} > 0.$$

We set

$$P_{tu} = p_{tt} + p_{t,t+1} + \ldots + p_{tu} \text{ when } t \leqslant u, \qquad P_{tu} = 0 \text{ when } t > u,$$

and use Σ_u to denote summation over

$$0 \leqslant p_{ju} \leqslant l_u - l_{u+1}, \quad i.e. \quad P_{j,u-1} \leqslant P_{ju} \leqslant l_u - l_{u+1} + P_{j,u-1},$$

subject to

$$P_{1u} > P_{2u} > \ldots > P_{uu} > 0.$$

Then (5) may be written

$$[P_{11}] = \Sigma_2 \left| (l_2, l_3 \between l_2 - l_3 + P_{\sigma 1} - P_{\tau 2}) \right| [P_{12}, P_{22}] \quad (\sigma, \tau = 1, 2).$$

Now $P_{1,u-1} > P_{s,u-1}$ for $s > 1$, and hence from (4)

$$b(l_u, l_u - l_t + r_t + P_{s,u-1}) = \sum_{w=0}^{l_u - l_{u+1} + P_{1,u-1}} (l_u, l_{u+1} \between l_u - l_{u+1} + P_{s,u-1} - w) \, b(l_{u+1}, l_{u+1} - l_t + r_t + w).$$

Then by a proof similar to that of Theorem 1 we obtain

$$[P_{1,u-1}, P_{2,u-1}, \ldots, P_{u-1,u-1}] = \Sigma_u \left| (l_u, l_{u+1} \between l_u - l_{u+1} + P_{\sigma,u-1} - P_{\tau,u}) \right| [P_{1u}, P_{2u}, \ldots, P_{uu}]$$
$$(\sigma, \tau = 1, 2, \ldots, u). \quad (6)$$

We find that if

$$P_{1u} > P_{2u} > \ldots > P_{uu} \geqslant 0,$$

then the coefficient of

$$[P_{1u}, P_{2u}, \ldots, P_{uu}] \quad (7)$$

in the expansion of (6) is

$$\left| (l_u, l_{u+1} \between l_u - l_{u+1} + P_{\sigma,u-1} - P_{\tau,u}) \right| \quad (\sigma, \tau = 1, 2, \ldots, u). \quad (8)$$

Then in (8),

$$P_{j,u-1} + l_u - l_{u+1} \geqslant P_{ju};$$

otherwise the first j elements of the last $n-j+1$ rows of (8) will be zero, and the determinant vanishes. Also

$$P_{ju} \geqslant P_{j,u-1};$$

otherwise the last $n-j+1$ elements of the first j rows will be zero and the determinant vanishes. If $P_{uu} = 0$, then rows u and $u+1$ of (7) are equal. Hence the expansion of (6) follows.

We set $\xi_\sigma = l_u - l_{u+1} + P_{\sigma, u-1} + n + \sigma$,

$\eta_\tau = P_{\tau, u} + n + \tau$;

then $\xi_\sigma \geqslant \xi_{\sigma+1}$ and $\eta_\tau \geqslant \eta_{\tau+1}$.

Hence $|(l_u, l_{u+1})(l_u - l_{u+1} + P_{\sigma, u-1} - P_{\tau, u})|$

is the S-function [**1**, 110]

$$\{\tilde{\xi}/\tilde{\eta}\} \tag{9}$$

of
$$\beta_{l_{u+1}+1}, \ldots, \beta_{l_u-1}, \beta_{l_u}, \tag{10}$$

where the tilde denotes conjugate partition.

Now
$$\{\tilde{\xi}/\tilde{\eta}\} = \Sigma\, g_{\zeta\eta\xi}\{\tilde{\zeta}\},$$

where the $g_{\zeta\eta\xi}$ are non-negative integers determined by

$$\{\zeta\}\{\eta\} = \Sigma\, g_{\zeta\eta\xi}\{\xi\}$$

[*cf*. **1**, 110, 91-96]. Since $\{\tilde{\zeta}\}$ itself may be expanded as a polynomial in (10) with positive integral coefficients[1], all the terms in the expansion of (6) as a polynomial in (10) have positive integral coefficients. Theorem 3 follows on repeated application of this argument.

We note that we may write in symbolic form:

$$|b(l_s, l_s - l_t + r_t)| = \prod_{u=1}^{n-1} \{\Sigma_u |(l_u, l_{u+1})(l_u - l_{u+1} + P_{\sigma, u-1} - P_{\tau, u})|\, |b(l_n, l_n - l_t + r_t + P_{s, n-1})|\}.$$

[1] This is well known [*cf*. **7**]. Aitken [**6**] gives a direct proof that (9) may be expanded with positive integral coefficients.

REFERENCES.

1. D. E. Littlewood, *Theory of Group Characters* (Oxford, 1940).

2. K. A. Hirsch, " A note on Vandermonde's determinant ", *Journal London Math. Soc.*, 24 (1949), 144–5.

3. A. C. Aitken, *Determinants and Matrices* (5th ed., Edinburgh, 1948).

4. ——— " On determinants of symmetric functions ", *Proc. Edinburgh Math. Soc.* (2), 1 (1929), 55.

5. ——— " Note on dual symmetric functions ", *Proc. Edinburgh Math. Soc.* (2), 2 (1931), 164.

6. ——— " The monomial expansion of determinantal symmetric functions ", *Proc. Royal Soc. Edinburgh* (A), 61 (1941–3), 300–310.

7. O. H. Mitchell, " Note on determinants of powers ", *American Journal Math.*, 4 (1881), 341–4.

8. H. O. Foulkes, " Modified bialternants and symmetric function identities ", *Journal London Math. Soc.*, 25 (1950), 268–75.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ABERDEEN.