CONVOLUTIONS AS BILINEAR AND LINEAR OPERATORS

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1. Basic properties of convolution. Throughout this paper X denotes a fixed Hausdorff locally compact group with left Haar measure dx. Various spaces of functions and measures on X will recur in the discussion, so we name and describe them forthwith. All functions and measures on X will be scalar-valued, though it matters little whether the scalars are real or complex.

$C = C(X)$ is the space of all continuous functions on X, $C_c = C_c(X)$ its subspace formed of functions with compact supports. $M = M(X)$ denotes the space of all (Radon) measures on X, $M_c = M_c(X)$ the subspace formed of those measures with compact supports. In general we denote the support of a function or a measure ξ by $[\xi]$.

What we shall term the "natural topology on C" is that of convergence locally uniform on X; the corresponding dual is then M_c, the duality being defined by $\langle f, \mu\rangle = \int f d\mu$. The "natural topology on C_c" is more complicated; it is obtained by regarding C_c as the internal inductive limit of its subspaces

$$C_{c,K} = \{f \in C_c: [f] \subset K\},$$

K varying over any base for the compact subsets of X, and $C_{c,K}$ being made into a Banach space with the norm

$$||f|| = \sup_{x\epsilon X}|f(x)|.$$

The dual of C_c is M, the duality being defined as above. Associated with these dualities are the weak topologies $\sigma(M, C_c)$, $\sigma(M_c, C)$, $\sigma(C, M_c)$, and $\sigma(C_c, M)$, the first of which is "the vague topology of measures."

Convolution is fundamentally a bilinear operation on (suitably restricted) pairs of measures. If λ and μ are measures, the convolution $\lambda*\mu$ is said to exist if and only if

$$(1.1)\quad \int^* f(xy)d(|\lambda| \otimes |\mu|)(x, y) = \int^* d|\lambda|(x) \int^* f(xy)d|\mu|(y) = \int^* d|\mu|(y) \int^* f(xy)d|\lambda|(x) < +\infty$$

for each positive function $f \in C_c$, in which case $\lambda * \mu$ is the measure defined by setting

$$(1.2)\quad \int f d(\lambda * \mu) = \int^* f(xy)d(\lambda \otimes \mu)(x, y) = \text{etc.}$$

for each $f \in C_c$.

Received May 2, 1963.

https://doi.org/10.4153/CJM-1964-027-0 Published online by Cambridge University Press

In this paper we shall be directly concerned only with the case in which at least one of λ or μ has a compact support, in which case the existence of $\lambda * \mu$ is certain.

Functions are introduced into convolution formulae by means of the convention that each function f, locally integrable for dx, be thought of as the measure having f as its Radon-Nikodym derivative with respect to dx. One then finds that in all the cases we wish to consider one may write

$$(1.3)\qquad \begin{aligned} \mu * f(x) &= \int f(y^{-1}x)d\mu(y), \\ f * \mu(x) &= \int \Delta(y) f(xy^{-1})d\mu(y), \\ f * g(x) &= \int f(y)g(y^{-1}x)dy, \end{aligned}$$

f and g denoting functions and μ a measure. In the second formula, Δ denotes the modular function of X, so that one has the characteristic identities

$$\begin{aligned} \int f(xa)dx &= \Delta(a)\int f(x)dx, \\ \int f(x^{-1})dx &= \int \Delta(x)f(x)dx. \end{aligned}$$

If f is continuous and one of μ (or g) and f has a compact support, the integrals in (1.3) are defined for all x, and the formulae mean that (for example) the convolution of μ and the measure fdx (in that order) exists and coincides with the measure whose Radon-Nikodyn derivative is equal l.a.e. to the function given by the appropriate integral. Notice that under the conditions stated the functions appearing in (1.3) are actually continuous.

As we propose to show in due course, convolution is largely characterized by its relationship with the left- and right-translation operators L_x and R_x associated with group elements x. It is convenient to define the action of L_x and R_x on functions f by the formulae

$$(1.4)\qquad L_x f(y) = f(x^{-1}y), \qquad R_x f(y) = \Delta(x)f(yx^{-1}).$$

These actions can be consistently defined for measures μ by setting

$$(1.5)\qquad \begin{aligned} \int fd(L_x\mu) &= \int f(xy)d\mu(y), \\ \int fd(R_x\mu) &= \int f(yx)d\mu(y), \end{aligned}$$

f ranging over C_c in each case.

There is one formula, itself expressing a close relationship between convolution and the translation operators, which will be useful, namely

$$(1.6)\qquad \lambda * \mu = \int (L_x\mu)d\lambda(x) = \int (R_x\lambda)d\mu(x).$$

In this formula, each of $x \to L_x\mu$ and $x \to R_x\lambda$ is regarded as a continuous function on X with values in M (endowed with its vague topology $\sigma(M, C_c)$). The formula is easily verifiable when at least one of λ or μ has a compact support; it is true in other cases as well, but we shall not need this.

The elementary inclusion $[\lambda * \mu] \subset [\lambda].[\mu]$, valid whenever $\lambda * \mu$ is defined, permits one to write down the following inclusion relations:

$$\begin{aligned}
&M_c * M \subset M, \quad M * M_c \subset M, \quad M_c * M_c \subset M_c,\\
(1.7)\qquad &C_c * M \subset C, \quad M * C_c \subset C,\\
&C_c * M_c \subset C_c, \quad M_c * C_c \subset C_c, \quad C_c * C_c \subset C_c.
\end{aligned}$$

The notation is almost self-explanatory: $A * B$ means the set of (defined) convolutions $\alpha * \beta$ with $\alpha \in A$ and $\beta \in B$. These relations are of significance when related to our theorems.

In seeking to characterize convolution as a bilinear operation, there are many possible choices of domain, some only of which are suggested by (1.7). A reasonably minimal hypothesis is to take as domain the product $C_c \times C_c$. Consider then the bilinear operator B on $C_c \times C_c$ defined by

$$B(f, g) = f * g. \tag{1.8}$$

Of the many properties of B we focus attention on the following five:

(1) *B is positive, i.e. $B(f, g) \geqslant 0$ if f and g are positive functions in C_c.*

(2) *For fixed $f \in C_c$, $g \to B(f, g)$ commutes with the R_x.*

(3) *For fixed $g \in C_c$, $f \to B(f, g)$ commutes with the L_x.*

(4_0) *If $f \in C_c$, then $B(f, g)$ is the limit in C_c of finite linear combinations of the $L_x g$ with $x \in [f]$.*

(5) *B is continuous from $C_c \times C_c$ into C_c.*

Property (1) is evident; (2) and (3) follow most easily from (1.6). To prove (5) it is enough to verify that the restriction of B to $C_{c,k} \times C_{c,K}$ is continuous for each compact $K \subset X$, which is very simple. To establish (4_0) one may use a routine argument based on the pointwise representation of $f * g$ given in (1.3), or one may use the abstract approach based upon (1.6), the integrands being now continuous from X into C_c.

Our primary aim is to show that some or all of these properties (sometimes even in weakened forms) suffice already to ensure that the B appearing in them differs from $f * g$ by at most a constant multiplicative factor. While dealing with this question in Sections 2 and 3 it becomes apparent that the problem is very closely connected with that of representing as convolutions linear operators which commute with translations, and we include in Section 4 some specialized results bearing upon such representation theorems.

2. The main theorem.

THEOREM 1. *Suppose that B is a bilinear operator from $C_c \times C_c$ into M with properties* (1), (2), (3) *above and the following weakened form of* (4_0):

(4) *For given $f \in C_c$, $B(f, g)$ is the vague limit of finite linear combinations of the* $L_x g$ with $x \in [f]$. *Then there exists a number $c \geqslant 0$ such that*

$$B(f, g) = \text{c.f.} * g \tag{2.1}$$

for f and g in C_c.

The proof of this theorem will be based upon the following proposition, which has some intrinsic interest.

PROPOSITION 1. *Let T be any positive linear operator mapping C_c into M which commutes with the R_x. Then there exists a measure $\mu \in M$ such that $\mu \geqslant 0$ and*

$$Tf = \mu * f \tag{2.2}$$

for $f \in C_c$. There is an analogous assertion, with $f * \mu$ in place of $\mu * f$, if T is known to commute with the L_x.

Proof of Proposition 1. The positivity of T is easily seen to ensure that it is continuous for the natural topology on C_c and any vector space topology on M. If f and g belong to C_c we may write (cf. (1.6))

$$f * g = \int (R_x f) g(x) dx.$$

Since T is continuous and commutes with the R_x, it follows from this that

$$T(f * g) = \int (R_x Tf) g(x) dx = Tf * g. \tag{2.3}$$

Let (N_i) be a base of neighbourhoods of the neutral element e of X, i ranging over some directed set I. We may assume that all the N_i lie within some chosen compact $N_0 \subset X$. For each i choose a positive function f_i in C_c such that $[f_i] \subset N_i$ and

$$\int f_i(x) dx = 1.$$

Since $\lim_i (f_i * g) = g$ in C_c, it follows from (2.3) that

$$Tg = \lim_i (\mu_i * g), \tag{2.4}$$

provided we write μ_i for the positive measure Tf_i. Since also the $f_i * g$ remain bounded in C_c, continuity of T entails that the $\mu_i * g = T(f_i * g)$ remain bounded in M. This signifies that, if $h \in C_c$, the numbers $\int hd(\mu_i * g)$ remain bounded. Supposing that h and g are positive, this in turn leads to a majorization

$$\int d\mu_i(x) \int h(xy) g(y) dy \leqslant m_{h,g}, \tag{2.5}$$

where the number $m_{h,g}$ depends on h and g but not on i. Now, given any compact set $K \subset X$, we can choose positive h and g in C_c such that

$$\int h(xy) g(y) dy \geqslant 1$$

for all $x \in K$, whence it follows from (2.5) that

$$\sup_i \int_K d\mu_i < +\infty.$$

Since C_c is barrelled, the μ_i therefore fall into an equicontinuous subset of M (qua dual of C_c) and the directed family (μ_i) accordingly possesses a vague limiting point $\mu \in M$. Necessarily μ is positive. Moreover, the μ_i being equicontinuous, $\mu * g$ is a limiting point of the family $(\mu_i * g)$ relative to the topology of locally uniform convergence. This, combined with (2.4), implies that

$$Tg = \mu * g.$$

An exactly similar argument is available when one is given that T commutes with the L_x.

Remarks. (i) It is trivial to verify that if $\mu \in M$ is given, then $Tf = \mu * f$ $[Tf = f * \mu]$ maps C_c linearly into $C \subset M$ and that T so defined commutes with the R_x [the L_x]. Moreover, T is positive if and only if $\mu \geqslant 0$; and in any case T is continuous from C_c into C (with their natural topologies).

(ii) There is little difficulty in showing that any linear operator T mapping C_c into C, which commutes with the R_x [the L_x] and which is continuous for the natural topology on C_c and the topology of pointwise convergence on C, is expressible as $Tf = \mu * f$ $[Tf = f * \mu]$ for a suitably chosen measure μ.

(iii) In view of Proposition 1 and these remarks it is tempting to conjecture that any continuous linear operator T mapping C_c into M and commuting with left- or right-translations is expressible by convolution on the appropriate side with some measure. However, unless X is discrete, this conjecture is false. When X is the real line or the unit circle it suffices to consider the operator T defined by $Tf = A * f$, where A is a suitable distribution on X. One may even choose A to have a compact support and to be such that its Fourier transform $\tilde{A}$ is a bounded function; there exist such distributions which are not measures. [If $X = R^1$ it suffices to take for A the finite part, or principal value, of a function $h(x)/x$, where $h \in C_c$ takes the value 1 throughout some neighbourhood of $x = 0$; compare (**1**, 115, formula (VII, 7; 19)). The operator $f \to A * f$ is then structurally akin to the Hilbert transform.] If A be chosen in this fashion, then Tf will be defined as a distribution whenever f is a distribution. Also, if $f \in L^2_{\text{loc}}$ (the space of functions which are L^2 over each compact set) then $Tf \in L^2_{\text{loc}}$ as well; in particular, T maps C_c into M, and evidently does this in a continuous fashion. Clearly, T commutes with translations.

For a development of this train of thought, see Section 4 below.

In spite of this counter-example it is possible, as we shall see in Section 2, to frame a valid analogue of Theorem 1 in which positivity of B is replaced by continuity.

Proof of Theorem 1. Fix $f \geqslant 0$ in C_c. Then $g \to B(f, g)$ is a positive linear operator mapping C_c into M which, by (2), commutes with the R_x. According to Proposition 1, therefore, there exists a positive measure ξ such that

$$B(f, g) = \xi * g \tag{2.6}$$

for $g \in C_c$. The linearity of B in its first argument now shows that to any $f \in C_c$, positive or not, corresponds a measure $\xi \in M$ such that (2.6) is true for all $g \in C_c$. Since it is evident that (2.6) determines ξ uniquely for a given f, the mapping $f \to \xi$ is easily seen to be positive, linear, and—on account of (3)—to commute with the L_x. A second appeal to Proposition 1 yields the conclusion that a positive measure μ exists such that $\xi = f * \mu$. Thus

$$B(f, g) = f * \mu * g \tag{2.7}$$

for f and g in C_c. It remains to use condition (4) to show that $[\mu] \subset \{e\}$, for then μ must be of the form $c.\epsilon$, where c is a positive number and ϵ is the Dirac measure at e, in which case (2.7) reduces to (2.1).

Now (4) entails that if $u \in C_c$, $g \in C_c$, and $\int L_a g(x)u(x)dx = 0$ for $a \in [f]$, then $\int (f*\mu*g)(x)u(x)dx = 0$. This means that from the relations

$$u \in C_c, \quad g \in C_c, \quad \int g(a^{-1}x)u(x)dx = 0 \quad \text{for } a \in [f]$$

follows the relation

$$\int f*\mu(y)dy \int g(y^{-1}x)u(x)dx = 0.$$

As is easy to check, it must therefore be the case that

$$[f*\mu] \subset [f]$$

for each $f \in C_c$. Finally, this entails that $[\mu] \subset \{e\}$, as we wished to show.

3. Variants of Theorem 1. Several of the variants we have in mind involve, explicitly or otherwise, distributional concepts. For the sake of simplicity we shall therefore assume henceforth that X is a finite product of lines and/or circles.

(A) In condition (4) of Theorem 1 it is possible to replace "vague limit" by the weaker "distributional limit." To do this calls for only slight modifications in the preceding proof.

(B) If one makes use of Schwartz' theorem referred to in (C) below, it is not difficult to show that any positive bilinear operator B mapping $C_c^\infty \times C_c^\infty$ into D', which enjoys properties (2), (3), and (4) (the latter modified as in (A)), has the form (2.1). In the above statement C_c^∞ denotes the standard space of test functions (indefinitely differentiable and with compact supports) on X, and D' denotes the space of distributions on X. The permanence of (2.1) under these wider conditions hinges solely on the fact that a positive distribution is necessarily a measure.

(C) Let us consider what can be said if B is continuous and bilinear from $C_c \times C_c$ into C (the latter with the topology of pointwise convergence) and enjoys properties (2), (3), and (4). Note in passing that, since C_c is an inductive limit of Banach spaces, continuity of B is equivalent to separate continuity.

According to Remark (ii) following the proof of Proposition 1, we can be sure that to each $f \in C_c$ corresponds a measure ξ such that

$$B(f, g) = \xi * g$$

for all $g \in C_c$. As before, this determines ξ uniquely in terms of f, whence it follows that the operator $T: f \to \xi$ maps C_c linearly into M and commutes with translations. The continuity of $f \to B(f, g)(0)$ shows that T is continuous for the vague topology on M. Since C_c is barrelled, this entails that T is continuous for the strong topology $\beta(M, C_c)$ on M, i.e. that for each compact set K, the semi-norm $f \to \int_K d|Tf|$ is continuous on C_c.

If now we temporarily restrict T to C_c^∞ and appeal to Schwartz' theorem (**1**, pp. 53–53), we conclude that there exists a distribution μ such that

$$(3.1) \qquad Tf = \mu * f$$

for $f \in C_c^\infty$. The continuity of T shows that this relation continues to hold for $f \in C_c$.

Next, if we use (4) as before, it appears that $[\mu] \subset \{0\}$. Consequently $\mu = P(D)\epsilon$, where $P(D)$ denotes a linear partial differential operator with constant coefficients. Thus

$$(3.2) \qquad B(f, g) = P(D)(f*g),$$

while (3.1) shows that $P(D)f \in M_c$ whenever $f \in C_c$. Since this entails that the Fourier transform of $P(D)f$, namely $P(z)\tilde{f}(z)$, where

$$\tilde{f}(z) = \int f(x)e^{-2\pi i z.x}dx,$$

is bounded for $z \in \tilde{X}$, the lemma to follow forces the conclusion that the polynomial P is bounded and therefore reduces to a constant. The equation (3.2) then reduces to the form (2.1), so that one may conclude that *any continuous bilinear B from $C_c \times C_c$ into C with properties* (2), (3), *and* (4) *is necessarily of the form* (2.1).

The auxiliary result called upon in the last paragraph may be phrased in somewhat more general terms, as follows.

LEMMA. *Let X be any locally compact abelian group with dual $\tilde{X}$, A any non-void open subset of $\tilde{X}$, and P a complex-valued function on $\tilde{X}$. Suppose that*

$$(3.3) \qquad \sup_{\tilde{x}\in\tilde{X}}|P(\tilde{x})\tilde{f}(\tilde{x})| < +\infty$$

for each $f \in C_c(X)$ with $[f] \subset A$. Then

$$(3.4) \qquad \sup_{\tilde{x}\in\tilde{X}}|P(\tilde{x})| < +\infty.$$

In (3.3), *$\tilde{f}$ is the Fourier transform of f:*

$$\tilde{f}(\tilde{x}) = \int f(x).\overline{(x, \tilde{x})}\, dx.$$

Proof. Take any compact set $K \subset A$ having a non-void interior, and denote by F the set of $f \in C_c(X)$ satisfying $[f] \subset K$. F is a Banach space under the norm

$$||f|| = \sup_{x\in X}|f(x)|.$$

By (3.3), $N: f \to \sup |P\tilde{f}|$ is a semi-norm on F. Since convergence in F evidently entails convergence in L^1, $f \to \tilde{f}(\tilde{x})$ is, for each $\tilde{x} \in \tilde{X}$, continuous on F. So N is lower semicontinuous on F. Since F is a Banach space, N must be continuous on F. This signifies the existence of a number k such that

$$(3.5) \qquad |P(\tilde{x})\tilde{f}(\tilde{x})| \leqslant k.||f||$$

for $\tilde{x} \in \tilde{X}$ and $f \in F$.

Now, if (3.4) were false, one could extract from $\tilde{X}$ a sequence $(\tilde{x}_n)$ such that $|P(\tilde{x}_n)| \geqslant n$ for all n. Take any $f_0 \in F$ for which $\int f_0 dx = c \neq 0$, which is possible since K has interior points. Put $f_n(x) = (x, \tilde{x}_n)f_0(x)$. Then $||f_n|| = ||f_0||$ and $\tilde{f}_n(\tilde{x}_n) = c$. Hence (3.5) gives for all n

$$n|c| \leqslant |P(\tilde{x}_n)\tilde{f}_n(\tilde{x}_n)| \leqslant k||f_n|| = k||f_0||,$$

which is absurd. This contradiction shows that (3.4) must be true and completes the proof.

Remark. If X is compact one may alternatively merely "thin out" $(\tilde{x}_n)$ to the extent of arranging that $|P(\tilde{x}_n)| \geqslant n^3$ and consider the continuous sum-function f of the absolutely convergent series

$$\sum_{n=1}^{\infty} n|P(\tilde{x}_n)|^{-1}(x, \tilde{x}_n),$$

for which $|P(\tilde{x}_n)\tilde{f}(\tilde{x}_n)| = n$. This shows that the negation of (3.4) denies (3.3).

(D) As a final variant we remark that the same conclusion follows if B is bilinear from $C_c \times C_c$ into M, enjoys properties (2), (3), and (4), and is continuous when we regard C_c as the internal inductive limit of the spaces $C_{c,K}$ endowed with the norms induced by that of L^1.

4. Linear operators from C into M commuting with translations. The remarks following the proof of Proposition 1 may be supplemented by two further results belonging to the representation theory for continuous linear operators which commute with translations. These have been deferred until now since they are not directly involved in the characterization of convolution as a bilinear operator.

Let us suppose again that X is a finite product of lines and/or circles, and let us denote by $D = D(X)$ the set of all distributions μ on X for which $[\mu]$ is compact and for which the Fourier transform $\tilde{\mu}$ is a bounded function on X.

PROPOSITION 2. (*a*) *The continuous linear operators T mapping C into M which commute with translations are precisely those of the form*

$$Tf = \mu * f, \tag{4.1}$$

where $\mu \in D$.

(*b*) *The assertion* (*a*) *remains true if therein we replace C and M by C_c and M_c respectively, M_c being endowed with the topology $\sigma(M_c, C_c)$ induced on it by the vague topology of measures.*

Proof. The verification that, if $\mu \in D$, then (4.1) defines an operator T of the prescribed type in each case is virtually contained in Remark (iii) following the proof of Proposition 1.

On the other hand if the converse portion of (*a*) is known, it is simple to derive that of (*b*) by considering the adjoint operator. For this it must first be observed that if T is a linear operator from C_c into M_c which is continuous for $\sigma(M_c, C_c)$, then it is continuous for the stronger topology $\sigma(M_c, C)$: the justification for this remark is analogous to that used to a similar end in (*C*) of Section 3, once account is taken of the formula

$$|\int g d(Tf)| \leqslant \sup|\int gu.d(Tf)|$$

for $g \in C$ and u ranging over those members of C_c satisfying $|u| \leqslant 1$. Consequently the adjoint T' of T will be continuous from C into M. If T commutes with translations, so too will T', so that (*a*) will apply to T'. From this it is trivial to deduce (*b*).

Thus all depends on showing that if T satisfies the hypotheses laid out in (*a*), then it is representable as (4.1) with a suitably chosen $\mu \in D$.

Now, according to the theorem of Schwartz already cited, there exists a distribution T such that (4.1) is true for $f \in C_c^\infty$, and an easy continuity argument shows that (4.1) continues to hold for $f \in C_c$. If one can show that $[\mu]$ is compact, both sides of (4.1) will be continuous in f for the natural topology on C and continuity will show that (4.1) indeed holds for all $f \in C$. The proof will then be completed by appeal to the Lemma in Section 3. It will now be shown by *reductio ad absurdum* that $[\mu]$ is indeed compact.

In the contrary case there would exist a sequence (a_n) extracted from $[\mu]$ such that $|a_n| \to \infty$. Let (c_n) be an arbitrary sequence of scalars and let $f \in C_c$. The set $\{c_n L_{a_n} f\}$ is bounded in C and is therefore transformed by T into a bounded subset of M. By (4.1), this entails that the set $\{c_n(L_{a_n} \mu * f)\}$ is bounded distributionally. Therefore (**1**, p. 51, Théorème XXII) the set $\{c_n.L_{a_n} \mu\}$ is bounded distributionally. Now $0 \in [L_{a_n} \mu]$ for all n, so that if K is any compact neighbourhood of 0, one can, for each n, choose $f_n \in C_{c,K}^\infty$ for which $\langle f_n, L_{a_n} \mu\rangle = k_n$ is non-zero. The space $C_{c,K}^\infty$ being metrizable, one may multiply each f_n by a strictly positive scalar p_n in such a way as to arrange that the set $\{p_n f_n\}$ is bounded in $C_{c,K}^\infty$. But then the boundedness of $\{c_n L_{a_n} \mu\}$ would entail the boundedness of the set $\{\langle p_n f_n, c_n L_{a_n} \mu\rangle\}$. In other words, the set $\{c_n p_n k_n\}$ would be bounded, no matter how the sequence (c_n) is chosen. Since $p_n k_n \neq 0$ for each n, this conclusion is plainly absurd. The implied contradiction shows that $[\mu]$ must be compact, as we wished to show.

Remark. Whatever the (locally compact) group X, if T is a linear operator mapping C_c into M which commutes with left translations and which is continuous for $\sigma(M, C_c)$, then (1.6) shows that

$$T(f*g) = f*Tg; \tag{4.2L}$$

and if "left" be replaced by "right," then

$$T(f*g) = Tf*g, \tag{4.2R}$$

in each case for $f, g \in C_c$.

It is perhaps worth remarking that, at least if X is sigma-compact, any linear

https://doi.org/10.4153/CJM-1964-027-0 Published online by Cambridge University Press

operator T from C_c into M which satisfies (4.2L) and (4.2R) is necessarily continuous (even for the toplogy $\beta(M, C_c)$).

In order to see this we note first that $\beta(M, C_c)$ makes M into a Fréchet space, if X is sigma-compact, a defining system of semi-norms being $\mu \rightarrow \int_K d|\mu|$ when K ranges over a base for the compact subsets of X. On the other hand, to verify continuity of T it suffices to verify that of $T|C_{c,K}$ for each compact $K \subset X$; and to do this it suffices to show that $T|C_{c,K}$ has a graph closed in $C_{c,K} \times M$. That this is the case follows readily from (4.2L) and (4.2R), which together show that if $f_n \rightarrow 0$ in $C_{c,K}$ and $\mu_n = Tf_n \rightarrow \mu$ in M, then for each $g \in C_c$ one has

$$f_n * Tg = T(f_n * g) = Tf_n * g.$$

Now $f_n * Tg \rightarrow 0$ and $Tf_n * g \rightarrow \mu * g$ pointwise, so that $\mu * g$ must vanish for all $g \in C_c$. Hence $\mu = 0$ and the graph is closed.

We end by recording two corollaries of Proposition 2. The first is a restatement of part (b) as a result about Fourier factor functions. The most interesting case is that in which $X = R^n$.

COROLLARY 1. *Let F be a complex-valued function on R^n. In order that $F\tilde{f}$ be the Fourier transform of a measure in $M_c(R^n)$ whenever $f \in C_c(R^n)$, it is necessary and sufficient that $F = \tilde{\mu}$ for some $\mu \in D$.*

According to the theorem of Paley-Wiener-Schwartz (**1**, p. 128, Théorème XVI) it is equivalent to demand that F be bounded on R^n and be the restriction to R^n of some entire function of exponential type of n complex variables.

Proof. Sufficiency is evident. The necessity is established by introducing the linear operator T from C_c into M_c defined by putting $Tf = \lambda$ just when $f \in C_c$, $\lambda \in M_c$, and $\tilde{\lambda} = F.\tilde{f}$. It is then almost evident that $T(f*g) = Tf*g$ for $f, g \in C_c$. Continuity of T follows the preceding Remark, and an application of Proposition 2(b) shows that $Tf = \mu * f$ for some $\mu \in D$ and all $f \in C_c$. On taking the Fourier transform it appears that $F = \tilde{\mu}$.

The second corollary is an analogue of a result given by Wells (**2**, Theorem 2) for bounded measures on a half-line. Actually each portion of Proposition 2 leads to such an analogue, as also does the result stated in Remark (ii) following the proof of Proposition 1. We confine ourselves to that analogue stemming from (b) of Proposition 2.

COROLLARY 2. *Let α and β be distributions and suppose that*

$$\alpha * C_c \subset \beta * M_c. \tag{4.3}$$

Then to each $\gamma \in M_c$ belonging to the distributional closure V of the set of finite linear combinations of translates of β there corresponds some $\mu \in D$ such that

$$\alpha * \gamma = \beta * \mu. \tag{4.4}$$

In particular, if $\epsilon \in V$ (as is the case if $X = R^n$ and $\beta \neq 0$ has a compact support), *then $\alpha = \beta * \mu$ for some $\mu \in D$, which relation implies* (4.3).

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Proof. To each $f \in C_c$ corresponds at least one $\xi \in M_c$ satisfying $\alpha * f = \beta * \xi$. This ξ may not be uniquely determined by f, but we can show that $\gamma * \xi$ is so uniquely determined. For this it suffices to show that, if $\xi \in M_c$ satisfies $\beta * \xi = 0$, then $\gamma * \xi = 0$ also. Now, if $\beta * \xi = 0$, then also $\beta * \xi * h = 0$ for each $h \in C_c^\infty$. The relation $\beta * \xi * h = 0$ remains true whenever β is replaced by any translate thereof. Since $\xi * h \in C_c^\infty$, the said relation remains valid in the limit when β is replaced by any element of V, hence in particular when β is replaced by γ. Thus $\gamma * \xi * h = 0$ for all $h \in C_c^\infty$, and so $\gamma * \xi = 0$.

Accordingly we may define a linear operator T from C_c into M_c by writing $Tf = \gamma * \xi$. The defining property of T is thus

$$\alpha * f * \gamma = \beta * Tf. \tag{4.5}$$

This formula makes it plain that $T(f * g) = Tf * g$ for f and g in C_c. Hence, as in the proof of Corollary 1, we may infer from Proposition 2(b) that there exists some $\mu \in D$ such that (4.1) holds. Substituting from (4.1) into (4.5), one obtains

$$\alpha * f * \gamma = \beta * \mu * f$$

for all f in C_c, whence follows (4.4).

It remains only to verify the assertion in parentheses, namely that $\epsilon \in V$ if $\beta \neq 0$ has a compact support. In fact, under these conditions, V contains *all* distributions. For, by the Hahn-Banach theorem and translation-invariance of V, it suffices to show that if $g \in C_c^\infty$ satisfies $\beta * h = 0$, then $h = 0$. Since β has a compact support, the Fourier transform $\tilde{\beta}$ is, like $\tilde{h}$, an entire analytic function of complex variables. The relation $\beta * h = 0$ entails $\tilde{\beta}.\tilde{h} = 0$, which, since $\tilde{\beta}$ does not vanish identically, shows that $\tilde{h} = 0$ everywhere. Thus $h = 0$, as we wished to show.

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https://doi.org/10.4153/CJM-1964-027-0 Published online by Cambridge University Press