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STARLIKE UNIVALENT FUNCTIONS BOUNDED ON THE REAL AXIS

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Introduction. We denote by E the open unit disc in **C** and by $H(E)$ the class of all analytic functions f on E with $f(0) = 0$. Let (see [**3**] for more complete definitions)

$$S = \{f \in H(E) | f \text{ is univalent on } E\}$$

$$S_0 = \{f \in H(E) | f \text{ is starlike univalent on } E\}$$

$$T_{\mathbf{R}} = \{f \in H(E) | f \text{ is typically real on } E\}.$$

The uniform norm on $(-1, 1)$ of a function $f \in H(E)$ is defined by

$$|f| = \sup_{-1<x<1} |f(x)|.$$

Our first goal in this paper is to obtain various estimates (in terms of $|f|$) for the functions f in S_0. Our main motivation is the following results recently obtained by Rahman and Ruscheweyh [**4**]:

THEOREM A. *Let*

$$f(z) \equiv \sum_{n=1}^{\infty} a_n z^n \in T_{\mathbf{R}}$$

with $|f| < \infty$. *Then,*

1° *f is subordinate to* $\dfrac{2|f|z}{1 + z^2}$;

2° $|a_n| \leqq 2|f|$ *for* $n = 1, 2, \ldots$;

3° $|f(x)| \leqq \dfrac{2|f|\,|x|}{1 + x^2}$ *for* $x \in (-1, 1)$.

We shall prove:

THEOREM 1. *Let $f \in S_0$ be an odd function with $|f| < \infty$. Then, for some $\theta \in [0, 2\pi)$, f is subordinate to*

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$$\frac{2|f|e^{i\theta}z}{1 + z^2}.$$

Theorem 1 admits several corollaries. We obtain:

COROLLARY 1.1. *Let* $f \in S_0$ *and* $f'(0) = 1$. *Then* $|f| \geqq 1/2$ *and equality is possible if and only if*

$$f(z) \equiv \frac{z}{(1 + iz)^{2\lambda}(1 - iz)^{2(1-\lambda)}} \quad \text{where } \lambda \in [0, 1].$$

COROLLARY 1.2. *Let*

$$f(z) \equiv \sum_{n=1}^{\infty} a_n z^n \in S_0$$

with $|f| < \infty$. *Then* $|a_n| \leqq 2|f|n$ *for* $n = 1, 2, \dots$. *If* $n = 1$, *equality is possible if and only if*

$$f(z) \equiv \frac{\alpha z}{(1 + iz)^{2\lambda}(1 - iz)^{2(1-\lambda)}}$$

where $\alpha \in \mathbf{C}$ *and* $\lambda \in [0, 1]$. *If* $n > 1$, *equality is possible if and only if*

$$f(z) \equiv \frac{\alpha z}{(1 \pm iz)^2}.$$

COROLLARY 1.3. *Let*

$$f(z) \equiv \sum_{n=1}^{\infty} a_{2n-1} z^{2n-1} \in S_0$$

with $|f| < \infty$. *Then* $|a_{2n-1}| \leqq 2|f|$ *for* $n = 1, 2, \dots$. *Equality is possible for some* $n \geqq 1$ *if and only if*

$$f(z) \equiv \frac{\alpha z}{1 + z^2} \quad \text{where } \alpha \in \mathbf{C}.$$

If $M > 0$ is a given positive number, there is no estimate of the type

$$|f(x)| \leqq M|f|\,|x|, \quad x \in (-1, 1)$$

valid over the whole class $H(E)$; a simple counter-example is given by the Chebyshev polynomial $T_n(z)$ of the first kind and degree n, with odd n, for which $|T_n| = 1$, $T_n(0) = 0$ and $|T_n'(0)| = n$. We will however show that the result 3° of Rahman and Ruscheweyh quoted above remains valid for the class S_0. We prove

THEOREM 2. *Let* $f \in S_0$ *with* $|f| < \infty$. *Then*

$$|f(x)| \leqq \frac{2|f|\,|x|}{1 + x^2}, \quad x \in (-1, 1).$$

Our proof of Theorem 2 is based on a result about polynomials which we believe is interesting enough to be quoted separately. Let $0 < r \leqq 1$, $n \geqq 1$ and $\mathscr{P}_n(r)$ be the set of polynomials p of degree n having all their zeros on the circle $|z| = 1/r$ and such that

$$\min(|p(1)|, |p(-1)|) \geqq 1.$$

We shall prove

PROPOSITION 1. *Let $p \in \mathscr{P}_n(r)$ and $x \in (-1, 1)$. Then*

$$|p(x)| \geqq \left(\frac{1 + r^2x^2}{1 + r^2}\right)^{n/2}$$

and equality is possible for some $x \in (-1, 1)$ if and only if

$$p(z) \equiv p(0)(1 + riz)^k(1 - riz)^{n-k}$$

where

$$|p(0)| = \frac{1}{(1 + r^2)^{n/2}} \quad \text{and } 0 \leqq k \leqq n.$$

Since the functions in S_0 with real Taylor coefficients are typically real functions, our results are in some sense extensions of Theorem A. In the conclusion we will discuss an extension of our corollaries to a subclass of S larger than S_0 and add some more comments about the relation between non-vanishing polynomials in the unit disc and starlike functions.

Proof of Theorem 1. We assume first that the function f is continuous on $[-1, 1]$. Since for $r \in (0, 1)$ and $\theta \in [0, 2\pi)$,

$$\frac{d}{dr}\log|f(re^{i\theta})| = \frac{1}{r}\operatorname{Re}\left(re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right) > 0,$$

we obtain that $|f(re^{i\theta})|$ is an increasing function of $r \in [0, 1]$, for each fixed $\theta \in [0, 2\pi)$. In particular $|f(r)|$ increases with $r \in [0, 1]$ and because $f(-r) = -f(r)$ we obtain

$$(1) \quad |f| = \max_{x \in [-1,1]} |f(x)| = |f(1)| = |f(-1)|.$$

Let $f(1) = |f(1)|e^{i\theta_1}$. We will prove that

$$\rho e^{i\theta_1} \notin f(E) \quad \text{if } \rho \geqq |f(1)|.$$

It is sufficient to prove that $f(1) \notin f(E)$ because $f(E)$ is a starlike domain with respect to the origin. If $f(1) \in f(E)$, there exists $z_0 \in E$ with $f(z_0) = f(1)$. It follows from (1) that z_0 is not real and because f is an open mapping, there must exist $\delta > 0$ such that

(2) $\{z|\ |z - z_0| < \delta\} \subset E,\quad \{z|\ |z - z_0| < \delta\} \cap (-1, 1) = \emptyset$

and $f(\{z|\ |z - z_0| < \delta\})$ is an open neighbourhood of $f(z_0) = f(1)$. Let

$$\{x_n\} \subset (0, 1) \quad \text{with} \lim_{n\to\infty} x_n = 1.$$

Then

$$\lim_{n\to\infty} f(x_n) = f(1)$$

and for n large enough,

$$f(x_n) \in f(\{z|\ |z - z_0| < \delta\}).$$

By (2) this is impossible because f is univalent. Therefore $f(1) \notin f(E)$ and the function f must omit the radial half-line $(f(1), \infty)$. The function f being odd, it must also omit the radial half-line $(-f(1), \infty)$. This means, in view of (1), that

(3) f is subordinate to $F(z) \equiv \dfrac{2|f|e^{i\theta_1}z}{1 + z^2}$.

More generally let f be an odd function in S_0 and define

$$g_n(z) \equiv f\left(\left(1 - \frac{1}{n}\right)z\right) \quad \text{for } n \geqq 2.$$

Then $g_n \in S_0$, g_n is odd and continuous on $[-1, 1]$ and

$$|g_n| = \left|f\left(1 - \frac{1}{n}\right)\right|.$$

By (3) there must exist, for each $n \geqq 2$, an analytic function w_n on E such that

(4) $|w_n(z)| \leqq |z|,\ z \in E$ and

$$g_n(z) \equiv \frac{2\left|f\left(1 - \frac{1}{n}\right)\right|e^{i\theta_n}w_n(z)}{1 + w_n^2(z)},$$

for some sequence $\{\theta_n\} \subset [0, 2\pi)$. The sequence $\{\theta_n\}$ admits a convergent subsequence $\{\theta_{n_j}\}$. The sequence of functions $\{w_{n_j}\}$ is uniformly bounded on E and also admits a subsequence converging to a function w analytic on E with $|w(z)| \leqq |z|$. Finally, since

$$|f| = \lim_{n\to\infty} \left|f\left(1 - \frac{1}{n}\right)\right|,$$

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we obtain by a passage to the limit in (4),

$$f(z) \equiv 2|f|e^{i\theta}\frac{w(z)}{1 + w^2(z)}$$

for some $\theta \in [0, 2\pi)$. This completes the proof of Theorem 1.

Proof of the corollaries. We need two lemmas. The first one depends on a well-known result about the coefficients of analytic functions with positive real part on E. The second one is purely computational. The proofs are omitted.

LEMMA 1. *Let* $\varphi \in [0, 2\pi)$ *and* $f \in S_0$ *such that*

$$\sqrt{-f(z)f(-z)} \equiv \frac{z}{1 + e^{2i\varphi}z^2}.$$

Then

$$f(z) \equiv \frac{z}{(1 - ie^{i\varphi}z)^{2\lambda}(1 + ie^{i\varphi}z)^{2(1-\lambda)}} \quad \textit{where } \lambda \in [0, 1].$$

LEMMA 2. *Let* $\varphi \in [0, 2\pi)$ *and* $f \in S_0$ *as in Lemma* 1. *Then* $|f| = 1/2$ *if and only if* $e^{i\varphi} = \pm 1$.

Proof of Corollary 1.1. We may assume that $|f| < \infty$. Define

$$g(z) \equiv \frac{f(z)}{|f|} \quad \text{and} \quad h(z) \equiv \sqrt{-g(z)g(-z)}.$$

For a suitable choice of the root, the function h is well-defined and

$$h'(0) = \frac{1}{|f|}.$$

Since

$$z\frac{h'(z)}{h(z)} \equiv \frac{1}{2}\left(z\frac{g'(z)}{g(z)} + -z\frac{g'(-z)}{g(z)}\right),$$

it follows easily that h is an odd starlike function. Since

$$|h| = \lim_{r\to 1} \sqrt{|g(r)|\,|g(-r)|} \leqq \lim_{r\to 1} \max(|g(r)|, |g(-r)|) = |g| = 1,$$

we obtain from Theorem 1 that

$$h(z) \equiv \frac{1}{|f|}z + \dots \text{ is subordinate to } \frac{2e^{i\theta}z}{1 + z^2}$$

for some $\theta \in [0, 2\pi)$. Therefore (see [**3**], page 192) $1/|f| \leqq 2$ and equality is possible if and only if

$$h(z) \equiv \sqrt{-g(z)g(-z)} \equiv \frac{2e^{i(\theta+\varphi)}z}{1 + e^{2i\varphi}z^2}$$

for some $\varphi \in [0, 2\pi)$. Since $h'(0) = 1/|f| = 2$ we must have

$$\sqrt{-f(z)f(-z)} \equiv \frac{z}{1 + e^{2i\varphi}z^2}$$

and the case of equality now follows from Lemma 1 and Lemma 2.

Proof of Corollary 1.2. Let

$$f(z) \equiv \sum_{n=1}^{\infty} a_n z^n \in S_0$$

with $|f| < \infty$. Then $g(z) \equiv f(z)/a_1$ verifies the hypothesis of Corollary 1.1 and therefore

$$|g| = \frac{|f|}{|a_1|} \geqq \frac{1}{2},$$

i.e., $|a_1| \leqq 2|f|$, and the equality is possible if and only if

$$f(z) \equiv \frac{a_1 z}{(1 + iz)^{2\lambda}(1 - iz)^{2(1-\lambda)}}$$

where a_1 is an arbitrary complex number and $\lambda \in [0, 1]$. The function g is normalized starlike univalent function; we know (see [**3**], page 44) that

$$\left|\frac{g^{(n)}(0)}{n!}\right| = \frac{|a_n|}{|a_1|} \leqq n$$

with equality only if g is a rotation of the Koebe function. From the reasoning above it follows that

$$|a_n| \leqq n|a_1| \leqq 2|f|n$$

with equality if and only if

$$f(z) \equiv \frac{a_1 z}{(1 \pm iz)^2} \quad \text{where } a_1 \in \mathbf{C}.$$

Proof of Corollary 1.3. Let

$$f(z) \equiv \sum_{n=1}^{\infty} a_{2n-1} z^{2n-1} \in S_0$$

with $|f| < \infty$. Then $g(z) \equiv f(z)/a_1$ verifies the hypothesis of Corollary 1.1 and therefore $|a_1| \leqq 2|f|$. Since the functions

$$\frac{z}{(1+iz)^{2\lambda}(1-iz)^{2(1-\lambda)}}$$

are odd if and only if $\lambda = 1/2$, we obtain that the equality is possible if and only if

$$f(z) \equiv \frac{a_1 z}{1+z^2}, \quad \text{where } a_1 \in \mathbf{C}.$$

The function g is a normalized odd starlike function. Therefore ([**3**], page 110)

$$\left|\frac{g^{(2n-1)}(0)}{(2n-1)!}\right| = \frac{|a_{2n-1}|}{|a_1|} \leqq 1$$

with the equality only if g is a rotation of $z/(1+z^2)$. It follows that $|a_{2n-1}| \leqq 2|f|$ and the case of equality follows at once.

Proof of Proposition 1.

LEMMA 3. *Let* $x \in (-1, 1)$. *There exists a polynomial* $q(z) \in \mathcal{P}_n(r)$ *such that*

$$\inf_{p\in\mathcal{P}_n(r)} |p(x)| = |q(x)| \leqq \left(\frac{1+r^2x^2}{1+r^2}\right)^{n/2} \quad \textit{and} \quad q(0) > 0.$$

Proof. There exists a sequence $\{p_j\} \in \mathcal{P}_n(r)$ with

$$\lim_{j\to\infty} |p_j(x)| = \inf_{p\in\mathcal{P}_n(r)} |p(x)|.$$

Assume that

$$p_j(z) = p_j(0) \prod_{k=1}^{n} (1 + re^{i\theta_{k,j}}z).$$

There is an integer J_0 such that

$$j \geqq J_0 \Rightarrow |p_j(x)| \leqq \inf_{p\in\mathcal{P}_n(r)} |p(x)| + 1$$

and therefore

$$|p_j(0)| \leqq \frac{\inf_{p\in\mathcal{P}_n(r)} |p(x)| + 1}{(1 - r|x|)^n} \quad \text{if } j \geqq J_0.$$

It follows that the sequences $\{p_j(0)\}$, $\{\theta_{1,j}\}$, $\{\theta_{2,j}\}, \ldots, \{\theta_{n,j}\}$ are bounded and the sequence $\{p_j\}$ admits a subsequence converging to a polynomial q; since

$$|p_j(0)| \geqq \frac{1}{(1 + r)^n},$$

we obtain that $|q(0)| > 0$ and $q \in \mathcal{P}_n(r)$. Clearly

$$|q(x)| = \inf_{p \in \mathcal{P}_n(r)} |p(x)|.$$

The class $\mathcal{P}_n(r)$ is closed under multiplication by unimodular constants, so that we can suppose $q(0) > 0$. Further,

$$\tilde{q}(z) \equiv \left(\frac{1 + riz}{1 + ri}\right)^n \in \mathcal{P}_n(r)$$

and therefore

$$|q(x)| \leqq |\tilde{q}(x)| = \left(\frac{1 + r^2x^2}{1 + r^2}\right)^{n/2}.$$

LEMMA 4. *Let*

$$q(z) \equiv \rho(1 - rz)^k(1 + rz)^{n-k} \in \mathcal{P}_n(r),$$

where $0 < \rho$ *and* $0 \leqq k \leqq n$. *Then*

$$\inf_{p \in \mathcal{P}_n(r)} |p(x)| < |q(x)|, \quad \textit{if } x \in (-1, 1).$$

Proof. The cases where $r = 1$ or $0 = k$ or $0 = n$ are trivial and we shall prove Lemma 4 under the hypothesis $0 < r < 1$ and $0 < k < n$. Suppose that the result is false; then, for some $x \in (-1, 1)$,

$$(6) \qquad \inf_{p \in \mathcal{P}_n(r)} |p(x)| = |q(x)|.$$

Since $q \in \mathcal{P}_n(r)$ we have

$$1 \leqq |q(1)| \leqq |q(-1)| \quad \text{or} \quad 1 \leqq |q(-1)| \leqq |q(1)|,$$

i.e.,

$$(7) \qquad 1 \leqq \rho(1 - r)^k(1 + r)^{n-k} \leqq \rho(1 + r)^k(1 - r)^{n-k}$$

or

$$(8) \qquad 1 \leqq \rho(1 + r)^k(1 - r)^{n-k} \leqq \rho(1 - r)^k(1 + r)^{n-k}.$$

We shall discuss below the case where (7) is valid. A similar reasoning is available if instead of (7) we consider (8).

By (7) we obtain

$$\left(\frac{1 + r}{1 - r}\right)^{n-k} \leqq \left(\frac{1 + r}{1 - r}\right)^k$$

and because $0 < r < 1$ we have

(9) $\quad n - k \leqq k.$

It also follows from (7) that

$$\frac{1}{(1+r)^k(1-r)^{n-k}} \leqq \frac{1}{(1-r)^k(1+r)^{n-k}} \leqq \rho$$

and in view of (6) we must have

$$\frac{1}{(1-r)^k(1+r)^{n-k}} = \rho \quad \text{and} \quad q(z) \equiv \left(\frac{1-rz}{1-r}\right)^k\left(\frac{1+rz}{1-r}\right)^{n-k}.$$

Then, by (9),

$$\begin{aligned} |q(x)|^2 &= \left(\frac{1-rx}{1-r}\right)^{2k}\left(\frac{1+rx}{1+r}\right)^{2(n-k)} \\ &\geqq \left(\frac{1-rx}{1-r}\right)^{2(n-k)}\left(\frac{1+rx}{1+r}\right)^{2(n-k)} \\ &= \left(\frac{1-r^2x^2}{1-r^2}\right)^{2(n-k)} \\ &> \left(\frac{1+r^2x^2}{1+r^2}\right)^{n}. \end{aligned}$$

This last inequality, together with (6), contradicts the result of Lemma 3. This completes the proof of Lemma 4.

We are now in position to translate our problem into the language of mathematical programming. Proposition 1 is equivalent to the following statement:

"Let $(\rho, \theta_1, \theta_2, \ldots, \theta_n)$ be a solution to the problem

$$(10) \quad \begin{cases} \underset{\rho,\theta_1,\ldots,\theta_n}{\text{minimize}}\ F(\rho, \theta_1, \ldots, \theta_n) \equiv \log(\rho) + \operatorname{Re}\left(\sum_{j=1}^{n} \log(1 + re^{i\theta_j}x)\right) \\ \text{under the constraints:} \\ H_1(\rho, \theta_1, \ldots, \theta_n) \equiv -\log(\rho) - \operatorname{Re}\left(\sum_{j=1}^{n} \log(1 + re^{i\theta_j})\right) \leqq 0, \\ H_2(\rho, \theta_1, \ldots, \theta_n) \equiv -\log(\rho) - \operatorname{Re}\left(\sum_{j=1}^{n} \log(1 - re^{i\theta_j})\right) \leqq 0, \\ H_3(\rho, \theta_1, \ldots, \theta_n) \equiv -\rho \leqq 0. \end{cases}$$

Then $\rho = (1 + r^2)^{-n/2}$ and $\{e^{i\theta_j}\} \subseteq \{i, -i\}$."

It follows from Lemma 3 that the above problem has a solution for each $x \in (-1, 1)$ with $\rho > 0$. Clearly the corresponding maximization problem has no solution. We compute

$$\frac{\partial H_1}{\partial \rho} = -\frac{1}{\rho}, \quad \frac{\partial H_1}{\partial \theta_j} = \frac{r \sin(\theta_j)}{|1 + re^{i\theta_j}|^2}, \quad j = 1, 2, \ldots, n.$$

$$\frac{\partial H_2}{\partial \rho} = -\frac{1}{\rho}, \quad \frac{\partial H_2}{\partial \theta_j} = \frac{-r \sin(\theta_j)}{|1 - re^{i\theta_j}|^2}, \quad j = 1, 2, \ldots, n.$$

Remark that

$$\frac{\partial H_1}{\partial \theta_j} = \frac{\partial H_2}{\partial \theta_j} \Leftrightarrow \sin \theta_j = 0.$$

Moreover we know by Lemma 4 that if $(\rho, \theta_1, \ldots, \theta_n)$ is a solution point of (10), there must exist an index $j \in (1, 2, \ldots, n)$ such that $\sin(\theta_j) \neq 0$. Therefore the gradients ∇H_1 and ∇H_2, evaluated at any solution point of (10), are linearly independent. We can apply the Fritz John Theorem (**[1]**, page 34) to obtain that there exist multipliers μ_1 and μ_2 with the following properties (all functions are evaluated at a solution point):

$$(11) \quad \begin{cases} \nabla F + \mu_1 \nabla H_1 + \mu_2 \nabla H_2 = 0 \\ \mu_1 \geqq 0, \quad \mu_2 \geqq 0, \quad \mu_1 H_1 + \mu_2 H_2 = 0 \end{cases}$$

and (11) is easily seen to be equivalent to

$$(12) \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0, \quad \mu_1 + \mu_2 = 1,$$

$$(13) \quad \frac{-rx \sin(\theta_j)}{|1 + re^{i\theta_j}x|^2} + \mu_1 \frac{r \sin(\theta_j)}{|1 + re^{i\theta_j}|^2} - \mu_2 \frac{r \sin(\theta_j)}{|1 - re^{i\theta_j}|^2} = 0, \; j = 1, \ldots, n,$$

$$(14) \quad \mu_1 > 0 \Rightarrow H_1 = 0; \quad \mu_2 > 0 \Rightarrow H_2 = 0.$$

We now obtain

LEMMA 5. *Let* $x \in (-1, 1)$ *and* $q \in \mathscr{P}_n(r)$ *with*

$$|q(x)| = \inf_{p \in \mathscr{P}_n(r)} |p(x)|.$$

Then $|q(1)| = |q(-1)| = 1$.

Proof. By (14) it is sufficient to verify that $\mu_1 > 0$ and $\mu_2 > 0$. Assume for example that $\mu_1 = 0$; then by (12) $\mu_2 = 1$ and after selecting an index j for which $\sin(\theta_j) \neq 0$, it follows from (13) that

$$\frac{-x}{|1 + re^{i\theta_j}x|^2} = \frac{1}{|1 - re^{i\theta_j}|^2}.$$

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This is obviously impossible when $x \geqq 0$. It is also impossible when $-1 < x < 0$ because

$$\frac{-x}{|1 + re^{i\theta_j}x|^2} = \frac{|x|}{|1 - re^{i\theta_j}|x|\,|^2}$$

and

$$\frac{d}{dt}\left(\frac{t}{|1 - re^{i\theta_j}t|^2}\right) = \frac{1}{|1 - re^{i\theta_j}t|^2}\,\mathrm{Re}\left(\frac{1 + re^{i\theta_j}t}{1 - re^{i\theta_j}t}\right) > 0$$

if $0 < t < 1$. Therefore $\mu_1 > 0$. A similar reasoning would show that $\mu_2 > 0$. This completes the proof of Lemma 5.

We are now able to prove an extension of Lemma 4:

LEMMA 6. *Let*

$$q(z) = \rho(1 - rz)^{k_1}(1 + rz)^{k_2}\tilde{q}(z) \in \mathscr{P}_n(r)$$

where $0 < \rho$, $0 < \min(k_1, k_2)$ *and* $\tilde{q}(z)$ *is a polynomial of degree* $n - k_1 - k_2$. *Then*

$$\inf_{p \in \mathscr{P}_n(r)} |p(x)| < |q(x)|, \quad \textit{for all } x \in (-1, 1).$$

Proof. We may assume that $0 < r < 1$ and $k_1 = k_2 = 1$. If the conclusion is false then, by Lemma 5, $|q(1)| = |q(-1)| = 1$ and we can write, for some $\theta \in [0, 2\pi)$,

$$q(z) \equiv e^{i\theta}\left(\frac{1 - r^2z^2}{1 - r^2}\right)\frac{\tilde{q}(z)}{\tilde{q}(1)} \quad \text{and} \quad |\tilde{q}(1)| = |\tilde{q}(-1)|.$$

We define a polynomial q^* of degree n by

$$q^*(z) \equiv \left(\frac{1 + irz}{1 + ir}\right)^2\frac{\tilde{q}(z)}{\tilde{q}(1)}.$$

The polynomial q^* has its zeros on $|z| = 1/r$ and since

$$q^*(1) = 1 \quad \text{and} \quad |q^*(-1)| = \frac{|\tilde{q}(-1)|}{|\tilde{q}(1)|} = 1$$

we obtain $q^* \in \mathscr{P}_n(r)$. On the other hand

$$\begin{aligned} |q^*(x)| &= \left(\frac{1 + r^2x^2}{1 + r^2}\right)\frac{|\tilde{q}(x)|}{|\tilde{q}(1)|} \\ &< \left(\frac{1 - r^2x^2}{1 - r^2}\right)\frac{|\tilde{q}(x)|}{|\tilde{q}(1)|} \\ &= |q(x)| \end{aligned}$$

$$= \inf_{p \in \mathscr{P}_n(r)} |p(x)|.$$

This is impossible and the conclusion of the lemma must hold.

The Fritz John Theorem also gives some important information concerning the non-real zeros of an extremal polynomial. Let

$$q(z) = q(0) \prod_{j=1}^{n} (1 + re^{i\theta_j} z) \in \mathscr{P}_n(r)$$

and assume that

$$|q(x)| = \inf_{p \in \mathscr{P}_n(r)} |p(x)|$$

for some $x \in (-1, 1)$ and $q(0) > 0$. Let

$$J = \{j | \sin(\theta_j) \neq 0\}.$$

We know that J is non-empty and by (12), (13)

$$\frac{-x}{|1 + re^{i\theta_j}x|^2} + \frac{\mu_1}{|1 + re^{i\theta_j}|^2} - \frac{1 - \mu_1}{|1 - re^{i\theta_j}|^2} = 0, \quad j \in J,$$

for some $\mu_1 \in (0, 1)$. This last equation is equivalent to

$$0 = (1 + r^2)[-x(1 + r^2) + (2\mu_1 - 1)(1 + r^2x^2)]$$
$$+ 2r[(\mu_1 - 1)x(1 + r^2) - (1 + r^2x^2)]\cos \theta_j$$

i.e., $\cos(\theta_j)$ is constant for all $j \in J$. Having in mind the results given by Lemma 5 and Lemma 6, the extremal polynomial $q(z)$ must be

$$(15) \quad q(z) \equiv \frac{(1 \pm rz)^k (1 \pm re^{i\alpha}z)^{k_1} (1 \pm re^{-i\alpha}z)^{n-k-k_1}}{(1 + r)^k |1 + re^{i\alpha}|^{n-k}}$$

where $0 \leqq k < n$, α is a real number such that $\sin(\alpha) \neq 0$ and

$$\left(\frac{1 - r}{1 + r}\right)^k \left|\frac{1 - re^{i\alpha}}{1 + re^{i\alpha}}\right|^{n-k} = 1.$$

In order to complete the proof of Proposition 1 we need to establish a certain inequality. Let $0 < k < n$, $0 < r < 1$ and $\alpha \in \mathbf{R}$ such that

$$(16) \quad \left(\frac{1 - r}{1 + r}\right)^k \left|\frac{1 - re^{i\alpha}}{1 + re^{i\alpha}}\right|^{n-k} = 1.$$

We obtain

LEMMA 7. *For all* $x \in (-1, 1)$,

$$\left(\frac{1 + r^2x^2}{1 + r^2}\right)^{n/2} < \left(\frac{1 + rx}{1 + r}\right)^k \left|\frac{1 + re^{i\alpha}x}{1 + re^{i\alpha}}\right|^{n-k}.$$

Proof. Define on $[-1, 1]$ the function

$$G(x) = \frac{1 + r^2x^2}{(1 + rx)^{2k/n}|1 + re^{i\alpha}x|^{2(1-(k/n))}}.$$

Then, by (16), $G(1) = G(-1)$ and the inequality in Lemma 7 is equivalent to

(17) $G(x) < G(1)$ for all $x \in (-1, 1)$.

The derivative of the non-constant function G has at least one zero in $(-1, 1)$, because $G(1) = G(-1)$. On the other hand

$$\frac{G'(x)}{G(x)} \equiv \frac{-2r(1 - rx)}{(1 + r^2x^2)(1 + rx)|1 + rxe^{i\alpha}|^2} \times (c_n + 2r(\cos\alpha)x + c_nr^2x^2)$$

where

$$c_n = \frac{k}{n} + \left(1 - \frac{k}{n}\right)\cos(\alpha).$$

Since $\cos\alpha \neq 0$,

$$G'(x) = 0 \Leftrightarrow \begin{cases} x = 0 & \text{if } c_n = 0, \\ 1 + \dfrac{2r\cos\alpha}{c_n}x + r^2x^2 = 0 & \text{if } c_n \neq 0. \end{cases}$$

Because $0 < r < 1$, we conclude that G' has exactly one zero on $(-1, 1)$ and that (17) is just equivalent to $G(0) < G(1)$; some computations show that $G(0) < G(1)$ if and only if

(18) $2(1 - r^2)^{2k/n-k} < (1 + r^2)^{k/n-k}[(1 - r)^{2k/n-k} + (1 + r)^{2k/n-k}]$

and the truth of (18) follows easily from the inequality

$$(1 - r^2)^2 < 1 - r^4,$$

valid for all $r \in (0, 1)$. This completes the proof of Lemma 7.

We can now prove Proposition 1. Let $q \in \mathscr{P}_n(r)$ such that

$$\inf_{p \in \mathscr{P}_n(r)} |p(x)| = |q(x)| \quad \text{and} \quad q(0) > 0.$$

By (15),

$$q(z) = \frac{(1 \pm rz)^k(1 \pm re^{i\alpha}z)^{k_1}(1 \pm re^{-i\alpha}z)^{n-k-k_1}}{(1 + r)^k|1 + re^{i\alpha}|^{n-k}}$$

where $0 \leqq k < n$ and (16) holds. If $r = 1$ then clearly $k = 0$; if $0 < r < 1$ and $0 < k$, we obtain from Lemma 7

$$\left(\frac{1 + r^2x^2}{1 + r^2}\right)^{n/2} < |q(x)|$$

and this contradicts the result of Lemma 3. Therefore k must vanish for all $r \in [0, 1]$ and by (16),

$$\left|\frac{1 - re^{i\alpha}}{1 + re^{i\alpha}}\right|^n = 1$$

which means that

$$e^{i\alpha} = \pm i \quad \text{and} \quad q(0) = (1 + r^2)^{-n/2}.$$

This completes the proof of Proposition 1.

Proof of Theorem 2. Let $0 < r < 1$ and $f \in S_0$; we shall prove

$$(19) \quad |f(rx)| \leqq \frac{(1 + r^2)|x|}{1 + r^2x^2} \max(|f(r)|, |f(-r)|).$$

Since, for any $f \in S_0$,

$$\lim_{r \to 1} \max(|f(r)|, |f(-r)|) = |f|,$$

it is clear that Theorem 2 is in fact a limiting case (let $r \to 1$) of (19).

The set of all functions g of the type

$$g(z) \equiv \frac{\alpha z}{\prod_{j=1}^{n} (1 + e^{i\theta_j}z)^{2\lambda_j}}.$$

where

$$\alpha \in \mathbf{C}, \quad \theta_j \in [0, 2\pi), \quad 0 \leqq \lambda_j \leqq 1 \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j = 1,$$

is well-known to be dense in S_0 endowed with the topology of uniform convergence on compact subsets of the unit disc; a simple application of Montel's criteria shows that this remains valid if the λ_j are assumed to be rational numbers in [0, 1]. In other words, the set $\widetilde{S}_0$ defined by

$$\widetilde{S}_0 = \left\{ f \,\middle|\, \begin{array}{l} f(z) \equiv \dfrac{z}{p(z)^{2/n}} \text{ where } n \geqq 1 \text{ and } p \text{ is a polynomial} \\ \text{of degree } n \text{ having its zeros on the unit circle} \end{array} \right\}$$

is dense in S_0. It will be sufficient to prove (19) assuming that $f \in \widetilde{S}_0$, i.e., we need to prove

$$(20) \quad \left(\frac{1 + r^2x^2}{1 + r^2}\right)^{n/2} \leqq \frac{|p(rx)|}{\min(|p(r)|, |p(-r)|)}, \quad x \in (-1, 1).$$

Since

$$\frac{p(rz)}{\min(|p(r)|, |p(-r)|)} \in \mathscr{P}_n(r)$$

for each polynomial p of degree n having its zeros on the unit circle, the truth of (20) follows from Proposition 1. This completes the proof of Theorem 2.

The result of Theorem 2 is sharp as can be seen from the functions

$$f(z) \equiv \frac{\alpha z}{(1 + iz)^{2\lambda}(1 - iz)^{2(1-\lambda)}}$$

with $\alpha \in \mathbf{C}$ and $\lambda \in [0, 1]$. We believe that equality can be attained only for these functions but we cannot prove it due to a limiting process used in the proof. It is however not difficult to prove that given $r \in [0, 1]$ and $x \in (-1, 1)$, the equality in (19) is possible within the class $\tilde{S}_0$ if and only if the function f is as above, with rational $\lambda \in [0, 1]$.

Conclusion. 1° Let

$$\tilde{S} = \{f \in S | \sqrt{-f(z)f(-z)} \in S\}.$$

We already noticed that $S_0 \subseteq \tilde{S}$ but in fact $\tilde{S}$ is larger than S_0 since it contains all odd univalent functions and the spirallike functions (see [**3**], page 52, for a suitable definition). Remark however that $\tilde{S} \neq S$: the Fekete-Szego Theorem ([**3**], page 104) asserts the existence of a function

$$f_0(z) \equiv z + \sum_{n=2}^{\infty} \alpha_n z^n \in S$$

such that

$$(21) \quad \left|\alpha_3 - \frac{1}{2}\alpha_2^2\right| > 1.$$

But

$$\sqrt{-f_0(z)f_0(-z)} \equiv z + \left(\alpha_3 - \frac{1}{2}\alpha_2^2\right)z^3 + \dots$$

is an odd analytic function and it cannot be univalent, by (21).

We point out that some of our results can be extended to $\tilde{S}$. Let $f \in \tilde{S}$ with $f'(0) = 1$. By the growth theorem for odd univalent functions we obtain

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$$|\sqrt{-f(z)f(-z)}\,| \geqq \frac{|z|}{1 + |z|^2}, \quad z \in E$$

and

$$\max(\,|f(z)\,|, |f(-z)\,|\,) \geqq \sqrt{|f(z)\,|\,|f(-z)\,|} \geqq \frac{|z|}{1 + |z|^2}, \quad z \in E.$$

This implies in particular that $|f| \geqq 1/2$, i.e., Corollary 1.1 can be extended to S. Now let

$$f(z) \equiv \sum_{n=1}^{\infty} a_n z^n \in \tilde{S} \quad \text{with } |f| < \infty.$$

As in the proof of Corollary 1.2, we obtain that $|a_1| \leqq 2|f|$. Moreover it follows from de Branges Theorem [2] that

$$|a_n| \leqq n|a_1| \leqq 2n|f|, \quad n \geqq 1$$

and Corollary 1.2 is valid for $\tilde{S}$. Of course no such simple generalization of Corollary 1.3 is possible.

2° The relation between starlike functions and non-vanishing polynomials in the unit disc is an interesting fact. We shall prove here the following result, apparently stronger than Proposition 1. In what follows let $0 < r \leqq 1$ and $\mathscr{P}_n^\circ(r)$ denote the set of all polynomials p of degree n, having all their zeros on or outside the circle $|z| = 1/r$, and such that

$$\min(\,|p(1)\,|, |p(-1)\,|\,) \geqq 1.$$

We obtain

PROPOSITION 2.

$$p \in \mathscr{P}_n^\circ(r) \Rightarrow |p(x)\,| \geqq \left(\frac{1 + r^2x^2}{1 + r^2}\right)^{n/2}, \quad x \in (-1, 1).$$

Proof. We put $p(z) \equiv q(rz)$ where the polynomial q has all its zeros outside the open unit disc. If

$$q(z) \equiv q(0) \prod_{j=1}^{n} (1 + r_j e^{i\theta_j} z) \quad \text{and} \quad f(z) \equiv \frac{z}{q(z)^{2/n}}$$

then

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) = \frac{1}{n} \sum_{j=1}^{n} \operatorname{Re}\left(\frac{1 - r_j e^{i\theta_j} z}{1 + r_j e^{i\theta_j} z}\right) > 0, \quad z \in E$$

because $0 < r_j \leqq 1$. Therefore $f \in S_0$ and by (19), for $x \in (-1, 1)$,

$$\frac{1}{|p(x)|^{2/n}} = \frac{1}{|q(rx)|^{2/n}} = \left|\frac{f(rx)}{rx}\right|$$
$$\leqq \frac{1 + r^2}{1 + r^2x^2} \max\left(\frac{|f(r)|}{r}, \frac{|f(-r)|}{r}\right)$$
$$= \frac{1 + r^2}{1 + r^2x^2} \max\left(\frac{1}{|p(1)|^{2/n}}, \frac{1}{|p(-1)|^{2/n}}\right)$$
$$\leqq \frac{1 + r^2}{1 + r^2x^2}.$$

This completes the proof of Proposition 2.

3° Finally we point out the following results that can be obtained just as Proposition 1 and Theorem 2 were. The proofs are omitted:

PROPOSITION 3. *Let $p(z)$ be a polynomial of degree $n \geqq 1$ having all its zeros outside the open unit disc. Then*

$$\min(|p(1)|, |p(-1)|)|p(x)| \leqq 2^{n/2}|p(0)|^2(1 + x^2)^{n/2}$$

for all $x \in (-1, 1)$. Equality is possible if and only if

$$p(z) \equiv p(0)(1 + iz)^k(1 - iz)^{n-k} \quad \text{where } 0 \leqq k \leqq n.$$

THEOREM 3. *Let $f \in S_0$ with $|f| < \infty$. Then*

$$\frac{|f'(0)|^2}{2|f|}\frac{|x|}{1 + x^2} \leqq |f(x)|, \quad x \in (-1, 1).$$

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