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From massless free scalar field to conformal field theories

In this chapter we analyze the simplest field theory, which is the theory of a free massless scalar field in two space-time dimensions, one space and one time.[1] The rich symmetry and algebraic structure of this theory encapsulates the basic concepts of two-dimensional conformal field theory, which will be the topic of the next chapter.

1.1 Complex geometry

It is convenient for the discussion of two-dimensional free scalar theory and later conformal field theories to introduce complex coordinates as follows:[2]

$$\xi = x^0 + ix^1 \quad \bar{\xi} = x^0 - ix^1. \tag{1.1}$$

We now take x^0 and x^1 to be in Euclidean space. Correspondingly we define the derivatives

$$\partial_\xi = \frac{1}{2}(\partial_0 - i\partial_1) \quad \partial_{\bar{\xi}} = \frac{1}{2}(\partial_0 + i\partial_1), \tag{1.2}$$

which is a special case of the decomposition to components of vectors, namely

$$\begin{aligned} A_\xi &= \tfrac{1}{2}(A^0 - iA^1) \quad & A_{\bar{\xi}} &= \frac{1}{2}(A^0 + iA^0) \\ A^\xi &= (A^0 + iA^1) \quad & A^{\bar{\xi}} &= (A^0 - iA^1). \end{aligned} \tag{1.3}$$

The metric of the flat Euclidean space-time $\mathrm{d}s^2 = \mathrm{d}{x^0}^2 + \mathrm{d}{x^1}^2$ translates into $\mathrm{d}s^2 = \mathrm{d}\xi d\bar{\xi}$, namely

$$g_{\xi\bar{\xi}} = g_{\bar{\xi}\xi} = \frac{1}{2}, \quad g^{\xi\bar{\xi}} = g^{\bar{\xi}\xi} = 2, \quad g_{\xi\xi} = g_{\bar{\xi}\bar{\xi}} = g^{\xi\xi} = g^{\bar{\xi}\bar{\xi}} = 0. \tag{1.4}$$

With this metric at hand the scalar product of two vectors takes the form

$$A^\mu B_\mu = A^\xi B_\xi + A^{\bar{\xi}} B_{\bar{\xi}} = \frac{1}{2}(A^\xi B^{\bar{\xi}} + A^{\bar{\xi}} B^\xi). \tag{1.5}$$

Complex components of higher-order tensors relate in a similar manner to the real components, in particular for a symmetric two-tensor (like the

[1] The content of this chapter comprises the basics of massless scalar fields in two dimensions. This is covered in many textbooks.

[2] The use of complex coordinates in the context of the bosonic string theory is described by Polyakov in [177].

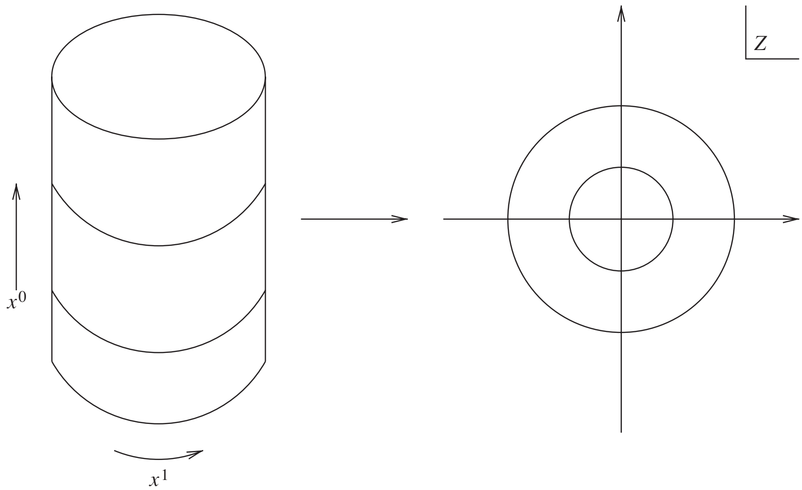

Fig. 1.1. The map between ξ and z.

energy-momentum tensor),

$$
\begin{aligned}
T \equiv T_{\xi\xi} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\
\bar{T} \equiv T_{\bar{\xi}\bar{\xi}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\
T_{\bar{\xi}\xi} = T_{\xi\bar{\xi}} &= \frac{1}{4}(T_{00} + T_{11}). \qquad (1.6)
\end{aligned}
$$

Often, especially in the context of string theory, the space direction is no longer $\mathcal{R}$, but rather is compactified on S^1 so that $x^1 \equiv x^1 + 2\pi$. For such a geometry it is convenient to introduce the following conformal map:

$$\xi \to z = e^{\xi} = e^{x^0 + ix^1},$$

which maps the cylinder to the complex plane (see Fig. 1.1).

In particular the past $x^0 = -\infty$ is mapped into the origin and the future $x^0 = \infty$ into a circle with an infinite radius. It is clear that the relations between $(\xi, \bar{\xi})$ and (x^0, x^1) derived above hold also between $(z, \bar{z})$ and $(Real(z), Im(z))$. The holomorphic and anti-holomorphic derivatives with respect to z will be denoted by $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$.

1.2 Free massless scalar field

The action S of the free massless scalar field $\hat{X}(z, \bar{z})$ is

$$
\begin{aligned}
S &= \int d^2x \mathcal{L} = \frac{1}{8\pi} \int d^2x \partial_\nu \hat{X} \bar{\partial}^\nu \hat{X} \\
&= \frac{1}{4\pi} \int d^2\xi \partial_\xi \hat{X} \partial_{\bar{\xi}} \hat{X} = \frac{1}{4\pi} \int d^2z \partial\hat{X} \bar{\partial}\hat{X}, \qquad (1.7)
\end{aligned}
$$

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where $\mathcal{L}$ is the Lagrangian density. The factor $\frac{1}{4\pi}$ is used to match the normalization of the bosonic string theory (with $\alpha' = 2$). In the complex coordinate notation $(\xi, \bar{\xi})$ and $(z, \bar{z})$ the measure of the integral is $\mathrm{d}^2\xi = (i/2)d\xi \wedge \mathrm{d}\bar{\xi}$ and $\mathrm{d}^2 z = (i/2)\mathrm{d}z \wedge \mathrm{d}\bar{z}$, respectively. Note that $\mathcal{L}$ is a local expression and thus is the same for the Euclidean plane or for any compact two-surface.

Varying the scalar field $\hat{X}(z,\bar{z}) \to \hat{X}(z,\bar{z}) + \delta\hat{X}(z,\bar{z})$ induces a variation in the action of the form

$$\delta S = -\frac{1}{2\pi}\int \mathrm{d}^2 z (\partial\bar{\partial}\hat{X})\delta\hat{X}. \tag{1.8}$$

The action is thus extremized by configurations that solve the corresponding equation of motion

$$\partial\bar{\partial}\hat{X} = 0. \tag{1.9}$$

It is thus clear that $\partial\hat{X}$ is a holomorphic function and $\bar{\partial}\hat{X}$ is an anti-holomorphic function, and the most general solution takes the form

$$\hat{X}(z,\bar{z}) = [X(z) + \bar{X}(\bar{z})]. \tag{1.10}$$

1.3 Symmetries of the classical action

By construction the action is invariant under translations and $SO(2)$ rotations. Translations in x^0 and x^1 translate in complex coordinates to

$$z \to z + a; \quad \bar{z} \to \bar{z} + \bar{a}, \tag{1.11}$$

where a is a constant complex number, and the $SO(2)$ rotations, in infinitesimal form, to

$$\delta z = -i\epsilon z; \quad \delta\bar{z} = i\epsilon\bar{z}, \tag{1.12}$$

where ϵ is an infinitesimal real parameter.

When we go back to Minkowski space, the $SO(2)$ rotations turn into $SO(1,1)$ transformations. In addition it is easy to realize that a shift of the field by a constant A,

$$\hat{X}(z,\bar{z}) \to \hat{X}(z,\bar{z}) + A, \tag{1.13}$$

leaves the Lagrangian invariant. It is a special feature of two dimensions that the symmetry group of the action is in fact much richer since one can replace the constant A with $A(z)$ and the constant $\bar{A}$ with $\bar{A}(\bar{z})$, which are arbitrary holomorphic and anti-holomorphic functions, respectively,

$$\hat{X}(z,\bar{z}) \to \hat{X}(z,\bar{z}) + A(z); \quad \hat{X}(z,\bar{z}) \to \hat{X}(z,\bar{z}) + \bar{A}(\bar{z}). \tag{1.14}$$

These are the *affine current algebra transformations.*[3]

[3] Affine Lie algebras describing a physical system were first discussed in [27]. More references will be given in the next two chapters.

In a similar manner the space-time translations (1.11) can also be elevated to holomorphic and anti-holomorphic transformations,

$$z \longrightarrow f(z); \quad \bar{z} \longrightarrow \bar{f}(\bar{z}), \tag{1.15}$$

referred to as two-dimensional *conformal transformations*. Affine current algebra transformations and conformal transformations will be further discussed in Sections 1.10 and 1.11

1.4 Mode expansion

The mode expansion of the classical solution depends on the boundary conditions. For the case where the underlying two-dimensional manifold is the infinite plane, a standard Fourier transform is used:

$$\hat{X}(x^0, x^1) = \int \frac{\mathrm{d}k^1}{\sqrt{2\pi}\sqrt{k^0}} [a(k^1)\mathrm{e}^{-ik\cdot x} + a^\dagger(k^1)\mathrm{e}^{ik\cdot x}]. \tag{1.16}$$

If the range of the space coordinate is bounded, one may impose two types of boundary conditions, associated with closed and open strings. In the case of *closed strings* the boundary conditions

$$\hat{X}(x^0, x^1) = \hat{X}(x^0, x^1 + 2\pi) \tag{1.17}$$

are automatically obeyed by $\hat{X}(z, \bar{z})$. For this case the mode expansion is expressed in terms of a Laurent series,

$$\partial X = -i \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{z^{n+1}} \quad \bar{\partial}\bar{X} = -i \sum_{n=-\infty}^{\infty} \frac{\bar{\alpha}_n}{\bar{z}^{n+1}}. \tag{1.18}$$

Integrating this expansion we get

$$\hat{X}(z, \bar{z}) = \mathcal{X} - i\mathcal{P}\ln(z\bar{z}) + i \sum_{m=-\infty,\ m\neq 0}^{\infty} \left(\frac{\alpha_m}{m} z^{-m} + \frac{\bar{\alpha}_m}{m} \bar{z}^{-m} \right), \tag{1.19}$$

with $\mathcal{X}$ a constant and

$$\mathcal{P} = \alpha_0 = \bar{\alpha}_0. \tag{1.20}$$

For *open strings* the boundary conditions are of Neumann type, namely

$$\partial_1 \hat{X}(x^0, x^1 = 0) = \partial_1 \hat{X}(x^0, x^1 = \pi) = 0 \Longrightarrow \partial\hat{X}(z, \bar{z} = z) = \bar{\partial}\hat{X}(z, \bar{z} = z). \tag{1.21}$$

The corresponding mode expansion takes the form

$$\hat{X}(z, \bar{z}) = \mathcal{X} - i\mathcal{P}\ln(z\bar{z}) + i \sum_{m=-\infty,\ m\neq 0}^{\infty} \frac{\alpha_m}{m} \left(z^{-m} + \bar{z}^{-m} \right). \tag{1.22}$$

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1.5 Noether currents and charges

Associated with the symmetries (1.14) and (1.15) are conserved Noether currents and charges. In the Noether procedure one is instructed to elevate the global parameters of transformations into local ones and extract the associated currents from the variation of the action, namely $\delta S \sim \int d^2 x J_\mu \partial^\mu \epsilon$. Let us apply this procedure first to the affine current algebra transformations so that we vary the action with respect to $\delta \hat{X}(z,\bar{z}) = \epsilon(z,\bar{z})$ yielding

$$\delta S = \frac{1}{4\pi}\int d^2 z[\partial\epsilon(z,\bar{z})\bar{\partial}\hat{X}(z,\bar{z}) + \bar{\partial}\epsilon(z,\bar{z})\partial\hat{X}(z,\bar{z})]. \tag{1.23}$$

Unlike the situation in more than two dimensions, and due to the fact that the symmetries (1.14) are in fact not only global ones but rather "half local", the currents

$$J \equiv \partial X; \quad \bar{J} \equiv \bar{\partial}\bar{X} \tag{1.24}$$

are holomorphic and anti-holomorphic conserved,

$$\bar{\partial}J \equiv \bar{\partial}\partial X = 0; \quad \partial\bar{J} \equiv \partial\bar{\partial}\bar{X} = 0. \tag{1.25}$$

The classical currents are determined up to an overall constant.

A similar situation occurs with respect to the conformal transformation. Replacing in the infinitesimal version of (1.15) $\delta z \to \epsilon(z,\bar{z})$ and $\delta\bar{z} \to \bar{\epsilon}(z,\bar{z})$ one finds,

$$\delta S = \frac{1}{2\pi}\int d^2 z[\partial\bar{\epsilon}(z,\bar{z})\bar{\partial}\hat{X}(z,\bar{z})\bar{\partial}\hat{X}(z,\bar{z}) + \bar{\partial}\epsilon(z,\bar{z})\partial\hat{X}(z,\bar{z})\partial\hat{X}(z,\bar{z})]. \tag{1.26}$$

The associated holomorphic and anti-holomorphic conserved energy-momentum tensor components are

$$T = -\frac{1}{2}\partial X\partial X; \quad \bar{T} = -\frac{1}{2}\bar{\partial}\bar{X}\bar{\partial}\bar{X}, \tag{1.27}$$

where the coefficients were chosen in a way that will turn out to be convenient when discussing the corresponding quantum generators.

1.6 Canonical quantization

Prior to imposing the canonical quantization condition one has to identify the time direction. There are several options. Using x^0 as the time direction, the

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corresponding conjugate momentum of $\hat{X}(z,\bar{z})$ is

$$\Pi = \frac{\delta\mathcal{L}}{\delta x_0 \hat{X}} = \frac{1}{4\pi}\partial_0 \hat{X},$$

and the standard quantization conditions are

$$\begin{aligned}
[\hat{X}(x^0,x^1),\Pi(y^0,y^1)]_{x^0=y^0} &= i\delta(x^1-y^1)\\
[\hat{X}(x^0,x^1),\hat{X}(y^0,y^1)]_{x^0=y^0} &= 0\\
[\Pi(x^0,x^1),\Pi(y^0,y^1)]_{x^0=y^0} &= 0.
\end{aligned} \tag{1.28}$$

These conditions yield the standard algebra of the creation and annihilation operators for (1.16),

$$[a(k^1),a^\dagger(p^1)] = \delta(k^1-p^1); \quad [a(k^1),a(p^1)] = [a^\dagger(k^1),a^\dagger(p^1)] = 0. \tag{1.29}$$

Substituting the mode expansion (1.16) into the expressions of the Noether charges associated with the symmetries of the action (1.7) one finds that the energy-momentum operators are proportional to $a^\dagger(k)a(k) + a(k)a^\dagger(k)$ and hence their vacuum expectation values are proportional to $\delta(0) \sim L$, where L is the size of the space direction. It is thus clear that for the infinite Euclidean plane (or a Minkowski space-time with space $\mathcal{R}$) these expectation values diverge. One then defines the **normal ordered** operators:

$$:\mathcal{O}: \equiv \mathcal{O} - <0|\mathcal{O}|0> . \tag{1.30}$$

For free fields this is equivalent to ordering annihilation operators to the right of creation operators, and sufficient to make $:\mathcal{O}:$ finite.

Using the algebra of the creation and annihilation operators and the normal ordered Hamiltonian, the construction of the Fock space is standard. One defines the vacuum state $|0>$ such that

$$a(k^1)|0> = 0. \tag{1.31}$$

The states in the Fock space are

$$\prod_i a^\dagger(k_i)^{ni}|0>, \tag{1.32}$$

and their energies, by applying the Hamiltonian,

$$H|\prod a^\dagger(k_i)^{ni}|0> = \sum_i (k_j^0)n_i(k_i)\prod a^\dagger(k_i)^{ni}|0> . \tag{1.33}$$

The canonical quantization for the scalar field on a compact space direction, with the boundary conditions of open or closed string, (1.21) and (1.17), respectively, follows very similar steps. Imposing the quantization conditions (1.28) above implies the following algebra for the α_n operators of the open string and for the α_n and $\bar{\alpha}_n$ operators for the closed string:

$$\begin{aligned}
[\alpha_m, \alpha_n] &= m\delta_{m+n} \\
[\bar{\alpha}_m, \bar{\alpha}_n] &= m\delta_{m+n} \\
[\alpha_m, \bar{\alpha}_n] &= 0.
\end{aligned} \tag{1.34}$$

It is thus clear that α_n operators are related to the $a(k)$ operators as

$$\alpha_m = \sqrt{m}a(m), m > 0; \quad \alpha_{-m} = \sqrt{m}a^\dagger(m), m > 0. \tag{1.35}$$

1.7 Radial quantization

For the case of a cylinder-like two-dimensional manifold, namely, where the space direction is compactified so that $x^1 \equiv x^1 + 2\pi$, it is natural to use the $z = \mathrm{e}^{x^0 + ix^1}$ coordinates. Space translations $x^1 \to x^1 + a$ take the form of multiplying by a phase factor $z \to \mathrm{e}^{ia}z$, and time translations $x^0 \to x^0 + a$ turn into dilatations $z \to \mathrm{e}^a z$. Rotations $(x^0 + ix^1) \to (c + is)(x^0 + ix^1)$, go into $z \to z^{(c+is)}$, with $(c + is) = \mathrm{e}^{i\theta}$, θ the rotation angle. Correspondingly the generators of these transformations change their geometrical operation. For instance the Hamiltonian obviously goes into the dilatation generator. Moreover, generators which are Noether charges transform into contour integrals. Recall that the Noether charge is $Q = \int \mathrm{d}x^1 J_0(x^1)$ which in the new coordinates reads $Q = \int \mathrm{d}\theta J_r(\theta)$ so that we can write,

$$Q = \frac{1}{2\pi i} \oint [\mathrm{d}z J(z) + \mathrm{d}\bar{z}\bar{J}(\bar{z})], \tag{1.36}$$

where the contour integral is performed at some radius and the sign convention we adopt is that both the $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ integral are taken to be positive for the counter-clockwise sense.

The infinitesimal transformation of an operator generated by the Noether charge Q is given by:

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{O} = \frac{1}{2\pi i} \oint [\mathrm{d}z J(z)\epsilon(z), \mathcal{O}(w, \bar{w})] + \mathrm{d}\bar{z}[\bar{J}(\bar{z})\bar{\epsilon}(\bar{z}), \mathcal{O}(w, \bar{w})]. \tag{1.37}$$

Define a product R of two operators $A(z)B(w)$ as taken radially, namely[4]

$$R(A(z)B(w)) = A(z)B(w), |z| > |w|; \quad B(w)A(z), |w| > |z|. \tag{1.38}$$

In Fig. 1.2 we show the two contour integrals that lead to a contour integral around w, the location of the operator $\mathcal{O}$, so that the infinitesimal transformation is given by,

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{O} = \frac{1}{2\pi i} \oint [\mathrm{d}z\epsilon(z)R(J(z)\mathcal{O}(w, \bar{w})) + \mathrm{d}\bar{z}\bar{\epsilon}(\bar{z})R(J(\bar{z})\mathcal{O}(w, \bar{w}))]. \tag{1.39}$$

[4] The notion of radial quantization was introduced in [104]. This construction was used in the context of complex geometry in [93].

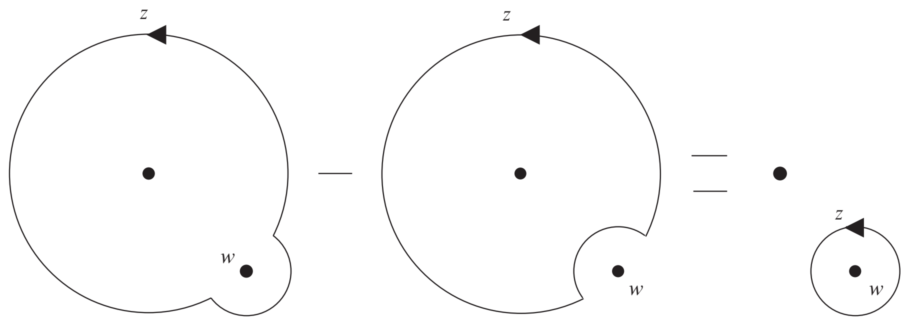

Fig. 1.2. A contour around w from the commutator.

We now apply this formulation to the symmetry generators (discussed in Section 2.1):

(i) The infinitesimal affine current algebra transformation $\hat{X}(z,\bar{z}) \to \hat{X}(z,\bar{z}) - \epsilon(z)$ is generated by the holomorphic current $J(z) = \partial X$ via

$$\begin{aligned} \delta_\epsilon \hat{X}(w,\bar{w}) &= \frac{1}{2\pi i}\oint \mathrm{d}z\epsilon(z) R(\partial X(z)\hat{X}(w,\bar{w})) \\ &= \frac{1}{2\pi i}\oint \mathrm{d}z \frac{-1}{z-w}\epsilon(z) = -\epsilon(w), \end{aligned} \tag{1.40}$$

where we have used for the product of operators,

$$R(X(z)X(w)) = -\log(z-w) + \text{finite terms}. \tag{1.41}$$

This is an example of the concept of operator product expansion, which is addressed in the next section.

(ii) In a similar manner we can compute the transformation of ∂X generated by the energy momentum tensor T

$$\begin{aligned} \delta_\epsilon \partial X(w) &= \frac{1}{2\pi i}\oint \mathrm{d}z\epsilon(z) R\left(-\frac{1}{2} : \partial X(z)\partial X(z) : \partial X(w)\right) \\ &= \frac{1}{2\pi i}\oint \mathrm{d}z \frac{1}{(z-w)^2}\partial X(z)\epsilon(z) = \partial\epsilon(w)\partial X(w) + \epsilon(w)\partial^2 X(w), \end{aligned} \tag{1.42}$$

which is indeed the infinitesimal transformation of the holomorphic current $J = \partial X(z)$. The generator T is normal ordered using the following expression:

$$T(w) = -\frac{1}{2} : \partial X(z)\partial X(w) :\equiv -\frac{1}{2}\lim_{z\to w}\left[\partial X(z)\partial X(w) + \frac{1}{(z-w)^2}\right]. \tag{1.43}$$

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1.8 Operator product expansion

In computing the contour integrals associated with infinitesimal transformations we have made use of the operator product expansions of pairs of operators.[5] The singularities that occur when the points are taken to approach one another are captured in the notion of *operator product expansion (OPE)*,

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k c_{ij}^k(x-y)\mathcal{O}_k(y), \tag{1.44}$$

where $c_{ij}^k(x-y)$ are the *coefficient functions* which are singular in the limit of $x \to y$. Such expansions were proven to hold in renormalizable field theories. The OPEs are an essential tool in exploring quantum field theories. Recall that all of the information on the QFT is encoded into the values of all possible correlation functions of the complete set of local operators $\mathcal{O}_i(x)$, namely, $< \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n) >$. In particular, one is interested in the behavior of these correlation functions when two or more points approach each other, which is encapsulated in the OPEs. For all applications discussed here the OPEs are treated as asymptotic expansions and only their singular terms will be specified. For the present case of two-dimensional free massless scalar field theory the OPE converges and in fact, as will be discussed in Section 3.7.2, a similar situation occurs in all 2d CFTs.

The OPEs of the free massless scalar can be deduced from its propagator, which can be evaluated from the solution. It takes the form:

$$< \hat{X}(z\bar{z})\hat{X}(w\bar{w}) >= -log|z-w|^2. \tag{1.45}$$

In terms of the separation of the solution into holomorphic and anti-holomorphic parts the two propagators read:

$$< X(z)X(w) >= -\log(z-w); \quad < \bar{X}(\bar{z})\bar{X}(\bar{w}) >= -\log(\bar{z}-\bar{w}). \tag{1.46}$$

By differentiating the last relation with respect to z and to w one finds the short distance expansion of other operators like $J(z)$, $T(z)$ etc. In particular the OPE of the currents is

$$J(z)J(w) = \partial X(z)\partial X(w) = -\frac{1}{(z-w)^2} + \text{finite terms}, \tag{1.47}$$

with a similar result for the anti-holomorphic currents.

A different, though equivalent, approach is to write the OPE as a Taylor expansion in $(z-w)$ and $(\bar{z}-\bar{w})$ in the following form:

$$\hat{X}(z,\bar{z})\hat{X}(w\bar{w}) = -log|z-w|^2 + \sum_{k=1}^{\infty}\frac{1}{k!}[(z-w)^k : (\partial^k\hat{X}(w,\bar{w}))\hat{X}(w,\bar{w}) : \\ +(\bar{z}-\bar{w})^k : (\bar{\partial}^k\hat{X}(w,\bar{w}))\hat{X}(w,\bar{w}) :]. \tag{1.48}$$

[5] Wilson introduced for the first time the concept of an operator product expansion [219]. It was used for two-dimensional conformal field theories in [33].

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This form of expansion is based on the property that the normal ordered product of the scalar fields,

$$: \hat{X}(z,\bar{z})\hat{X}(w\bar{w}) := \hat{X}(z,\bar{z})\hat{X}(w,\bar{w}) + log|z-w|^2, \tag{1.49}$$

obeys the equation of motion, namely,

$$\partial\bar{\partial} : \hat{X}(z,\bar{z})\hat{X}(w,\bar{w}) := 0, \tag{1.50}$$

and hence can be decomposed to holomorphic and anti-holomorphic functions and thus is non singular.

In the previous subsection we used two OPEs to determine the symmetry transformation of X and ∂X. We will work out now two additional examples of OPEs, involving the operator which will later be found to be very useful $: \mathrm{e}^{i\alpha X(w)} :$.

(i) The conformal properties of $: \mathrm{e}^{i\alpha X(w)}$ are being determined by its OPE with $T(z)$ which takes the form

$$\begin{aligned} T(z) : \mathrm{e}^{i\alpha X(w)} : &= -\frac{1}{2}(: \partial X(z)\partial X(z) :)(: \mathrm{e}^{i\alpha X(w)} :) \\ &= \frac{(\frac{\alpha^2}{2})}{(z-w)^2}\mathrm{e}^{i\alpha X(w)} + \frac{1}{(z-w)}\partial \mathrm{e}^{i\alpha X(w)}. \end{aligned} \tag{1.51}$$

In language that will be developed in Section 2.2 this result will mean that $: \mathrm{e}^{i\alpha X(w)} :$ has a conformal dimension of $\frac{\alpha^2}{2}$.

(ii) The OPE of two operators of the form $: \mathrm{e}^{i\alpha X(w)}$ is

$$(: \mathrm{e}^{i\alpha X(z)} :)(: \mathrm{e}^{-i\beta X(w)} :) = \frac{: \mathrm{e}^{i\alpha X(w)}\mathrm{e}^{-i\beta X(w)} :}{(z-w)^{\alpha\beta}}. \tag{1.52}$$

1.9 Path integral quantization

So far we have been using canonical quantization. Before proceeding to the general structure of affine current algebra and Virasoro algebra we introduce the quantization of a free massless scalar field using the Euclidean path integral approach. As usual the functional integration $D\hat{X}(z,\bar{z})$ can be approximated by discretizing the two-dimensional space and representing the functional integral by products of ordinary integrals. Expectation values of operators $\mathcal{O}(X)$ constructed from X are given by,

$$< \mathcal{O}(\hat{X}) >= \int D\hat{X}(z,\bar{z})\mathcal{O}(\hat{X})\mathrm{e}^{-S} = \int D\hat{X}(z,\bar{z})\mathcal{O}(\hat{X})\mathrm{e}^{-\frac{1}{2\pi}\int \mathrm{d}^2 z\partial\hat{X}\bar{\partial}\hat{X}}. \tag{1.53}$$

Correlation functions have to obey the equation

$$\partial\bar{\partial} < \hat{X}(z,\bar{z})\hat{X}(w,\bar{w}) >= -2\pi\delta^2(z-w,\bar{z}-\bar{w}), \tag{1.54}$$

as can be deduced by using the fact that the path integral of a total derivative vanishes:

$$\begin{aligned} 0 &= \int D\hat{X}(z,\bar{z}) \frac{\delta}{\delta \hat{X}(z,\bar{z})} [\mathrm{e}^{-S} \hat{X}(w,\bar{w})] \\ &= \int D\hat{X}(z,\bar{z}) \mathrm{e}^{-S} \left[-\frac{\delta S}{\delta \hat{X}(z,\bar{z})} \hat{X}(w,\bar{w}) + \delta^2(z-w,\bar{z}-\bar{w}) \right] \\ &= \frac{1}{2\pi} < \partial\bar{\partial}\hat{X}(z,\bar{z})\hat{X}(w,\bar{w}) > + < \delta^2(z-w,\bar{z}-\bar{w}) > . \end{aligned} \tag{1.55}$$

Alternatively one can use (1.45) and (1.46) directly. Note that in that case care must be exercised, as naively we would get zero rather than the delta function, since the expression is a sum of two terms, one depending on z only and the other on $\bar{z}$ only. The point is that the expressions (1.46) cannot be taken over at the origin. A working rule is:

$$\bar{\partial}\left(\frac{1}{z}\right) = \partial\left(\frac{1}{\bar{z}}\right) = (2\pi)\delta^2(z,\bar{z})$$

This can be derived by going over from $\frac{1}{z}$ to $\frac{\bar{z}}{z\bar{z}+\epsilon^2}$, to regulate the singularity at the origin.

1.10 Affine current algebra

As was shown in Section 1.3 the classical action is invariant under both affine current algebra transformations and conformal transformations. We would like to study the algebraic structure of the generators of these symmetries. We start with the invariance under affine current algebra transformations. Recall that the corresponding generators are the holomorphic and anti-holomorphic currents J and $\bar{J}$, given in (1.24). Expanding the currents in Laurant series,

$$J = \sum_{n=-\infty}^{\infty} J_n z^{-(n+1)}; \quad \bar{J} = \sum_{n=-\infty}^{\infty} \bar{J}_n \bar{z}^{-(n+1)}, \tag{1.56}$$

it is obvious from the mode expansion (1.18) that the algebra of the currents is related to that of the α_n operators, namely

$$[J_m, J_n] = -m\delta_{m+n}; \quad [\bar{J}_m, \bar{J}_n] = -m\delta_{m+n}. \tag{1.57}$$

This form of algebra will be shown in the next chapter to be associated with level $k = 1$ abelian affine current algebra (or Kac–Moody algebra as it is sometimes referred to).

This algebra translates into the following algebra of the currents:

$$[J(z), J(w)] = \delta'(z-w). \tag{1.58}$$

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Using the technique developed in Section 1.7 we can derive this result also from the operator product expansion of two currents,

$$J(z)J(w) = \frac{1}{(z-w)^2} + \text{finite terms}. \tag{1.59}$$

1.11 Virasoro algebra

Next we address the algebraic structure of the generators of conformal transformations (1.27). Upon inserting (1.18) into the Laurent expansion of the energy momentum tensor,

$$T = \sum_{n=-\infty}^{\infty} L_n z^{-(n+2)}; \quad \bar{T} = \sum_{n=-\infty}^{\infty} \bar{L}_n \bar{z}^{-(n+2)}, \tag{1.60}$$

one finds that for L_n with $n \neq 0$,

$$L_n = 1/2 \sum_{m=-\infty}^{\infty} : \alpha_{n-m}\alpha_m : . \tag{1.61}$$

For $n \neq 0$ the operators α_{n-m} and α_m commute, and so the product equals the normal ordered one. The situation is different for L_0. Here one encounters an infinity in the product of chiral fields, which normal ordering removes, resulting in,

$$L_0 = 1/2\mathcal{P}^2 + \sum_{1}^{\infty} \alpha_{-m}\alpha_m . \tag{1.62}$$

We shall later see that it is sometimes necessary to shift L_0 by a constant. Using the commutation relation of α_n one finds the following "naive" expression for the commutator of L_n operators:

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{4}\sum_{k,l}[\alpha_{m-k}\alpha_k, \alpha_{n-l}\alpha_l] \\
&= \frac{1}{2}\sum_k k\alpha_{m-k}\alpha_{k+n} + \frac{1}{2}\sum_k (m-k)\alpha_{m-k+n}\alpha_k \\
&= (m-n)L_{m+n},
\end{aligned} \tag{1.63}$$

where to get to the third line we have changed a variable in the first sum from $k \to k-n$. This is the *classical Virasoro algebra*,[6] and in fact in the quantum theory it is further corrected. The correction appears only for the case $m+n=0$, so for $m \neq -n$ the classical form (1.63) is exact. For generators with $m+n=0$ the two sums in the second line of (1.63) have to be brought to normal order. As re-ordering means using the commutator, one gets divergent series for which, in the case at hand, one cannot shift the variable of summation without changing

[6] The Virasoro algebra was presented in [212]. More references will be given in the next two chapters.

the result. Taking this into account, one gets a c-number shift in the commutation rule,

$$[L_m, L_n] = (m-n)L_{m+n} + \mathcal{A}(m)\delta_{m+n}. \tag{1.64}$$

To compute the anomaly term $\mathcal{A}(m)$ we introduce a cutoff function $f_\Lambda(k)$, which tends to 1 in the limit of infinite regulator Λ for any k, but for every finite Λ goes to zero sufficiently rapidly at infinite k. Thus we view the operators L_n as regularized sums,

$$L_n = 1/2 \sum_{m=-\infty}^{\infty} : \alpha_{n-m}\alpha_m : f_\Lambda(m), \tag{1.65}$$

to replace (1.61). With this regularized expression, a direct computation gives for the anomaly,

$$\begin{aligned}\mathcal{A}(m) = 1/4 \sum_{k=1}^{\infty} \{ & k(m-k) f_\Lambda(m-k)[f_\Lambda(k-m) + f_\Lambda(-k)] \\ & + k(m+k) f_\Lambda(-k)[f_\Lambda(k) + f_\Lambda(-m-k)]\}.\end{aligned} \tag{1.66}$$

If we now take $f_\Lambda(k)$ to 1, without being careful, we get the divergent sum,

$$\mathcal{A}(m) \to m \sum_{k=1}^{\infty} k. \tag{1.67}$$

Using ζ-function regularization, namely replacing k by k^{-s}, we get a convergent sum for any $s > 1$, and then we continue analytically to $s = -1$, to get $-m/12$ for the right-hand side of the last equation.

To compute $\mathcal{A}(m)$ with the regulators f_Λ, we now look at,

$$\begin{aligned}\mathcal{A}(m) + \frac{m}{12} = 1/4 \sum_{k=1}^{\infty} \{ & k(m-k) f_\Lambda(m-k)[f_\Lambda(k-m) + f_\Lambda(-k)] \\ & + k(m+k) f_\Lambda(-k)[f_\Lambda(k) + f_\Lambda(-m-k)] - 4mk\}.\end{aligned} \tag{1.68}$$

Only large k is relevant now, as for any finite k we can take Λ to infinity first, obtaining zero on the right-hand side. We now take,

$$f_\Lambda(q) \approx |q|^{-p} \quad |q| \to \infty, \tag{1.69}$$

with $p \to 0$ as $\Lambda \to \infty$. Expanding in powers of $\frac{m}{k}$, and recalling that $\zeta(s)$ has a pole only at $s = 1$, we get by summing first and then letting $p \to 0$, the result,

$$\mathcal{A}(m) + \frac{m}{12} = \frac{m^3}{12}. \tag{1.70}$$

The anomaly term $\mathcal{A}(m)$ can also be determined using the Jacobi identity $[L_k, [L_m, L_n]] + [L_m, [L_n, L_k] + [L_n, [L_k, L_m] = 0$. One finds that for $k + m + n = 0$ the anomaly term obeys $(m-n)\mathcal{A}(k) + (n-k)\mathcal{A}(m) + (k-n)\mathcal{A}(m) = 0$. Recall also that $\mathcal{A}(0) = 0$ and $\mathcal{A}(m) = -\mathcal{A}(-m)$ so it is enough to determine $\mathcal{A}(m)$ for positive m. The relation derived from the Jacobi identity can be used to get a recursion relation which is determined by values of $\mathcal{A}(1)$ and $\mathcal{A}(2)$. In fact the general solution is of the form $\mathcal{A}(n) = b_3 n^3 + b_1 n$. The coefficient b_1 is correlated with the normal ordering ambiguity constant of L_0. One can determine the coefficients b_1 and b_3 by computing the vacuum expectation values of $<0|[L_1, L_{-1}]|0> = 0$ and $<0|[L_2, L_{-2}]|0> = \frac{1}{2}$, so that altogether one finds $\mathcal{A}(n) = \frac{1}{12}n(n^2 - 1)$ and the full Virasoro algebra associated with the massless free scalar field is,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m+n}. \tag{1.71}$$

In the next chapter the Virasoro algebra will be discussed in a broader perspective. In that context it will become clear that the algebra of (1.71) associated with the massless free scalar is characterized by a $c = 1$ Virasoro anomaly.