15 Theories with more than four conserved supercharges

In theories with more than four conserved supercharges (extended supersymmetry), the supersymmetry generators obey the relations

$$\left\{Q^I_\alpha, Q^J_{\dot\alpha}\right\} = \not{p}\,\delta^{IJ}, \quad \left\{Q^I_\alpha, Q^J_\beta\right\} = Z^{IJ}\epsilon_{\alpha,\beta}. \tag{15.1}$$

The quantities Z^{IJ} are known as central charges. We will see that these can arise in a number of physically interesting ways.

In theories with four supersymmetries, we saw in Chapters 13 and 14 supersymmetry provides powerful constraints on the possible dynamics. Theories with more than four supercharges ($N > 1$ in four dimensions) are not plausible as models of the real world but they do have a number of remarkable features. As in some of our $N = 1$ examples, these theories typically have exact moduli spaces. Gauge theories with $N = 4$ supersymmetry exhibit an exact duality between electricity and magnetism. Theories with $N = 2$ supersymmetry have a rich – and tractable – dynamics, closely related to important problems in mathematics. In all these cases supersymmetry provides remarkable control over the dynamics, allowing one to address questions which are inaccessible in theories without supersymmetry. Supersymmetric theories in higher dimensions generally have more than four supersymmetries, and a number of the features of the theories we study in this chapter will reappear when we come to higher-dimensional field theories and string theory.

15.1 $N = 2$ theories: exact moduli spaces

Theories with $N = 1$ supersymmetry are tightly constrained, but theories with more supersymmetry are even more highly constrained. We have seen that often, in perturbation theory, $N = 1$ theories have moduli; non-perturbatively, sometimes, these moduli are lifted. In theories with $N = 1$ supersymmetry, a detailed analysis is usually required to determine whether the moduli acquire potentials at the quantum level. For theories with more supersymmetries ($N > 1$ in four dimensions; $N \geq 1$ in five or more dimensions), one can show rather easily that the moduli space is exact. Here we consider the case of $N = 2$ supersymmetry in four dimensions. These theories can also be described by a superspace, in this one case built from two Grassmann spinors, θ and $\tilde\theta$. There are two basic types of superfields: vectors and hypermultiplets. The vectors are chiral with respect to both D_α and

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$\tilde{D}_\alpha$ and have an expansion, in the case of a $U(1)$ field,

$$\psi = \phi + \tilde{\theta}^\alpha W_\alpha + \tilde{\theta}^2 \bar{D}^2 \phi^\dagger, \tag{15.2}$$

where ϕ is an $N = 1$ chiral multiplet and W_α is an $N = 1$ vector multiplet. The fact that $\phi^\dagger$ appears as the coefficient of the $\tilde{\theta}^2$ term is related to an additional constraint satisfied by ψ. This expression can be generalized to non-Abelian symmetries; the expression for the highest component of ψ is then somewhat more complicated but we will not need it here.

The theory possesses an $SU(2)$ R-symmetry under which θ and $\tilde{\theta}$ form a doublet. Under this symmetry, the scalar component of ϕ and the gauge field are singlets, while ψ and λ form a doublet.

We will not describe the hypermultiplets in detail except to note that, from the perspective of $N = 1$, they consist of two chiral multiplets. The two chiral multiplets transform as a doublet of the $SU(2)$ group. The superspace description of these multiplets is more complicated.

In the case of a non-Abelian theory, the vector field ψ^a is in the adjoint representation of the gauge group. For these fields the Lagrangian has a very simple expression as an integral over half the superspace:

$$\mathcal{L} = \int d^2\theta d^2\tilde{\theta}\ \psi^a \psi^a, \tag{15.3}$$

or, in terms of $N = 1$ components,

$$\mathcal{L} = \int d^2\theta\ W_\alpha^2 + \int d^4\theta \phi^\dagger e^V \phi. \tag{15.4}$$

The theory with vector fields alone has a classical moduli space, given by the values of the fields for which the scalar potential vanishes. Here this just means that the D fields vanish. Written as a matrix we have

$$D = [\phi, \phi^\dagger], \tag{15.5}$$

which vanishes for diagonal ϕ, i.e. for

$$\phi = \frac{a}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15.6}$$

For many physically interesting questions one can focus on the effective theory for the light fields. In the present case the light field is the vector multiplet ψ. Roughly,

$$\psi \approx \psi^a \psi^a = a^2 + a\delta\psi^3 + \cdots . \tag{15.7}$$

What kind of effective action can we write for ψ? At the level of terms with up to four derivatives, the most general effective Lagrangian has the form[1]

$$\mathcal{L} = \int d^2\theta d^2\tilde{\theta} f(\psi) + \int d^8\theta\, \mathcal{H}(\psi, \psi^\dagger). \tag{15.8}$$

[1] This, and essentially all the effective actions we will discuss, should be thought of as Wilsonian effective actions, obtained by integrating out heavy fields and high-momentum modes.

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Terms with covariant derivatives correspond to terms with more than four derivatives when written in terms of ordinary component fields.

The first striking result we can read off from this Lagrangian, with no knowledge of $\mathcal{H}$ and f, is that there is no potential for ϕ, i.e. the moduli space is exact. This statement is true both perturbatively and non-perturbatively.

One can next ask about the function f. This function determines the effective coupling in the low-energy theory and is an object studied by Seiberg and Witten, which we will discuss in Section 15.4.

15.2 A still simpler theory: $N = 4$ Yang–Mills

The $N=4$ Yang–Mills theory is interesting in its own right: it is finite and conformally invariant. It also plays an important role in our current understanding of non-perturbative aspects of string theory. The $N=4$ Yang–Mills has 16 supercharges and is even more tightly constrained than the $N=2$ theories. First, we will describe the theory. In the language of $N=2$ supersymmetry, it consists of one vector multiplet and one hypermultiplet. In terms of $N=1$ superfields, it contains three chiral superfields, ϕ_i and a vector multiplet. The Lagrangian is

$$\mathcal{L} = \int d^2\theta W_\alpha^2 + \int d^4\theta\, \phi_i^\dagger e^V \phi_i + \int d^2\theta\, \phi_i^a \phi_j^b \phi_k^c \epsilon_{ijk} \epsilon^{abc}. \tag{15.9}$$

In the above description there is a manifest $SU(3)\times U(1)$ R-symmetry. Under this symmetry the ϕ_is have $U(1)_R$ charge $2/3$ and form a triplet of $SU(3)$. But the real symmetry is larger – it is $SU(4)$. Under this symmetry, the four Weyl fermions form a 4-dimensional representation, while the six scalars transform in the 6-dimensional representation. Later, our studies of the toroidal compactifications of the heterotic string (Chapter 25) will later give us an heuristic understanding of this $SU(4)$ symmetry: it reflects the $O(6)$ symmetry of the compactified six dimensions. In string theory this symmetry is broken by the compactification manifold; this reflects itself in higher-derivative, symmetry-breaking, operators.

In the $N = 4$ theory there is, again, no modification of the moduli space, perturbatively or non-perturbatively. This can be understood in a variety of ways. We can use the $N = 2$ description of the theory, defining the vector multiplet to contain the $N = 1$ vector and one (arbitrarily chosen) chiral multiplet. Then an identical argument to that given above ensures that there is no superpotential for the chiral multiplet alone. The $SU(3)$ symmetry then ensures that there is no superpotential for any chiral multiplet. Indeed, we can make an argument directly in the language of $N = 1$ supersymmetry. If we tried to construct a superpotential for the low-energy theory in the flat directions, it would have to be an $SU(3)$-invariant holomorphic function of the ϕ_is. But there is no such object.

Similarly, it is easy to see that there are no corrections to the gauge couplings. For example, in the $N = 2$ language, we want to ask what sort of function f is allowed in

$$\mathcal{L} = \int d^2\theta d^2\tilde{\theta} f(\psi). \tag{15.10}$$

The theory has a $U(1)$ R-invariance under which

$$\psi \to e^{2/3i\alpha}\psi, \quad \theta \to e^{i\alpha\theta}, \quad \tilde{\theta} \to e^{-i\alpha}\tilde{\theta}. \tag{15.11}$$

Already, then,

$$\int d^2\theta d^2\tilde{\theta}\, \psi\psi \tag{15.12}$$

is the unique structure which respects these symmetries. Now we can introduce a background dilaton field, τ. Classically the theory is invariant under shifts in the real part of τ, $\tau \to \tau+\beta$. This ensures that there are no perturbative corrections to the gauge couplings. With a little more work one can show that there are no non-perturbative corrections either.

One can also show that the quantity $\mathcal{H}$ in Eq. (15.8) is unique in this theory, again using the symmetries. The expression

$$\mathcal{H} = c \ln\psi \ln\psi^\dagger \tag{15.13}$$

respects all the symmetries. At first sight it might appear to violate scale invariance; given that ψ is dimensionful one would expect a scale Λ sitting in the logarithm. However, it is easy to see that if one integrates over the full superspace, any Λ-dependence disappears since ψ is chiral. Similarly, if one considers the $U(1)$ R-transformation, the shift in the Lagrangian vanishes after the integration over superspace. To see that this expression is not renormalized, one merely needs to note that any non-trivial τ-dependence spoils these two properties. As a result, in the case of $SU(2)$ the four derivative terms in the Lagrangian are not renormalized. Note that this argument is non-perturbative. It can be generalized to an even larger class of higher-dimensional operators.

15.3 A deeper understanding of the BPS condition

In our study of monopoles we saw that, under certain circumstances, the complicated second-order non-linear differential equations reduce to first-order differential equations. The main condition is that the potential should vanish. We are now quite used to the idea that supersymmetric theories often have moduli, and we have seen that this is an exact feature of $N = 4$ and many $N = 2$ theories. In the case of an $N = 2$ supersymmetric gauge theory the potential is just that arising from the D term, and one can construct a Prasad–Sommerfield solution. We will now see that the Bogomol'nyi–Prasad–Sommerfield (BPS) condition is not simply magic but is a consequence of the extended supersymmetry of the theory. The resulting mass formula, as a consequence, is *exact*; it is not simply a feature of the classical theory but a property of the full quantum theory. This sort of BPS condition is relevant not only to the study of magnetic monopoles but to topological objects in various dimensions and contexts, particularly in string theory. Here we will give the flavor of the

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argument without worrying about factors of two. More details can be worked out in the exercises; see also the references.

First, we show that the electric and magnetic charges enter in the supersymmetry algebra of this theory as central charges. Thinking of this as an $N = 1$ theory, we have seen that the supercurrents take the form

$$S^{\mu}_{\alpha} = \sigma^{\mu}_{\alpha\dot{\beta}}(\sigma^{\rho\sigma})^{\dot{\beta}\dot{\gamma}}F_{\rho\sigma}\lambda_{\dot{\gamma}} + \partial_{\rho}\phi^{i}\sigma^{\rho}_{\alpha\dot{\beta}}(\sigma^{\mu})^{\dot{\beta}\gamma}\psi^{i}_{\gamma} + F\text{-term contributions.} \quad (15.14)$$

In this theory, however, there is an $SU(4)$ symmetry and the supercurrents should transform as a 4 representation. It is not hard to guess the other three currents

$$S^{i}_{\mu\alpha} = (\sigma_{\mu})_{\alpha\dot{\beta}}(\sigma^{\rho\sigma})^{\dot{\beta}\dot{\gamma}}F_{\rho\sigma}\psi^{i}_{\dot{\gamma}} + \epsilon_{ijk}\partial_{\rho}\phi^{j}\sigma^{\rho}_{\alpha\dot{\beta}}(\sigma_{\mu})^{\dot{\beta}\gamma}\psi^{k}_{\gamma} + F\text{-term contributions.} \quad (15.15)$$

We are interested in proving bounds on the mass. It is useful to define Hermitian combinations of the charges $Q_{\alpha i} = \int d^3 \times S_{\alpha i}$, since we want to study positivity constraints. In this case, it is more convenient to write a four-component expression, using a Majorana (real) basis for the γ matrices. Taking an $N = 2$ subgroup and carefully computing the commutators of the charges, we obtain

$$\{Q_{\alpha i}, Q_{\beta j}\} = \delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu} + \epsilon_{ij}(\delta_{\alpha\beta}U_k + (\gamma_5)_{\alpha\beta}V_k). \quad (15.16)$$

Here

$$\begin{aligned} U_k &= \int d^3x\partial_i(\phi^{a}_{\text{re }k}E^{a}_{i} + \phi^{a}_{\text{im }k}B^{a}_{i}), \\ V_k &= \int d^3x\partial_i(\phi^{a}_{\text{im }k}E^{a}_{i} + \phi^{a}_{\text{re }k}B^{a}_{i}). \end{aligned} \quad (15.17)$$

In the Higgs phase the integrals are, by Gauss's theorem, of electric and magnetic charges multiplied by the Higgs expectation value. From these relations we can derive bounds on masses, using the fact that Q^2_{α} is a positive operator. Taking the expectations of both sides we have, for an electrically neutral system of mass M in its rest frame,

$$M \pm Q_{\text{m}}v \geq 0. \quad (15.18)$$

This bound is saturated when Q annihilates the state. Examining the form of Q_{α}, this is just the BPS condition.

15.3.1 $N = 4$ Yang–Mills theories and electric–magnetic duality

The $N = 4$ theory contains, from the point of view of $N = 1$ supersymmetry, a gauge multiplet and three chiral multiplets in the adjoint representation. In addition to the interactions implied by the gauge symmetry, there is a superpotential

$$W = \frac{1}{6}f_{abc}\epsilon_{ijk}\Phi^{a}_{i}\Phi^{b}_{j}\Phi^{c}_{k}. \quad (15.19)$$

We have normalized the kinetic terms for the fields Φ with a $1/g^2$ factor. So, this interaction has a strength related to the strength of the gauge interactions. This theory has a global $SU(4)$ symmetry. Under this symmetry, the four adjoint fermions transform as a 4, the scalars transform as a 6 and the gauge bosons are invariant. The theory has a large set of

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flat directions. If we simply take all the Φ fields, regarded as matrices, to be diagonal then the potential vanishes. As a result, this theory has monopoles of the BPS type.

This theory has a symmetry even larger than the Z_2 duality symmetry that we contemplated when we examined Maxwell's equations; the full symmetry is $SL(2, Z)$. We might have guessed this by remembering that the coupling constant is part of the holomorphic variable

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \tag{15.20}$$

Thus in addition to our conjectured $e \to 1/e$ symmetry there is a symmetry $\theta \to \theta + 2\pi$. So, in terms of τ we have the two symmetry transformations

$$\tau \to -\frac{1}{\tau}, \quad \tau \to \tau + 1. \tag{15.21}$$

Together, these transformations generate the group $SL(2, Z)$:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \tag{15.22}$$

Now we can look at our BPS formula. To understand whether it respects the $SL(2, Z)$ symmetry we need to understand how this symmetry acts on the states. Writing

$$M = eQ_{\rm e}v + \frac{Q_{\rm m}v}{e}, \tag{15.23}$$

with

$$Q_{\rm e} = n_{\rm e} - n_{\rm m}\frac{\theta}{2\pi}, \quad Q_{\rm m} = 4\pi\frac{n_{\rm m}}{e}, \tag{15.24}$$

the spectrum is invariant under the $SL(2, Z)$ transformation of τ accompanied by

$$\begin{pmatrix} n_{\rm e} \\ n_{\rm m} \end{pmatrix} \to \begin{pmatrix} d & -b \\ c & -d \end{pmatrix} \begin{pmatrix} n_{\rm e} \\ n_{\rm m} \end{pmatrix}. \tag{15.25}$$

Because it follows from the underlying supersymmetry the mass formula is exact, so this duality of the spectrum of BPS objects is a non-perturbative statement about the theory.

15.4 Seiberg–Witten theory

We have seen that $N=4$ theories are remarkably constrained, and this allowed us, for example, to explore an exact duality between electricity and magnetism. Still, these theories are not nearly as rich as field theories with $N \leq 1$ supersymmetry. The $N = 2$ theories are still quite constrained, but exhibit a much more interesting array of phenomena. They illustrate the power provided by supersymmetry over non-perturbative dynamics. They will also allow us to study phenomena associated with magnetic monopoles in a quite non-trivial way. In this section, we will provide a brief introduction to *Seiberg–Witten theory*. This subject has applications not only in quantum field theory but also for our understanding of string theory and, perhaps most dramatically, in mathematics.

It is convenient to describe the $N = 2$ theories in $N = 1$ language. The basic $N = 2$ multiplets are the vector multiplet and the tensor (or hyper) multiplet. From the point of view of $N = 1$ supersymmetry, the $N = 2$ vector contains an $N = 1$ a vector multiplet and a chiral multiplet. The tensor contains two chiral fields. We will focus mainly on theories with only vector multiplets, with gauge group $SU(2)$. In the $N = 1$ description the fields are a vector multiplet V and a chiral multiplet ϕ, both in the adjoint representation. The Lagrangian density is

$$\mathcal{L} = \int d^4\theta \, \frac{1}{g^2} \phi^\dagger e^V \phi - \frac{i}{16\pi} \int d^2\theta \, \tau W^{a\alpha} W^a_\alpha + \text{h.c.} \tag{15.26}$$

Here

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}. \tag{15.27}$$

The $1/g^2$ in front of the chiral field kinetic term is somewhat unconventional, but it makes the $N = 2$ supersymmetry more obvious. As we indicated earlier, one way to understand the $N = 2$ supersymmetry is to note that the Lagrangian we have written down has a global $SU(2)$ symmetry. Under this symmetry the scalar fields ϕ^a and the gauge fields A^a_μ are singlets, while the gauginos λ^a and the fermionic components ψ^a of ϕ transform as a doublet. Acting on the conventional $N = 1$ generators, the $SU(2)$ symmetry produces four new generators. So, we have generators Q^A_α, with $A = 1, 2$.

As it stands, the model has flat directions, with

$$\phi = \frac{a}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15.28}$$

In these directions the spectrum consists of two massive gange bosons and one massless gauge boson, a massive complex scalar that is degenerate with the gauge bosons and a massive Dirac fermion as well as a massless vector and a massless chiral multiplet. The masses of all these particles are

$$M_W = \sqrt{2} a. \tag{15.29}$$

This is precisely the right number of states to fill an $N = 2$ multiplet. Actually, it is a BPS multiplet. It is annihilated by half the supersymmetry generators. The classical theory possesses, in addition to the global $SU(2)$ symmetry, an anomalous $U(1)$ symmetry,

$$\phi \to e^{i\alpha\phi}, \quad \psi \to e^{i\alpha} \psi. \tag{15.30}$$

Under this symmetry, we have

$$\theta \to \theta - 4\alpha \tag{15.31}$$

or

$$\tau \to \tau - 2\pi\alpha. \tag{15.32}$$

Because the physics is periodic in θ with period 2π, $\alpha = \pi/2$ is a symmetry, i.e. the theory has a Z_4 symmetry,

$$\phi \to e^{i\pi/2} \phi. \tag{15.33}$$

Note that ϕ is not gauge invariant. A suitable gauge-invariant variable for the analysis of this theory is

$$u = \langle \text{Tr}\, \phi^2 \rangle. \tag{15.34}$$

Under the discrete symmetry, we have $u \to -u$; at weak coupling

$$u \approx a^2. \tag{15.35}$$

The spectrum of this theory includes magnetic monopoles, in general with electric charges. At the classical level the monopole solutions in this theory are precisely those of Prasad and Sommerfield, with mass

$$M_\text{M} = 4\pi\sqrt{2}\frac{a}{g^2}. \tag{15.36}$$

As in the $N = 4$ theory, there is a BPS formula for the masses:

$$m = \sqrt{2}\,|aQ_\text{e} + a_\text{D}Q_\text{M}|. \tag{15.37}$$

At tree level,

$$a_\text{D} = \frac{4\pi}{g^2}ia = \tau a, \tag{15.38}$$

where the last equation holds if $\theta = 0$. The appearance of i in this formula is not immediately obvious. To see that it must be present, consider the case of dyonic excitations of monopoles. These should have energy of order the charge, with no factors of $1/g^2$. This is ensured by the relative phase between a and a_D. These formulas will receive corrections in perturbation theory and beyond; our goal is to understand the form of these corrections and their (dramatic) physical implications.

Equation (15.38) is not meaningful as it stands; τ is a function of scale. Instead, Seiberg and Witten suggested that

$$\tau = \frac{da_\text{D}}{da}. \tag{15.39}$$

They also proposed the existence of a duality symmetry, under which

$$a_\text{D} \leftrightarrow a, \quad \tau \to -\frac{1}{\tau}. \tag{15.40}$$

To formulate our questions more precisely and to investigate this proposal, it is helpful, as always, to consider a low-energy effective action. This action should respect the $N = 2$ supersymmetry; in $N = 1$ language this means that the Lagrangian should take the form

$$\mathcal{L} = \int d^4\theta\, K(a, \bar{a}) - \frac{i}{16\pi^2}\int d^2\theta\, \tau(a) W^\alpha W_\alpha. \tag{15.41}$$

The $N = 2$ supersymmetry implies a relation between K and τ; without it these would be independent quantities. Both quantities can be obtained from a holomorphic function called the *prepotential*, $\mathcal{F}(a)$:

$$\tau = \frac{d^2\mathcal{F}}{da^2}, \quad K = \frac{1}{4\pi}\frac{d\mathcal{F}}{da}a^*. \tag{15.42}$$

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From

$$\tau = \frac{da_D}{da} = \frac{d}{da}\left(\frac{d\mathcal{F}}{da}\right) \tag{15.43}$$

we have

$$\frac{d\mathcal{F}}{da} = ia_D, \tag{15.44}$$

so that

$$K = \frac{1}{4\pi}\,\mathrm{Im}\, a_D a^*. \tag{15.45}$$

Our goal will be to obtain a non-perturbative description of $\mathcal{F}$. At weak coupling the beta function of this theory is obtained from $b_0 = 3N - N = 2N = 4$, so

$$\tau = \frac{i}{\pi}\ln\frac{u}{\Lambda^2}. \tag{15.46}$$

As a check on this formula note that, under $u \to e^{2i\alpha}u$, $\theta \to \theta - 4\alpha$, we have

$$\tau = \frac{\theta}{2\pi} + 4\pi i g^{-2} \to \tau - \frac{2i\alpha}{\pi}, \tag{15.47}$$

and this is precisely the behavior of the formula Eq. (15.46).

This is similar to phenomena we have seen in $N = 1$ theories. But, when we consider the monopoles of the theory, the situation becomes more interesting. First note that, using the leading-order result for τ,

$$a_D = \frac{2i}{\pi}\left(a\ln\frac{a}{\Lambda} - a\right). \tag{15.48}$$

So, under the transformation $u \to e^{i\alpha u}$ of u,

$$a_D \to e^{i\alpha/4}\left(a_D - \frac{\alpha}{2\pi}a\right). \tag{15.49}$$

Our BPS mass formula transforms to

$$m \to \sqrt{2}\left|a\left(Q_e - \frac{4\alpha}{2\pi}Q_m\right) + a_D Q_M\right|. \tag{15.50}$$

This is the Witten effect, which we discussed earlier: in the presence of θ, the coefficient of $F\tilde{F}$, of (7.39), a magnetic monopole acquires an electric charge. More generally, the spectrum of dyons is altered.

Consider now what happens when we do a full 2π change of θ ($u \to -u$); it should be a symmetry. It is in this case, but in a subtle way: the spectrum of the dyonic excitations of the theory is unchanged but the charges of the dyons have shifted by one fundamental unit. This, in turn, is related to the branched structure of τ.

At the non-perturbative level the structure is even richer. We might expect that

$$\tau(u) = \frac{i}{\pi}\ln\frac{u}{\Lambda^2} + \alpha\exp\left(-\frac{8\pi^2}{g^2}\right) + \beta\exp\left(-\frac{8\pi^2}{g^2}\right) + \cdots. \tag{15.51}$$

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Note that, interpreting $\exp(-8\pi^2/g^2)$ as $\exp(2\pi i\tau)$, each term in this series has the correct periodicity in θ. Moreover,

$$\exp(2\pi i\tau) = \frac{\Lambda^2}{u^2}. \tag{15.52}$$

These corrections have precisely the structure required for them to be instanton corrections, and these instanton corrections have been computed. But, following Seiberg and Witten, we can be bolder and consider what happens when g becomes large. Naively, we might expect that some monopoles become light. Associated with this, τ may have a singularity at some point $u_0 = \gamma\Lambda^2$, where Λ is the renormalization-group-invariant mass of the theory. In light of the Z_2 symmetry there must also be a singularity at $-u_0$. Such a singularity arises because a particle is becoming massless. If we think of $\tau_{\rm D}$ as the dual of τ then there is an electrically charged light field of unit charge; more precisely, there must be two particles of opposite charge in order that they can gain mass. So $\tau_{\rm D}$ has the following structure:

$$\tau_{\rm D} = -\frac{2i}{2\pi}\ln m_{\rm M}. \tag{15.53}$$

Assuming that $a_{\rm D}$ has a simple zero,

$$a_{\rm D} \approx b(u-u_0), \quad m_{\rm M} = \sqrt{2}a_{\rm D}, \tag{15.54}$$

then

$$\tau_{\rm D} = -\frac{i}{\pi}\ln(u-u_0) = -\frac{1}{\tau(u)}. \tag{15.55}$$

Starting with the relation

$$\frac{da}{da_{\rm D}} = -\tau_{\rm D} = -\frac{i}{\pi}\ln a_{\rm D}, \tag{15.56}$$

we have

$$a = \frac{i}{\pi}(a_{\rm D}\ln a_{\rm D} - a_{\rm D}). \tag{15.57}$$

Similarly, we can consider the behavior at the point $-u_0$. This is the mirror image of the previous case, but we must be careful about the relation of a and $a_{\rm D}$. They are connected by the symmetry transformation

$$\tilde{a} = ia, \quad \tilde{a}_{\rm D} = i(a_{\rm D} - a). \tag{15.58}$$

Now,

$$\tau_{\rm D} = -\frac{1}{\tau(u)} = -\frac{i}{\pi}\ln(u+u_0) \tag{15.59}$$

and

$$\tilde{a} = \frac{1}{\pi}(\tilde{a}_{\rm D}\ln\tilde{a}_{\rm D} - \tilde{a}_{\rm D}). \tag{15.60}$$

Going around the singularities, at u_0 we have

$$a \to a - 2a_{\rm D}, \quad a_{\rm D} \to a_{\rm D}, \tag{15.61}$$

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while at $-u_0$

$$a \to 3a - 2a_D, \quad a_D \to 2a - a_D. \tag{15.62}$$

This should be compared with the effect of going around 2π at large u, when $a \to -a$ and $a_D \to -(a_D - a)$. Assuming that these are the only singularities, we can, from this information, reconstruct τ. We will not give the full solution of Seiberg and Witten here, but the basic idea is to note that $\tau(u)$ is the modular parameter of a two-dimensional torus and to reconstruct the torus.

This analysis has allowed us to study the theory deep in the non-perturbative region. Seiberg and Witten uncovered a non-trivial duality, a limit in which monopoles become massless, and they provided insight into confinement. These sorts of ideas have been extended to other theories and to theories in higher dimensions and have provided insight into many phenomena in string theory, quantum gravity and pure mathematics.

Suggested reading

The lectures by Lykken (1996) provide a brief introduction to aspects of $N > 1$ supersymmetry. Olive and Witten (1978) first clarified the connection between the BPS condition and extended supersymmetry, in a short and quite readable paper. Harvey (1996) provides a more extensive introduction to monopoles and the BPS condition. The original paper of Seiberg and Witten (1994) is quite clear; Peskin's lectures, from which we have borrowed extensively here, provide a brief and very clear introduction to the subject.

Exercises

(1) Check the supersymmetry commutators in extended supersymmetry, Eq. (15.16).
(2) Rewrite these supersymmetry commutators in a real basis for the Dirac matrices. Using this, verify the BPS inequality.
(3) Check that the spectrum of monopoles and dyons in Eq. (15.23) is invariant under $SL(2, Z)$ transformations.

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