AN ANALOGUE OF A PROBLEM OF J. BALÁZS AND P. TURÁN

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1. Introduction. In 1955, J. Surányi and P. Turán (**8**) initiated the problem of existence and uniqueness of interpolatory polynomials of degrees less than or equal to $2n - 1$ when their values and second derivatives are prescribed on n given nodes. This kind of interpolation was termed (0, 2)-interpolation. Later, Balázs and Turán (**1**) gave the explicit representation of the interpolatory polynomials for the case when the n given nodes (n even) are taken to be the zeros of $\pi_n(x) = (1 - x^2)P_{n-1}'(x)$, where $P_{n-1}(x)$ is the Legendre polynomial of degree $n - 1$. In this case the explicit representation of interpolatory polynomials turns out to be simple and elegant.

Balázs and Turán (**2**) proved the convergence of these polynomials when $f(x)$ has a continuous first derivative satisfying certain conditions of modulus of continuity. They noted (**1**) that a significant application of lacunary interpolation could possibly be given in the theory of a differential equation of the form $y'' + A(x)y = 0$. Let

$$(1.1)\qquad -1 = x_{n+2} < x_{n+1} < \ldots < x_2 < x_1 = 1$$

be the zeros of $w_n(x) = (1 - x^2)P_n(x)$. Here we are interested in determining the interpolatory polynomials $R_n(x)$ of degree less than or equal to $2n + 1$ satisfying the following conditions:

$$(1.2)\qquad \begin{aligned} R_n(x_\nu) &= \alpha_\nu, \qquad \nu = 1, 2, \ldots, n + 2, \\ R_n''(x_\nu) &= \beta_\nu, \qquad \nu = 2, 3, \ldots, n + 1. \end{aligned}$$

It turns out that these polynomials are unique for n even, but for n odd there does not exist, in general, a unique polynomial of degree less than or equal to $2n + 1$ satisfying the conditions of (1.2). Obviously,

$$(1.3)\qquad R_n(x) = \sum_{\nu=1}^{n+2} \alpha_\nu r_\nu(x) + \sum_{\nu=2}^{n+1} \beta_\nu \rho_\nu(x),$$

where $r_\nu(x)$ and $\rho_\nu(x)$ are as in Theorem 2.1 of § 2. Our main aim is to prove the following theorem concerning $R_n(x)$.

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THEOREM 1.1. *Let $f(x)$ be a continuous function satisfying the Zygmund condition*

$$|f(x+h) - 2f(x) + f(x-h)| = o(h)$$

in $[-1, 1]$ *and let* $|\beta_\nu| \leqq o(n)/\sqrt{(1-x_\nu^2)}$. *Then the sequence of interpolatory polynomials* $R_n(x, f)$ (with $\alpha_\nu = f(x_\nu)$) *converges uniformly to* $f(x)$ *in* $[-1, 1]$.

A comparison of the above theorem with the corresponding theorem of G. Freud (**5**) shows that as far as convergence of (0, 2) interpolatory polynomials is concerned, the zeros of $(1-x^2)P_{n-1}'(x)$ or $(1-x^2)P_n(x)$ are equally good. However, a comparison (**10**) indicates that the zeros of $(1-x^2)T_n(x)$ are not as good as the zeros of $(1-x^2)P_n(x)$ or $(1-x^2)P_{n-1}'(x)$ since their convergence theorem, although proved best possible, in a sense requires $f'(x) \in \operatorname{Lip}\alpha, \alpha > \frac{1}{2}$.

2. THEOREM 2.1. *For n even we have, for* $2 \leqq \nu \leqq n+1$, *that*

$$\rho_\nu(x) = \frac{(1-x^2)^{1/2}P_n(x)}{2P_n'(x_\nu)}\left[A_\nu \int_{-1}^{x} \frac{P_n(t)}{\sqrt{(1-t^2)}}\,dt + \int_{-1}^{x} \frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt\right], \tag{2.1}$$

where

$$A_\nu \int_{-1}^{1} \frac{P_n(t)}{\sqrt{(1-t^2)}}\,dt = -\int_{-1}^{1} \frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt. \tag{2.2}$$

Further,

$$r_1(x) = \frac{1+x}{2}P_n^2(x) - \frac{1-x^2}{2}P_n(x)P_n'(x) - \frac{(1-x^2)^{1/2}}{2}P_n(x)\int_{-1}^{x}\frac{P_n'(t)}{\sqrt{(1-t^2)}}\,dt, \tag{2.3}$$

$$r_{n+2}(x) = \frac{1-x}{2}P_n^2(x) + \frac{(1-x^2)}{2}P_n(x)P_n'(x) - \frac{(1-x^2)^{1/2}}{2}P_n(x)\int_{-1}^{x}\frac{P_n'(t)}{\sqrt{(1-t^2)}}\,dt, \tag{2.4}$$

and for $2 \leqq \nu \leqq n+1$, *we have that*

$$r_\nu(x) = \frac{1-x^2}{2(1-x_\nu^2)}l_\nu^2(x) + \frac{(1-x^2)P_n'(x)l_\nu(x)}{2(1-x_\nu^2)P_n'(x_\nu)} + \frac{P_n(x)(1-x^2)^{1/2}}{2(1-x_\nu^2)P_n'(x_\nu)}\left[B_\nu\int_{-1}^{x}\frac{P_n(t)}{\sqrt{(1-t^2)}}\,dt - \int_{-1}^{x}\frac{tl_\nu'(t)}{\sqrt{(1-t^2)}}\,dt\right] + c_\nu\rho_\nu(x), \tag{2.5}$$

where

$$c_\nu = \frac{n(n+1)}{1-x_\nu^2} - \frac{x_\nu^2}{(1-x_\nu^2)^2} \tag{2.6}$$

and

$$B_\nu\int_{-1}^{1}\frac{P_n(t)}{\sqrt{(1-t^2)}}\,dt = \int_{-1}^{1}\frac{tl_\nu'(t)}{\sqrt{(1-t^2)}}\,dt. \tag{2.7}$$

The proof of this theorem could be obtained on the same lines as in (**10**) and is omitted here.

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3. Preliminaries. We shall later make use of the following well-known results about Legendre polynomials; see Szegö (**9**). For $-1 \leqq x \leqq 1$, we have that

$$n^{1/2}(1-x^2)^{1/4}|P_n(x)| \leqq \sqrt{(2/\pi)}, \tag{3.1}$$

$$(1-x^2)^{3/4}|P_{n-1}'(x)| \leqq \sqrt{2n}, \tag{3.2}$$

$$|P_n'(x)| \leqq \tfrac{1}{2}n(n+1), \tag{3.3}$$

$$|P_n(x)| \leqq 1, \tag{3.4}$$

$$|(1-x^2)^{1/2}P_n'(x)| \leqq n, \tag{3.5}$$

$$\sum_{\nu=2}^{n+1} \frac{(1-x^2)l_\nu^2(x)}{(1-x_\nu^2)} = 1 - P_n^2(x); \qquad \text{see } (\mathbf{4}). \tag{3.6}$$

(3.7) follows from (3.6) by comparing the coefficient of x^{2n} on both sides.

$$\sum_{\nu=2}^{n+1} \frac{1}{(1-x_\nu^2)[P_n'(x_\nu)]^2} = 1. \tag{3.7}$$

From (**9**, p. 236, formula (8.9.2)), we have that

$$|P_n'(\cos\theta_\nu)| \sim (\nu-1)^{-3/2}n^2, \qquad \nu = 2, 3, \ldots, \tfrac{1}{2}n+1, \tag{3.8}$$

$$|P_n'(\cos\theta_\nu)| \sim (n-\nu+2)^{-3/2}n^2, \qquad \nu = \tfrac{1}{2}n+2, \ldots, n+1, \tag{3.9}$$

$$(1-x_\nu^2) > (\nu-1)^2/n^2, \qquad \nu = 2, 3, \ldots, \tfrac{1}{2}n+1, \tag{3.10}$$

$$(1-x_\nu^2) > (n-\nu+2)^2/n^2, \qquad \nu = \tfrac{1}{2}n+2, \ldots, n+1. \tag{3.11}$$

From (**7**) we have that

$$\left|\int_{-1}^{x} \frac{P_n(t)}{t-x_\nu}\,dt\right| < \frac{4}{\sqrt{\pi}n^{3/2}(x_\nu - x)} \quad \text{for } x < x_\nu < 1 \tag{3.12}$$

and

$$\left|\int_{1}^{x} \frac{P_n(t)}{t-x_\nu}\,dt\right| < \frac{4}{\sqrt{\pi}n^{3/2}(x - x_\nu)} \quad \text{for } -1 < x_\nu < x. \tag{3.13}$$

4. In order to prove our main theorem we need the following lemmas.

LEMMA 4.1 (G. Freud (**5**)). *Let $f(x)$ be a continuous function satisfying the Zygmund condition in $[-1, 1]$; then there exists a sequence of polynomials $\Phi_n(x)$ of degree less than or equal to n with the following properties:*

$$|f(x) - \Phi_n(x)| = o(n^{-1})[(1-x^2)^{1/2} + n^{-1}] \tag{4.1}$$

and

$$|\Phi_n''(x)| = o(n)\min[(1-x^2)^{-1/2}, n] \tag{4.2}$$

which hold uniformly in $[-1, 1]$.

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LEMMA 4.2. *For n even we have that*

$$\frac{2}{n+1} < \int_{-1}^{1} \frac{P_n(t)}{\sqrt{(1-t^2)}}\,dt < \frac{2}{n}, \tag{4.3}$$

where $P_n(x)$ is the Legendre polynomial of degree n. Further,

$$\left| \int_{-1}^{1} \frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt \right| \leqq \frac{12}{(1-x_\nu^2)^{7/4}[P_n'(x_\nu)]^2}. \tag{4.4}$$

Proof. Since (see Sansone (**6**, p. 200))

$$P_{2k}(\cos\theta) = 2\sum_{i=0}^{k-1} \alpha_i\alpha_{2k-i}\cos(2k-2i)\theta + \alpha_k^2, \tag{4.5}$$

where

$$\alpha_i = \sqrt{\frac{2}{\pi}}\left(\sqrt{\frac{1}{2i+\theta_1}}\right), \qquad 0 < \theta_1 < 1, \qquad \alpha_0 = 1, \tag{4.6}$$

we have, on integrating (4.5) from 0 to π, that

$$\int_0^\pi P_n(\cos\theta)\,d\theta = \frac{2}{2n+\theta_1},$$

whence we obtain (4.3).

In order to prove (4.4) we first observe that

$$\int_{-1}^{1} \frac{[P_{2r}(x) - P_{2r+2}(x)]}{\sqrt{(1-x^2)}}\,dx = (\alpha_r^2 - \alpha_{r+1}^2)\pi \leqq \frac{1}{r^2}. \tag{4.7}$$

Now, using the well-known Christoffel formula (Sansone (**6**, p. 179)), we have that

$$l_\nu(x) = \frac{1}{(1-x_\nu^2)[P_n'(x_\nu)]^2}\left[\sum_{r=1}^{n-1} P_r'(x_\nu)(P_{r-1}(x) - P_{r+1}(x)) + P_n'(x_\nu)P_{n-1}(x) + P_{n+1}'(x_\nu)P_n(x)\right]. \tag{4.8}$$

Hence, from (4.7) and (4.8) we obtain

$$\left| \int_{-1}^{1} \frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt \right| \leqq \frac{1}{(1-x_\nu^2)[P_n'(x_\nu)]^2}\left[\sum_{r=1}^{\frac{1}{2}n-1} \frac{|P_{2r+1}'(x_\nu)|}{r^2} + \frac{2|P_{n+1}'(x_\nu)|}{n}\right].$$

Using formula (3.2) and observing that

$$\sum_{r=1}^{\frac{1}{2}n-1} \frac{1}{r^{3/2}} < \sum_{r=1}^{\infty} \frac{1}{r^{3/2}} = \text{constant},$$

we obtain the required result.

LEMMA 4.3. *We have that*

$$\left| \int_{-1}^{x} \frac{P_n(t)}{\sqrt{(1-t^2)}}\,dt \right| < \frac{7}{n}.$$

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Proof. This lemma is an immediate consequence of Theorem 8.21.13 of Szegö's book (**8**).

LEMMA 4.4. *We have that*

$$\left|(1-x^2)^{1/4}P_n(x)\int_{-1}^{x}\frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt\right| \leqq \frac{|l_\nu(x)|}{n^{3/2}(1-x_\nu^2)^{1/2}} + \frac{6}{n(1-x_\nu^2)|P_n'(x_\nu)|}.$$

Proof. On substituting $t = \cos\gamma$ and $x = \cos\theta$, we have that

$$I = \int_{-1}^{x}\frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt = \frac{1}{\sin\theta_\nu}\int_{\theta}^{\pi}\sin\theta_\nu l_\nu(\cos\gamma)\,d\gamma.$$

Since

$$\sin\theta_\nu l_\nu(\cos\gamma) = \cot\left(\frac{\theta_\nu+\gamma}{2}\right)\frac{P_n(\cos\gamma)}{P_n'(\cos\theta_\nu)} + \sin\gamma l_\nu(\cos\gamma),$$

we have that

$$I = I_1 + I_2, \quad \text{say}.$$

Since $\sin\theta_\nu \leqq \sin\theta_\nu + \sin\gamma \leqq 2\sin\frac{1}{2}(\theta_\nu+\gamma)$ $(0 \leqq \theta \leqq \pi, 0 < \theta_\nu < \pi)$ we have, using (3.1), that

$$|I_1| \leqq \frac{6}{n^{1/2}(1-x_\nu^2)|P_n'(x_\nu)|}.$$

From the result of Saxena (**7**) we have that

$$|I_2| \leqq \frac{1}{\sin\theta_\nu}\int_{\theta}^{\pi}\sin\gamma l_\nu(\cos\gamma)\,d\gamma$$
$$\leqq \frac{1}{n^{3/2}(1-x_\nu^2)^{1/2}|P_n'(x_\nu)|\,|x-x_\nu|} \quad (\text{for } x < x_\nu < 1 \text{ and } -1 < x_\nu < x).$$

Therefore, finally, we have that

$$\left|(1-x^2)^{1/4}P_n(x)\int_{-1}^{x}\frac{l_\nu(t)}{\sqrt{(1-t^2)}}\,dt\right| \leqq \frac{6}{n(1-x_\nu^2)|P_n'(x_\nu)|} + \frac{(1-x^2)^{1/4}|l_\nu(x)|}{n^{3/2}(1-x_\nu^2)^{1/2}}$$

from which the lemma follows.

Estimation of the fundamental polynomials of the second kind. The above lemmas lead us to formulate the following lemma.

LEMMA 4.5. *For n even and for all x such that* $-1 \leqq x \leqq 1$, *we have that*

$$(4.9)\quad |\rho_\nu(x)| \leqq \frac{48}{n^{1/2}(1-x_\nu^2)^{7/4}|P_n'(x_\nu)|^3} + \frac{3}{n(1-x_\nu^2)[P_n'(x_\nu)]^2} + \frac{(1-x^2)^{1/4}|l_\nu(x)|}{2n^{3/2}|P_n'(x_\nu)|(1-x_\nu^2)^{1/2}}$$

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and

$$\sum_{\nu=2}^{n+1} (1 - x_\nu^2)^{-1/2}|\rho_\nu(x)| \leqq \frac{105}{n}. \tag{4.10}$$

Proof. (4.9) follows from (2.1), Lemmas 4.2, 4.3, 4.4, and relation (3.1). With the help of (4.9), (3.8), (3.9), (3.10), and (3.11), we have that

$$\sum_{\nu=2}^{n+1} (1 - x_\nu^2)^{-1/2}|\rho_\nu(x)| \leqq 102 \sum_{\nu=2}^{\frac{1}{2}n+1} \frac{1}{n^2} + 2 \sum_{\nu=2}^{\frac{1}{2}n+1} \frac{(1 - x^2)^{1/4}|l_\nu(x)|}{2n^{3/2}(1 - x_\nu^2)^{1/2}|P_n'(x_\nu)|}.$$

Since

$$(1 - x^2)^{1/4}|l_\nu(x)| \leqq \frac{3n}{(1 - x_\nu^2)^{5/4}[P_n'(x_\nu)]^2} \tag{4.11}$$

(which can be easily obtained by the help of the Christoffel formula; see Sansone (**6**, p. 179)), we obtain (4.10).

5. Estimation of the fundamental polynomials of the first kind.

LEMMA 5.1. *We have that*

$$\left| \int_{-1}^{x} \frac{P_n'(t)}{\sqrt{(1 - t^2)}} dt \right| \leqq 21\, n.$$

Proof. The proof follows from Lemma 2.3 and the identity $P_n'(x) - P_{n-2}'(x) = (2n - 1)P_{n-1}(x)$.

Lemma 5.1 leads us to prove the following lemma.

LEMMA 5.2. *We have, for* $-1 \leqq x \leqq +1$, *that*

$$|r_1(x)| \leqq 13n^{1/2} \tag{5.1}$$

and

$$|r_{n+2}(x)| \leqq 13n^{1/2}. \tag{5.2}$$

To find the estimate for $r_\nu(x)$ $(2 \leqq \nu \leqq n + 1)$ we prove the following lemmas.

LEMMA 5.3. *We have that*

$$\left| \int_{-1}^{1} \frac{tl_\nu'(t)}{\sqrt{(1 - t^2)}} dt \right| \leqq \frac{12n^{3/2}}{(1 - x_\nu^2)^{7/4}[P_n'(x_\nu)]^2}. \tag{5.3}$$

Proof. Using the Christoffel formula, differentiating once, making use of formula (14) (Sansone (**6**, p. 178)) and integrating between the limits -1 and 1, we have that

$$\int_{-1}^{1} \frac{tl_\nu'(t)}{\sqrt{(1 - t^2)}} dt = \frac{1}{(1 - x_\nu^2)[P_n'(x_\nu)]^2} \Bigg[- \sum_{r=1}^{n-2} (2r + 1)P_r'(x_\nu)$$

$$\times \int_{-1}^{1} \frac{tP_r(t)}{\sqrt{(1 - t^2)}} dt + P_n'(x_\nu) \int_{-1}^{1} \frac{tP_{n-1}'(t)}{\sqrt{(1 - t^2)}} dt + P_{n-1}'(x_\nu) \int_{-1}^{1} \frac{tP_{n-2}'(t)}{\sqrt{(1 - t^2)}} dt \Bigg]$$

$$= I_1 + I_2 + I_3, \quad \text{say}.$$

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First we note that $I_2 = 0$ for n even, and by integration by parts and using the differential equation for $P_n(x)$ we have that

$$(5.4)\qquad \left| \int_{-1}^{1} \frac{tP_{n-2}'(t)}{\sqrt{(1-t^2)}}\,dt \right| \leqq 2n - 2.$$

Now, using (3.2) we have that

$$(5.5)\qquad |I_3| \leqq \frac{2^{3/2} n^{3/2}}{(1-x_\nu^2)^{7/4}[P_n'(x_\nu)]^2}.$$

From the recurrence relation for $P_r(x)$ we have, on using Lemma 4.2, that

$$(5.6)\qquad \left| \int_{-1}^{1} \frac{tP_{2r-1}(t)}{\sqrt{(1-t^2)}} \right| \leqq \frac{4}{4r-1}.$$

Therefore, using (3.2) and (5.6), we have that

$$\left| \sum_{r=1}^{n-2} (2r+1)P_r'(x_\nu) \int_{-1}^{1} \frac{tP_r(t)}{\sqrt{(1-t^2)}}\,dt \right| \leqq \frac{8n^{3/2}}{(1-x_\nu^2)^{3/4}}.$$

Hence,

$$(5.7)\qquad |I_1| \leqq \frac{8n^{3/2}}{(1-x_\nu^2)^{7/4}[P_n'(x_\nu)]^2}.$$

Thus, combining (5.5) and (5.7) we obtain the required result.

LEMMA 5.4. *We have that*

$$(5.8)\qquad \left| \int_{-1}^{x} \frac{tl_\nu'(t)}{\sqrt{(1-t^2)}}\,dt \right| < \frac{162n^{3/2}}{(1-x_\nu^2)^{7/4}[P_n'(x_\nu)]^2}.$$

Proof. The proof of this lemma follows that of Lemma 5.3, along with the use of Lemmas 4.3 and 5.1; we omit the details.

LEMMA 5.5. *For n even and for all x such that $-1 \leqq x \leqq 1$, we have that*

$$(5.9)\qquad |r_\nu(x)| \leqq \frac{432n^{3/2}}{(1-x_\nu^2)^{11/4}|P_n'(x_\nu)|^3} + \frac{1-x^2}{2(1-x_\nu^2)}\, l_\nu^2(x)$$
$$+ \frac{n(1-x^2)^{1/2}|l_\nu(x)|}{2(1-x_\nu^2)|P_n'(x_\nu)|} + \frac{9n}{(1-x_\nu^2)^2[P_n'(x_\nu)]^2} + \frac{3n^{1/2}|l_\nu(x)|(1-x^2)^{1/4}}{2(1-x_\nu^2)^{3/2}|P_n'(x_\nu)|}$$

$$(5.10)\qquad \sum_{\nu=2}^{n+1} |r_\nu(x)| \leqq 894\, n \log n,$$

and

$$(5.11)\qquad \sum_{\nu=2}^{n+1} (1-x_\nu^2)^{1/2}|r_\nu(x)| \leqq 894n.$$

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Proof. From (2.5) we have, for $2 \leqq \nu \leqq n + 1$, that

$$r_\nu(x) = \frac{1 - x^2}{2(1 - x_\nu^2)} l_\nu^2(x) + \frac{(1 - x^2)P_n'(x)l_\nu(x)}{2(1 - x_\nu^2)P_n'(x_\nu)} + \frac{P_n(x)(1 - x^2)^{1/2}B_\nu}{(1 - x_\nu^2)P_n'(x_\nu)} \int_{-1}^{x} \frac{P_n(t)}{\sqrt{(1 - t^2)}} dt - \frac{P_n(x)(1 - x^2)^{1/2}}{(1 - x_\nu^2)P_n'(x_\nu)} \int_{-1}^{x} \frac{tl_\nu'(t)}{\sqrt{(1 - t^2)}} dt + c_\nu \rho_\nu(x)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5, \quad \text{say.}$$

Now

$$|I_1| = \frac{(1 - x^2)}{2(1 - x_\nu^2)} l_\nu^2(x) \tag{5.12}$$

and using (3.5), we have that

$$|I_2| = \frac{(1 - x^2)|P_n'(x)|\,|l_\nu(x)|}{2(1 - x_\nu^2)|P_n'(x_\nu)|} \leqq \frac{n(1 - x^2)^{1/2}|l_\nu(x)|}{2(1 - x_\nu^2)|P_n'(x_\nu)|}. \tag{5.13}$$

Using Lemmas 4.2, 4.3, 5.3, and relation (3.1), we have that

$$|I_3| \leqq \frac{126n}{(1 - x_\nu^2)^{11/4}|P_n'(x_\nu)|^3}. \tag{5.14}$$

With the help of Lemma 5.4 and relation (3.1) we have that

$$|I_4| \leqq \frac{162n}{(1 - x_\nu^2)^{11/4}|P_n'(x_\nu)|^3}. \tag{5.15}$$

Further, since by (2.6), (3.10), and (3.11) we have that

$$|c_\nu| \leqq \frac{3n^2}{1 - x_\nu^2}, \tag{5.16}$$

whence, using (5.16) and Lemma 4.5, we obtain

$$|I_5| \leqq \frac{144n^2}{\sqrt{n}(1 - x_\nu^2)^{11/4}|P_n'(x_\nu)|^3} + \frac{3n^2|l_\nu(x)|(1 - x^2)^{1/4}}{2n^{3/2}(1 - x_\nu^2)^{3/2}|P_n'(x_\nu)|} + \frac{9n^2}{n(1 - x_\nu^2)^2[P_n'(x_\nu)]^2}. \tag{5.17}$$

From (5.12)–(5.15) and (5.17) we obtain the required result. With the help of (5.9), (4.11), and (3.6)–(3.11), we obtain (5.10) as well as (5.11).

6. Proof of Theorem 1.1. Owing to the uniqueness theorem, we have that

$$\Phi_n(x) = \sum_{\nu=1}^{n+2} \Phi_n(x_\nu)r_\nu(x) + \sum_{\nu=2}^{n+1} \Phi_n''(x_\nu)\rho_\nu(x), \tag{6.1}$$

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where $\Phi_n(x)$ is defined by Lemma 4.1, and by (1.3) we have that

$$R_n(x,f) = \sum_{\nu=1}^{n+2} f(x_\nu)r_\nu(x) + \sum_{\nu=2}^{n+1} \beta_\nu\rho_\nu(x). \tag{6.2}$$

Since

$$|R_n(x,f) - f(x)| \leqq |R_n(x,f) - \Phi_n(x)| + |\Phi_n(x) - f(x)|, \tag{6.3}$$

use of (6.1) and (6.2) yields

$$\begin{aligned} |R_n(x,f) - \Phi_n(x)| &\leqq \sum_{\nu=1}^{n+2} |f(x_\nu) - \Phi_n(x_\nu)|\,|r_\nu(x)| \\ &\quad + \sum_{\nu=2}^{n+1} |\beta_\nu|\,|\rho_\nu(x)| + \sum_{\nu=2}^{n+1} |\Phi_n''(x_\nu)|\,|\rho_\nu(x)| \\ &= |s_1| + |s_2| + |s_3|, \quad \text{say.} \end{aligned} \tag{6.4}$$

Now, using Lemmas 4.1, 5.2, and 5.5, we have that

$$\begin{aligned} |s_1| &= |f(x_1) - \Phi_n(x_1)|\,|r_1(x)| + \sum_{\nu=2}^{n+1} |f(x_\nu) - \Phi_n(x_\nu)|\,|r_\nu(x)| \\ &\quad + |f(x_{n+2}) - \Phi_n(x_{n+2})|\,|r_{n+2}(x)| \\ &= o(1) + o\left(\frac{1}{n}\right)[894\, n] + o\left(\frac{1}{n^2}\right)[894\, n \log n] \\ &= o(1). \end{aligned} \tag{6.5}$$

Further, using Lemma (4.5) and the estimate $|\beta_\nu| = o(n)/\sqrt{(1 - x_\nu^2)}$ we have that

$$|s_2| \leqq o(n) \sum_{\nu=2}^{n+1} (1 - x_\nu^2)^{-1/2}|\rho_\nu(x)| \leqq o(n)\,\frac{105}{n} = o(1). \tag{6.6}$$

Again using Lemmas 4.1 and 4.5, we obtain

$$|s_3| \leqq o(n) \sum_{\nu=2}^{n+1} (1 - x_\nu^2)^{-1/2}|\rho_\nu(x)| \leqq o(n)\,\frac{105}{n} = o(1). \tag{6.7}$$

Therefore from (6.4), (6.5), (6.6), and (6.7) we have that

$$|R_n(x,f) - \Phi_n(x)| = o(1)$$

and using Lemma 4.1, we have that

$$|R_n(x,f) - f(x)| = o(1).$$

This completes the proof of the theorem.

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