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THE LAPLACE TRANSFORM OF HITTING TIMES OF INTEGRATED GEOMETRIC BROWNIAN MOTION

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Abstract

In this note we compute the Laplace transform of hitting times, to fixed levels, of integrated geometric Brownian motion. The transform is expressed in terms of the gamma and confluent hypergeometric functions. Using a simple Itô transformation and standard results on hitting times of diffusion processes, the transform is characterized as the solution to a linear second-order ordinary differential equation which, modulo a change of variables, is equivalent to Kummer's equation.

Keywords: Integrated geometric Brownian motion; confluent hypergeometric function; hitting time; Laplace transform

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1. Introduction

The modest goal of this note is to compute the Laplace transform of the hitting time

$$\tau_a := \inf\{t \geq 0 \colon A_t = a\}, \tag{1.1}$$

where A_t is integrated geometric Brownian motion (IGBM). More precisely, $A_t = \int_0^t V_s \, \mathrm{d}s$, where V_t is a solution to $\mathrm{d}V_t = \mu V_t \, \mathrm{d}t + \sigma V_t \, \mathrm{d}W_t$. Here $\mu \in \mathbb{R}$, $\sigma > 0$, and W is standard Brownian motion.

Though much is known about IGBM (see [4] or [9] and [10] for excellent summaries, as well as [2], [3], or [5] for applications in finance), it does not appear that a closed-form expression for the function

$$u(a, v, \alpha, \sigma, \mu) := \mathbb{E}_v[\mathrm{e}^{-\alpha\tau_a}] \tag{1.2}$$

is explicitly available in the current literature, nor have we been able to locate such an expression in the compendium of [1]. It should be noted here that Kyprianou and Pistorius [6] derived the transform for hitting times of the related process $V_t^{-1}(A_t + k)$ for constant $k > 0$, using fluctuation results for Bessel processes.

The remainder of this note is dedicated to proving Theorem 1.1 below, which provides a closed-form expression for the function defined in (1.2). In contrast to much of the literature on IGBM our proof does not use Bessel processes. As discussed briefly in Section 3, this allows for a more concise and streamlined argument.

Before stating our main result we clarify notation. In (1.2) the domain of u is taken to be $\mathbb{R}_+^4 \times \mathbb{R}$, where $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{E}_v$ denotes the expectation conditioned upon $V_0 = v$.

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We denote the confluent hypergeometric function by $M(\lambda, \theta, z)$, recalling that it is defined as (the series converging for all $z \in \mathbb{C}$ and noninteger θ)

$$M(\lambda, \theta, z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\theta)_n} \frac{z^n}{n!},$$

where, for real x and integer $n \geq 0$, we define $(x)_n = \Gamma(x+n)/\Gamma(x)$ with Γ denoting the gamma function. The reader unfamiliar with the function M is encouraged to consult either [12], where it is denoted by ${}_1F_1$, or [7], where it is denoted by Φ.

Our main result is the following.

Theorem 1.1. *The function u defined in (1.2) is given by*

$$u(a, v, \alpha, \sigma, \mu) = \left(\frac{2v}{a\sigma^2}\right)^{\gamma} \frac{\Gamma(\gamma + 2\mu/\sigma^2)}{\Gamma(2\gamma + 2\mu/\sigma^2)} M\left(\gamma, 2\gamma + \frac{2\mu}{\sigma^2}, -\frac{2v}{a\sigma^2}\right),$$

where γ is the unique positive root of

$$\xi^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)\xi - \frac{2\alpha}{\sigma^2}, \qquad \xi \in \mathbb{R}.$$

Our proof of this theorem is quite straightforward and we discovered it through a need to understand the absorption (at the origin) time of the process

$$V_t\left[1 - c\int_0^t V_s^{-1}\,\mathrm{d}s\right],$$

which arises quite naturally in structural models of credit risk. The fact that V_t^{-1} remains a geometric Brownian motion motivates the transformation which begins our proof.

2. Proof of Theorem 1.1

To begin, we define $X_t = V_t^{-1}[1 - a^{-1}A_t]$. Using Itô's lemma, we find that X_t is a solution to

$$\mathrm{d}X_t = [(\sigma^2 - \mu)X_t - a^{-1}]\,\mathrm{d}t + \sigma X_t\,\mathrm{d}\bar{W}_t,$$

with initial condition $X_0 = V_0^{-1}$ and where $\bar{W} = -W$. Moreover, X_t strikes 0 at precisely the same moment as A_t strikes a. Hence, if we define

$$\tau := \inf\{t > 0 \colon X_t = 0\}$$

then $\tau = \tau_a$, where τ_a is given by (1.1) and is the hitting time of ultimate interest. We are thus led to consider hitting times of diffusion processes, in particular solutions to the stochastic differential equation

$$\mathrm{d}X_t = (\bar{\mu}X_t - \bar{c})\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \tag{2.1}$$

where $\bar{\mu} \in \mathbb{R}$ and $\bar{c} > 0$. The present problem corresponds to $\bar{c} = a^{-1}$ and $\bar{\mu} = \sigma^2 - \mu$.

Diffusion with drift and volatility specified in (2.1) is regular on $\mathbb{R}_+$ (but not necessarily on $\mathbb{R}$—solutions which begin in $\mathbb{R}_+$ can enter $\mathbb{R}_- = (-\infty, 0)$ and never return). Moreover, it is straightforward to show that, for solutions in $\mathbb{R}_+$, ∞ is a natural boundary while the origin is exit-not-entrance, in the sense of Section II.1.6 of [1]. It is prudent to note that Lewis [8] studied the pricing of option assets whose underlying dynamics are given by (2.1) and derived a closed-form (but ultimately unwieldy) expression for the cumulative distribution function of τ.

For fixed $\alpha \in \mathbb{R}_+$, define the function $u\colon \mathbb{R}_+ \mapsto [0, 1]$ by

$$u(x) = \mathbb{E}_x[\mathrm{e}^{-\alpha\tau}].$$

Regularity on $\mathbb{R}_+$ ensures that u will solve the second-order equation

$$\frac{\sigma^2 x^2}{2} u''(x) + (\bar{\mu}x - \bar{c})u'(x) - \alpha u(x) = 0, \qquad x \in (0, \infty). \tag{2.2}$$

Since the origin is exit-not-entrance, we have $u(0) = 1$, and since ∞ is natural, we have $u(\infty^-) = 0$, where $u(\infty^-) = \lim_{x\to\infty} u(x)$; see [1] for more details. Note that the origin is an irregular singular point for (2.2).

In order to solve (2.2), we bring it to Kummer's equation—note that Lewis [8] used a similar approach to solve a partial differential equation whose spatial component is similar to the left-hand side of (2.2). To this end, we use the change of variable $z = -2\bar{c}/\sigma^2 x$ to put (2.2) in standard form and turn the origin into a regular singular point. Indeed, setting $g(z) = u(x)$ leads to the equation

$$z^2 g'' + \left[2\left(1 - \frac{\bar{\mu}}{\sigma^2}\right)z - z^2\right]g' - \frac{2\alpha}{\sigma^2} g = 0, \qquad z \in (-\infty, 0). \tag{2.3}$$

If $g(z)$ is a solution to (2.3) subject to $g(0) = 0$ and $g(-\infty^+) = 1$, then $u(x) = g(-2\bar{c}/\sigma^2 x)$ yields the desired solution to (2.2). Note also that moving to the negative half-line, instead of the positive, simplifies the ensuing algebra.

In order to solve (2.3), we move to the complex plane and seek a function g which solves (2.3) for all $z \in \mathbb{C}$. When translated to the complex plane, our boundary conditions become $g(z) \to 0$ as $z \to 0$ and $g(z) \to 1$ as $|z| \to \infty$ along the negative real axis. As the origin is a regular singular point for (2.3), we are motivated to write g in the form $g(z) = z^\gamma f(z)$ for some constant γ. A routine calculation then shows that f solves the equation

$$z^2 f'' + z[\beta - z]f' + \left[\gamma^2 + \left(1 - \frac{2\bar{\mu}}{\sigma^2}\right)\gamma - \frac{2\alpha}{\sigma^2} - \gamma z\right]f = 0,$$

where $\beta = 2(\gamma + 1) - 2\bar{\mu}/\sigma^2$. If γ is taken to be a root of the quadratic

$$h(\xi) = \xi^2 + \left(1 - \frac{2\bar{\mu}}{\sigma^2}\right)\xi - \frac{2\alpha}{\sigma^2}, \qquad \xi \in \mathbb{R}, \tag{2.4}$$

then this clearly reduces to Kummer's equation:

$$zf''(z) + (\beta - z)f'(z) - \gamma f(z) = 0, \qquad z \in \mathbb{C}. \tag{2.5}$$

It is worth pausing here to briefly consider (2.4). Since $h(0) < 0$ (and the coefficient on ξ^2 is positive), (2.4) will always have two distinct real roots of opposite sign. If γ is taken to be the positive root then, since $h(2\bar{\mu}/\sigma^2 - 1) < 0$, we find that $2\bar{\mu}/\sigma^2 - 1 < \gamma$; hence, $\gamma - \beta + 1 < 0$. In what follows we take γ to be the positive root, noting that we would arrive at the same ultimate result were we to take γ to be the negative root.

A general solution to (2.5) can be expressed (see Section 9.10 of [7] or Chapter 1 of [12]) in terms of the confluent hypergeometric function M, and we may write

$$f(z) = AM(\gamma, \beta, z) + Bz^{1-\beta} M(\gamma - \beta + 1, 2 - \beta, z)$$

for constants A and B. Thus, g has the general form

$$g(z) = Az^{\gamma} M(\gamma, \beta, z) + Bz^{\gamma-\beta+1} M(\gamma - \beta + 1, 2 - \beta, z). \tag{2.6}$$

Since $M(\lambda, \theta, 0) = 1$ and $\gamma - \beta + 1 < 0 < \gamma$, the first term in (2.6) remains bounded as $z \to 0$ while the second term does not. Thus, in order to enforce the condition $g(0) = 0$, we must set $B = 0$.

In order to determine A, we use an asymptotic relation found in Section 4.1.1 of [12], namely,

$$M(\gamma, \beta, z) \sim \frac{(-z)^{-\gamma}\Gamma(\beta)}{\Gamma(\beta - \gamma)} \quad \text{as } |z| \to \infty,$$

which is valid for $\arg(z) \in (\pi/2, \pi]$. Thus, $g(z) \sim A(-1)^{-\gamma}\Gamma(\beta)/\Gamma(\beta - \gamma)$ as $|z| \to \infty$ along the negative real axis, and in order to enforce our second boundary condition, we must choose $A = (-1)^{\gamma}\Gamma(\beta - \gamma)/\Gamma(\beta)$.

The desired solution to (2.3) is thus given by

$$g(z) = (-z)^{\gamma} \frac{\Gamma(\beta - \gamma)}{\Gamma(\beta)} M(\gamma, \beta, z),$$

and since $u(x) = g(-2\bar{c}/\sigma^2 x)$ yields the desired solution to (2.2), we find that

$$u(x) = \left(\frac{2\bar{c}}{\sigma^2 x}\right)^{\gamma} \frac{\Gamma(\beta - \gamma)}{\Gamma(\beta)} M\left(\gamma, \beta, -\frac{2\bar{c}}{\sigma^2 x}\right).$$

Theorem 1.1 follows upon substituting the relations

$$x = \frac{1}{v}, \qquad \bar{c} = \frac{1}{a}, \qquad \bar{\mu} = \sigma^2 - \mu.$$

3. A brief note on the use of Bessel processes

As indicated in the introduction, Bessel processes are a common tool in the analysis of IGBM. The key to this approach is the Lamperti relation (see Exercise 1.28 of [11, Chapter XI]), which provides an auxiliary Bessel process of index $\nu = \mu - \sigma^2/2$ that can be used to learn about the IGBM of interest. A common difficulty with the Bessel-based approach is that arguments constructed for $\nu \geq 0$ frequently do not carry over to the case $\nu < 0$ in a straightforward way (if at all). See [2] for a lengthy and insightful discussion, noting in the interest of prudence that suitable modifications of pertinent arguments from that paper (Sections 12 and 13 in particular) could presumably be used to derive the transform of interest here. The underlying reason for this dichotomy is that, in terms of its behaviour near the origin, the Bessel process of nonnegative index is a fundamentally different process to that of negative index.

In light of this phenomenon it is natural to ask (i) what, if anything, happens to the auxiliary process used in this paper as ν varies and (ii) why this does not influence the argument of Section 2. Recall that the auxiliary processes used here are solutions to (2.1) on $\mathbb{R}^+$ with

$\bar{\mu} = \sigma^2 - \mu \in \mathbb{R}$ and $\bar{c} = a^{-1} > 0$. Note that $\nu \geq 0$ if and only if $\bar{\mu} \geq \sigma^2/2$ and that the restriction $\bar{c} > 0$ is crucial for the subsequent discussion.

The argument given in this paper is based on a differential equation (DE) related to the auxiliary process. The DE itself is determined by the generator of the process, while the boundary conditions for the DE are determined by the behaviour of the diffusion near these boundaries, in particular the nature of the boundaries as determined by the classification scheme in Section II.1.6 of [1]. These classifications are invariant with respect to the specific values of the underlying parameters. Under no circumstances can ∞ be reached in finite time (though it can be approached asymptotically as $t \to \infty$ when $\bar{\mu} > \sigma^2/2$, or, equivalently, $\nu < 0$) and under all circumstances, the origin can be reached in finite time (this occurs almost surely when $\bar{\mu} \leq \sigma^2/2$, or, equivalently, $\nu \geq 0$, and with positive probability otherwise). It is these facts which together dictate the boundary conditions for the DE of interest, and their stability with respect to the underlying parameters ensures that a single argument holds for all values of the parameters.

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