CERTAIN FOURIER TRANSFORMS OF DISTRIBUTIONS

EUGENE LUKACS AND OTTO SZÁSZ

1. Introduction. Fourier transforms of distribution functions are frequently studied in the theory of probability. In this connection they are called characteristic functions of probability distributions. It is often of interest to decide whether a given function $\varphi(t)$ can be the characteristic function of a probability distribution, that is, whether it admits the representation

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x). \tag{1}$$

Here $F(x)$ is a distribution function and $\varphi(t)$ its characteristic function. If $F(x)$ is absolutely continuous, $f(x) = \frac{dF}{dx}$ is called the frequency function (probability density). The frequency function $f(x)$ is non-negative and real for real x and $\int_{-\infty}^{+\infty} f(x)\, dx = 1$.

H. Cramér [2] gave a necessary and sufficient condition which a complex valued function $\varphi(t)$ of a real variable t must satisfy in order to be a characteristic function. His work is a simplification of an earlier result of Bochner [1]. Another criterion is due to Khintchine [4]. These general theorems are not easily applicable in practice. It is therefore desirable to derive conditions which are restricted to certain classes of functions but are applied more readily. Marcinkiewicz [6] and Lévy [5] derived necessary conditions for an entire function to be a characteristic function. A rather simple sufficient condition was given by Pólya [7]. In this paper another class of functions, namely the reciprocals of polynomials are studied.

The reciprocal of a polynomial with a single imaginary root is the characteristic function of the well known gamma distribution. The product of two characteristic functions is always a characteristic function [1, Satz 18] so that the reciprocal of a polynomial having only purely imaginary roots is always the characteristic function of a distribution. Employing the limit theorem of Lévy one can prove easily that the reciprocal of an entire function of genus 0 or 1, and having only imaginary roots is also a characteristic function. An analogous situation arose in a problem on the bilateral Laplace transform treated by Schoenberg [8].

In this note the following necessary condition is derived: If the reciprocal of a polynomial without multiple roots is a characteristic function then the following two conditions are satisfied:

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(a) The polynomial has no real roots. Its roots are located either all on the imaginary axis or in pairs $\pm b + ia$ symmetric with regard to this axis.

(b) If $b + ia$ (a, b, real; $a \neq 0$, $b \neq 0$) is a root of the polynomial then it has at least one root $i\alpha$ such that sgn α = sgn a and $|\alpha| \leqslant |a|$.

The assumption that the polynomial has no multiple roots is used only in deriving (b). These conditions indicate for example that functions like $\frac{1}{1+t^4}$ or $\frac{1}{(1+t^4)(1+t^2)}$ cannot be characteristic functions. A theorem due to Marcinkiewicz [6, theorem 3] shows this for the first of these functions but fails to reject the second.

2. Necessary conditions derived from elementary considerations. If a function $\varphi(t)$ is the characteristic function of a distribution then [3, p. 91]:

$$\varphi(0) = 1, \tag{2.11}$$

$$|\varphi(t)| \leqslant 1, \tag{2.12}$$

$$\varphi(-t) = \overline{\varphi(t)}. \tag{2.13}$$

If $\varphi(t)$ is the reciprocal of a polynomial of degree n one can write

$$\varphi(t) = \left\{\left(1 - \frac{it}{v_1}\right)\left(1 - \frac{it}{v_2}\right)\dots\left(1 - \frac{it}{v_n}\right)\right\}^{-1}, \tag{2.2}$$

where $v_1, v_2, \dots, v_n$ are complex numbers. The zeros of the polynomial are then given by

$$t_j = -iv_j \qquad (j = 1, 2, \dots, n). \tag{2.21}$$

Condition (2.11) is satisfied; from condition (2.13) one obtains

$$\frac{(v_1 + it)\ (v_2 + it)\dots(v_n + it)}{v_1\, v_2 \dots v_n} = \frac{(\bar{v}_1 + it)\ (\bar{v}_2 + it)\dots(\bar{v}_n + it)}{\bar{v}_1\, \bar{v}_2 \dots \bar{v}_n}.$$

By arranging both sides according to powers of (it) and comparing the coefficients, it is seen that the elementary symmetric functions of the n numbers $v_1, v_2, \dots, v_n$ are real, that is the numbers $v_1, \dots, v_n$ are the roots of an equation of degree n with real coefficients. Consequently the v_j are either real or occur in pairs of conjugate complex numbers. Moreover, no v_j can be a purely imaginary number. The corresponding pole of the characteristic function would be real (by 2.21) which would contradict (2.12). These elementary considerations establish part (a) of our necessary conditions.

3. Auxiliary formulae. Consider first a polynomial having only one imaginary root. The corresponding frequency function is obtained by means of the inverse Fourier transform and is

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$$(3.11)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}dt}{(1-it/\alpha)^{\lambda}} = \begin{cases}\dfrac{\alpha^{\lambda}}{\Gamma(\lambda)}x^{\lambda-1}e^{-\alpha x} & \text{if } x>0,\\ 0 & \text{if } x<0,\end{cases}$$

and

$$(3.12)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}dt}{(1+it/\beta)^{\lambda}} = \begin{cases}\dfrac{\beta^{\lambda}}{\Gamma(\lambda)}(-x)^{\lambda-1}e^{\beta x} & \text{if } x<0,\\ 0 & \text{if } x>0.\end{cases}$$

Here α and β are assumed to be real and positive, λ is a positive integer. If we put in (3.11) and (3.12) $t = \tau + \gamma$, where γ is a positive real constant, we obtain similar formulae for complex poles:

$$(3.21)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}dt}{(1-it/v)^{\lambda}} = \begin{cases}\dfrac{v^{\lambda}}{\Gamma(\lambda)}x^{\lambda-1}e^{-vx} & \text{for } x>0,\\ 0 & \text{for } x<0,\end{cases}$$

$$(3.22)\qquad \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}dt}{(1+it/w)^{\lambda}} = \begin{cases}\dfrac{w^{\lambda}}{\Gamma(\lambda)}(-x)^{\lambda-1}e^{wx} & \text{for } x<0,\\ 0 & \text{for } x>0.\end{cases}$$

In these formulae $v=\alpha-i\gamma$ and $w = \beta + i\gamma$. The first formula is applied if the imaginary part of the pole of the integrand is negative, and the second if it is positive.

4. Functions whose Fourier transform is a polynomial without multiple roots. The formulae (3.11), (3.12), (3.21), (3.22) can be used to find the functions whose Fourier transform is the reciprocal of a polynomial satisfying the necessary conditions derived in §2. In the following only polynomials without multiple roots are considered. The zeros of the polynomials can be divided into four groups:

(i) zeros $i\beta_h$ $(h = 1, 2, \ldots, \mu)$ on the positive imaginary axis $(\beta_h > 0)$;

(ii) zeros $-i\alpha_j$ $(j = 1, 2, \ldots, \nu)$ on the negative imaginary axis $(\alpha_j > 0)$;

(iii) p symmetric pairs of complex roots in the upper half planes iw_k and $i\overline{w}_k$ where $w_k = c_k + id_k$, $c_k > 0$, $d_k > 0$ $(k = 1, 2, \ldots, p)$;

(iv) n symmetric pairs of complex roots in the lower half planes $-iv_m$ and $-i\overline{v}_m$ where $v_m = a_m + ib_m$, $a_m > 0$, $b_m > 0$ $(m = 1, 2, \ldots, n)$.

The function (2.2) can then be written

(4.1)

$$\varphi(t) = \left\{\prod_{j=1}^{\nu}\left(1-\frac{it}{\alpha_j}\right)\prod_{h=1}^{\mu}\left(1+\frac{it}{\beta_h}\right)\prod_{k=1}^{p}\left(1+\frac{it}{w_k}\right)\left(1+\frac{it}{\overline{w}_k}\right)\prod_{m=1}^{n}\left(1-\frac{it}{v_m}\right)\left(1-\frac{it}{\overline{v}_m}\right)\right\}^{-1}.$$

If $\varphi(t)$ is decomposed into partial fractions it is seen that

$$(4.11)\qquad \varphi(t) = \sum_{j=1}^{\nu}\frac{A_j}{1-it/\alpha_j} + \sum_{h=1}^{\mu}\frac{B_h}{1+it/\beta_h} + \sum_{k=1}^{p}\left(\frac{D_k}{1+it/w_k}+\frac{\overline{D}_k}{1+it/\overline{w}_k}\right) + \sum_{m=1}^{n}\left(\frac{C_m}{1-it/v_m}+\frac{\overline{C}_m}{1-it/\overline{v}_m}\right).$$

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If $\varphi(t)$ is the Fourier transform of a distribution then the same is true for $\varphi(-t)$. It is seen from (4.1) that the zeros of $[\varphi(t)]^{-1}$ and of $[\varphi(-t)]^{-1}$ are symmetrical with respect to the real axis. This shows that it is sufficient to prove the statement for one of the half planes. In the following only the negative half plane will be considered. Applying formulæ (3.11), (3.12), (3.21), (3.22) the frequency function corresponding to $\varphi(t)$ is found to be

$$f(x) = \sum_{j=1}^{\nu} A_j a_j e^{-x a_j} + \sum_{m=1}^{n} (C_m v_m e^{-x v_m} + \bar{C}_m \bar{v}_m e^{-x \bar{v}_m}), \qquad \text{if } x > 0. \tag{4.2}$$

A similar expression is found for $f(x)$ if $x < 0$.

Introducing trigonometrical representation one has

$$v_m C_m = R_m e^{i\varphi_m} \qquad (m = 1, 2, \ldots, n) \tag{4.31}$$

and

$$v_m C_m e^{-v_m x} + \bar{v}_m C_m e^{-\bar{v}_m x} = 2R_m e^{-a_m x} \cos(\phi_m - b_m x). \tag{4.32}$$

Substituting (4.32) into (4.2) one has finally

$$f(x) = \sum_{j=1}^{\nu} A_j a_j e^{-a_j x} + 2 \sum_{m=1}^{n} R_m e^{-a_m x} \cos(\phi_m - b_m x), \qquad \text{if } x > 0. \tag{4.4}$$

5. Derivation of condition (b). We need the following lemma.

LEMMA. *A non-negative generalized trigonometric polynomial without constant term is identically zero.*

A function $g(x)$ is said to be a generalized trigonometric polynomial if

$$g(x) = \lambda_0 + \sum_{i=1}^{n} (\lambda_i \cos b_i x + \mu_i \sin b_i x), \tag{5.1}$$

where the numbers $b_1, \ldots, b_n$ are arbitrary real quantities but not necessarily integers.

Proof. Let $g(x)$ be a generalized trigonometric polynomial without constant term i.e. $\lambda_0 = 0$. Since $\int_0^x g(t)\, dt$ is bounded we have

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x g(t)\, dt = 0. \tag{5.2}$$

If $g(x_0) > 0$ for some x_0 then $g(x) > 2\epsilon$ for $|x - x_0| \leqslant \delta$. The function $g(x)$ is almost periodic, therefore there is an $L > 0$ such that in any interval $(y, y + L)$ a translation number τ exists such that $|g(x) - g(x + \tau)| < \epsilon$ for all x. Therefore in any interval $(y, y + L)$ there is a subinterval of length $\geqslant \delta$ where $|g(x)| \geqslant \epsilon$. But this implies

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x g(t)\, dt \geqslant \frac{\epsilon\delta}{L} > 0$$

in contradiction to (5.2). We proceed now to the proof of condition (b).

Denote the pairs of roots of $[\varphi(t)]^{-1}$ in the lower half plane by $-iv_m$ and $-i\bar{v}_m$ where $v_m = a_m + ib_m$ $(m = 1, 2, \ldots, n)$. If the polynomial has also ν roots

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on the negative imaginary axis we denote them by $-i\alpha_j$ $(j = 1, 2, \ldots \nu)$. The notation should be such that

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \tag{5.3}$$

and if $\nu > 0$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_\nu. \tag{5.4}$$

We observe that no term in (4.4) can vanish identically.

Let $\varphi(t)$ be a characteristic function; we assume first that $\nu > 0$ and show $\alpha_1 \leqslant a_1$. We carry the proof indirectly and suppose $a_1 < \alpha_1$. The frequency function $f(x)$ of $\varphi(t)$ as given by (4.4) is non-negative for $x > 0$. This may be written in the form

$$\sum_{j=1}^{\nu} A_j \alpha_j e^{-\alpha_j x} + \sum_{m=1}^{n} e^{-a_m x}(\lambda_m \cos b_m x + \mu_m \sin b_m x) \geqslant 0 \quad \text{for } x > 0. \tag{5.5}$$

We define ρ by $a_1 = a_2 = \ldots = a_\rho < a_{\rho+1}$ and put

$$g(x) = \sum_{m=1}^{\rho} (\lambda_m \cos b_m x + \mu_m \sin b_m x).$$

The function $g(x)$ cannot vanish identically and we have from (5.5)

$$g(x) + \sum_{j=1}^{\nu} A_j \alpha_j e^{-\alpha'_j x} + \sum_{m=\rho+1}^{n} e^{-a'_m x}(\lambda_m \cos b_m x + \mu_m \sin b_m x) \geqslant 0 \quad \text{for } x > 0. \tag{5.6}$$

Here $\alpha'_j = \alpha_j - a_\rho$ and $a'_m = a_m - a_\rho$ and $\alpha'_j > 0$, $a'_m > 0$ on account of (5.3) and (5.4). By the Lemma, $g(x)$ must assume negative values and we see (as in the proof of the Lemma) that $g(x) < -\epsilon$ for some $\epsilon > 0$ and arbitrarily large x. But this is in contradiction to (5.6). If $\nu = 0$ i.e. if there are no roots on the imaginary axis the term $\sum_{j=1}^{\nu} A_j \alpha_j e^{-\alpha'_j x}$ is missing in (5.6). The preceding argument can nevertheless be applied to show that $[\varphi(t)]^{-1}$ must have a root on the imaginary axis.

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