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LEFSCHETZ NUMBERS AND UNITARY GROUPS

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We give a formula for the Euler-Poincare characteristic of the fixed point set of the Cartan involution on the set of integral equivalence classes of integral unimodular hermitian forms, in terms of a product of special values of Riemann zeta functions and Dirichlet L-functions. This is done via reduction by Galois cohomology to a volume computation using the Tamagawa measure on the unitary groups.

1. INTRODUCTION

(1.1). Rohlfs studied in [7, 8] the Galois action on arithmetic groups and calculated the Lefschetz number of these actions. In the particular case when Γ is $SL(n, \mathbb{Z})$ and $g = \{1, \sigma\}$ is the group of order two with action given by $\sigma A = A^{t-1} (A \in \Gamma)$, the first non-abelian cohomology $H^1(\mathfrak{g}, \Gamma)$ is just the set of integral-equivalence classes of integral unimodular symmetric bilinear forms. In this note, we carry out the procedure of Rohlfs for unimodular hermitian forms.

(1.2). Let $\overline{a}$ denote the complex conjugate of an element a in the ring $\mathbb{Z}[\sqrt{-1}]$ of Gaussian integers. The non-trivial element σ of the group $\mathfrak{g}$ of order 2 acts on $SL(n, \mathbb{Z}[\sqrt{-1}])$ by

$$\sigma A = \overline{A}^{t-1}.$$

Let Γ be a subgroup of $SL(n, \mathbb{Z}[\sqrt{-1}])$. An element $H \in \Gamma$ determines a cocycle $(1, H)$ of the nonabelian cohomology set $H^1(\mathfrak{g}, \Gamma)$ if $1 = H.\sigma(H)$, that is, $H = \overline{H}^t$ is an integral hermitian matrix. Two cocycles $(1, H)$ and $(1, H')$ are Γ equivalent if there exists a $B \in \Gamma$ such that $B^t H \overline{B} = H'$. We can associate to a cocycle $(1, H)$ a sesqui-linear form

$$H(x, y) = x^t H \overline{y}.$$

Here $x, y \in (\mathbb{Z}[\sqrt{-1}])^n$ are column vectors. If for example $\Gamma = SL(n, \mathbb{Z}[\sqrt{-1}])$, then we get a bijection of $H^1(\mathfrak{g}, \Gamma)$ with the set of integral equivalence classes of integral unimodular hermitian forms.

To simplify the notation, we shall write H for the cohomology class represented by the cocycle $(1, H)$.

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(1.3). Let H be a hermitian matrix in $SL(n, \mathbb{Z}[\sqrt{-1}])$. We denote by G the special unitary group with respect to H, that is

$$G(\mathbb{Q}) = \{g \in SL(n, \mathbb{Q}(\sqrt{-1})) \mid g^t H \bar{g} = H\}.$$

Fix a maximal compact subgroup of $G(\mathbb{R})$. Let X_H denote the hermitian symmetric space $K \setminus G$.

Let Γ be a torsion-free congruence subgroup of $SL(n, \mathbb{Z}[\sqrt{-1}])$. Write Γ_H for $\Gamma \cap G(\mathbb{Q})$. Then Γ_H acts on X_H. We compute in this note the sum

$$\mathcal{L} = \sum_{H \in H^1(\mathfrak{g}, \Gamma)} \chi(H)$$

where
$$\chi(H) = \sum (-1)^i \dim H_i(X_H/\Gamma_H, \mathbb{R})$$

is the Euler-Poincare characteristics of X_H/Γ_H.

This computation begins with Harder's Gauss-Bonnet theorem which says that there exists an Euler–Poincare form ω_χ on X_H such that

$$\chi_{(H)} = \int_{X_H/\Gamma_H} \omega_\chi.$$

Then one uses Rohlfs' exact sequence of the Hasse map h:

$$1 \to C \to H^1(\mathfrak{g}\, \Gamma) \xrightarrow{h} \coprod_v H^1(\mathfrak{g}, \Gamma_v) \tag{1.4}$$

to reduce the calculation of the above integral to the computation of local volumes. Here $\Gamma_v = SL(n, \mathbb{Z}_v[\sqrt{-1}])$ for almost all v, and

$$\Gamma = \bigcap (\Gamma_v \cap SL(n, \mathbb{Q}(\sqrt{-1})))$$

and
$$C = SU(n, \mathbb{Q}) \setminus SU(n, \mathbb{A}) / \Pi(\Gamma_{I_n})_v$$

and
$$(\Gamma_{I_n})_v = SU(n, \mathbb{Z}_v) \cap \Gamma_v.$$

(1.5). Let ℓ be an odd prime, Γ be the congruence subgroup of $SL(n, \mathbb{Z}[\sqrt{-1}])$ of level ℓ. Write $L(s, \psi)$ for the Dirichlet L-function for the quadratic charactor $\psi = (-4/\cdot)$. Define $\lambda(n)$ as follows: if n is odd then

$$\lambda(n) = 2\ell^{n^2-1} \prod_{r=2}^{n} \left(1 - (\psi(\ell)\ell)^{-r}\right),$$

and if $n \equiv 2 \bmod 4$ then

$$\lambda(n) = -2^{n+1}(1-2^{-n})\ell^{n^2-1}\prod_{r=2}^{n}\left(1-(\psi(\ell)\ell)^{-r}\right).$$

Theorem 1.6.

$$\mathcal{L} = \lambda(n)\prod_{\substack{r=1\\ r\equiv 1(2)}}^{n-1}\zeta(-r)\prod_{\substack{r=1\\ r\equiv 0(2)}}^{n-1}L(-r,\psi).$$

(1.7). For example, for $\ell = 3$, we get

n	$\mathcal{L}$
2	$2^2 . 3$
3	$2^3 . 3^2 . 7$
4	$2^2 . 3^4 . 7 . 61$
5	$2^5 . 3^8 . 5 . 7 . 61$

(1.8). The paper is divided into four sections. The local volume computations are carried out in Section 2. The final result is assembled in Section 4.

2. Volume computations

In this section we calculate the volume of some of the local compact subgroups of the special unitary group G with respect to an integral hermitian form H of n variables over $\mathbb{Z}[\sqrt{-1}]$.

(2.1). Let $\mathcal{G}$ be the Lie algebra of G. Choose a Chevalley basis $e_1, \ldots, e_{n^2-1}$ of $\mathcal{G}_{\mathbb{Z}}$. Then $\omega = de_1 \wedge \ldots \wedge de_{n^2-1}$ is a form of highest degree on the semisimple group G. Moreover ω is bi-invariant.

We can use ω to define measure (see Weil [9], Harder [3]). For each place v of $\mathbb{Q}$, ω determines a bi-invariant measure ω_v on the locally compact group, $G(\mathbb{Q}_v)$. In particular, if ω_∞ is the measure belonging to the metric determined by the Killing form, and if p is a rational prime, V a sufficiently small neighbourhood of 0 in $\mathcal{G}(\mathbb{Q}_p)$ so that the exponential map exp is biannalytic then

$$\int_{\exp V}\omega_p = \int_V \omega.$$

Moreover, ω determines a bi-invariant measure $\Pi\omega_v$ on $G(\mathbb{A})$, which, by the product formula, is independent of the choice of the form ω.

(2.2). We first do a calculation at infinity. Assume the signature of the form H is (p, q) with $n = p + q$. In this subsection write G for $G(\mathbb{R}) = SU(p, q)$ and K for its maximal compact subgroup $S(U(p) \times U(q))$.

(2.2.1). We have a Cartan decomposition

$$\mathcal{G} = \mathcal{K} + \mathcal{P}$$

with

$$\mathcal{K} = \left\{ \begin{bmatrix} A & \\ & D \end{bmatrix} : A \in \mathfrak{u}(p),\ D \in \mathfrak{u}(q),\ tr\, A + tr D = 0 \right\}$$

$$\mathcal{P} = \left\{ \begin{bmatrix} & B \\ {}^t\overline{B} & \end{bmatrix} : B \in M(p \times q, \mathbb{C}) \right\}.$$

Let E_{rs} be the matrix $(\delta_{ir}\delta_{js})_{1 \leqslant i, j \leqslant n}$. Then $\mathcal{K}$ has a basis consisting of the following elements.

$$\sqrt{-1}(E_{rr} - E_{r+1,r+1}) \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \sqrt{-1} & & \\ & & & -\sqrt{-1} & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \qquad 1 \leqslant r \leqslant n = p + q$$

$$E_{rs} - E_{sr} = \begin{array}{c c} & \begin{array}{ccc} r & s & p \end{array} \\ \begin{array}{c} r \\ s \\ p \end{array} & \begin{bmatrix} & 1 & \\ -1 & & \\ & & \end{bmatrix} \end{array} \qquad 1 \leqslant r < s \leqslant p$$

$$\sqrt{-1}(E_{rs} + E_{sr}) = \begin{array}{c c} & \begin{array}{ccc} r & s & p \end{array} \\ \begin{array}{c} r \\ s \\ p \end{array} & \begin{bmatrix} & \sqrt{-1} & \\ \sqrt{-1} & & \\ & & \end{bmatrix} \end{array} \qquad 1 \leqslant r < s \leqslant p$$

$$E_{rs} - E_{sr} = \begin{array}{c c} \begin{bmatrix} & & \\ & & 1 \\ & -1 & \end{bmatrix} & \begin{array}{c} p+1 \\ r \\ s \end{array} \end{array} \qquad p + 1 \leqslant r < s \leqslant n$$

$$\sqrt{-1}(E_{rs} + E_{sr}) \begin{array}{c c} \begin{bmatrix} & & \\ & & \sqrt{-1} \\ & \sqrt{-1} & \end{bmatrix} & \begin{array}{c} p+1 \\ r \\ s \end{array} \end{array} \qquad p + 1 \leqslant r < s \leqslant n$$

$p+1 \quad r \quad s$

And $\mathcal{P}$ has a basis consisting of

$$E_{rs}+E_{sr}=\begin{array}{cc} \begin{array}{c} \qquad\qquad p+1 \end{array} & \\ \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} & p+1 \end{array} \qquad \begin{array}{c} 1\leqslant r\leqslant p \\ p<s\leqslant n \end{array}$$

$$\sqrt{-1}(E_{rs}-E_{sr})=\begin{array}{cc} \begin{array}{c} \qquad\qquad p+1 \end{array} & \\ \begin{bmatrix} & & \sqrt{-1} \\ & & \\ \sqrt{-1} & & \end{bmatrix} & p+1 \end{array} \qquad \begin{array}{c} 1\leqslant r\leqslant p \\ p<s\leqslant n \end{array}$$

(2.2.2). The Killing form is given by

$$B(X,Y)=2n\operatorname{tr}(X,Y).$$

We choose the metric to be

$$e_G(X,Y)=-\frac{1}{2}\operatorname{tr}(X,Y).$$

Let ω_G, ω_K, ω_X be the volumes form with respect to the metric e of G, K, $X=K\backslash G$. If we write the basis of $\mathcal{K}$ (in a suitable order) as $e_1,\ldots,e_{n^2-2pq-1}$ and the basis of $\mathcal{P}$ as $e_{n^2-2pq},\ldots,e_{n^2-1}$, then the matrix $((e_i,e_j))$, $1\leqslant i,j\leqslant n^2-1$ is

$$\begin{bmatrix} 1 & -\frac{1}{2} & & & & & & \\ -\frac{1}{2} & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & -\frac{1}{2} & & & \\ & & & -\frac{1}{2} & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} n^2-2pq$$

and

$$\begin{aligned} \omega_G&=\det\left(e(e_i,e_j)_{1\leqslant i,j\leqslant n^2-1}\right)^{1/2} e_1^*\wedge\ldots\wedge e_{n^2-1}^* \\ &=\sqrt{n/2^{n-1}}\qquad e_1^*\wedge\ldots\wedge e_{n^2-1}^*. \end{aligned}$$

Similarly we have

$$\begin{aligned} \omega_K&=n/2^{n-1}\cdot e_1^*\wedge\ldots\wedge e_{n^2-2pq-1}^*, \\ \omega_X&=e_{n^2-2pq}^*\wedge\cdots\wedge e_{n^2-1}^* \end{aligned}$$

(2.3). Suppose that the rational prime $p \equiv 3 \bmod 4$. Then p is unramified in $\mathbb{Q}(\sqrt{-1})$. Define

$$G_p(j) := \{g \in G(\mathbb{Z}_p) \mid g \equiv I \bmod p^j\};$$
$$\mathcal{G}_p(j) := \{A \in \mathcal{G}(\mathbb{Z}_p) \mid |A|_p \leqslant p^j\};$$

where
$$\mathcal{G}(\mathbb{Z}_p) = \left(\mathcal{G} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p\right) \cap M(n, \mathbb{Z}_p)$$

and
$$|(a_{ij})|_p = \max\{|a_{ij}|_p : 1 \leqslant i, j \leqslant n\}.$$

The following lemma is well known ([1], Chapter III.7).

Lemma 2.3.1. *For $j \geqslant 1$, the exponential map $\exp(A) = \sum_{r=0}^{\infty} A^r/r!$, defines an isomorphism*

$$\exp : \mathcal{G}_p(j) \to G_p(j)$$

of analytic manifolds.

Remark. The above lemma remains true for $p = 2$ and $j \geqslant 2$.

It follows from (2.3.1) that with respect to the measure ω_p defined by the Chevalley basis $e_1, \ldots, e_{n^2-1}$ we get the following formula.

Lemma 2.3.2.
$$\mathrm{vol}_{\omega_p}(G_p(j)) = p^{-j(n^2-1)}.$$

An immediate corollary is:

Lemma 2.3.3.

$$\mathrm{vol}_{\omega_p}(G(\mathbb{Z}_p)) = p^{-j(n^2-1)}[G(\mathbb{Z}_p) : G_p(j)].$$

Remark. The above formula is true for $p = 2$ if $j \geqslant 2$.

We have an isomorphism

$$G(\mathbb{Z}_p)/G_p(1) \approx G(\mathbb{Z}/p\mathbb{Z}).$$

The group is of type ${}^2A_{n-1}$. It is well known [2], that

$$|G(\mathbb{Z}/p\mathbb{Z})| = p^{n^2-1} \prod_{r=2}^{n} \left(1 - (-p)^{-r}\right).$$

Therefore

Lemma 2.3.4. *When $p \equiv 3 \bmod 4$,*

$$\mathrm{vol}_{\omega_p}(G(\mathbb{Z}_p)) = \prod_{r=2}^{n}\left(1-(-p)^{-r}\right).$$

(2.4). Now if $p \equiv 1 \bmod 4$, then p splits in $\mathbb{Q}(\sqrt{-1})$ $p = \mathcal{P}\overline{\mathcal{P}}$, $\mathcal{P} \neq \overline{\mathcal{P}}$, (say). In this case $G(\mathbb{Z}_p)$ is isomorphic with $SL(n, \mathbb{Z}_p)$. Well known formulas ([**2**]) give

Lemma 2.4.1. *When $p \equiv 1 (\bmod 4)$*

$$\mathrm{vol}_{\omega_p} G(\mathbb{Z}_p) = \prod_{r=2}^{n}(1-p^{-r}).$$

2.5. We come to the case $p = 2$. It is well-known that a hermitian matrix H with coefficients over $\mathbb{Z}[\sqrt{-1}]$ is equivalent to one of the following three matrices

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & \sqrt{-1} & & & & & & \\ \sqrt{-1} & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}$$

(See Lee [**6**]). As in (2.3), it reduces to the computation of the order of the finite group $SU(H, \mathcal{O}/2^j\mathcal{O})$. Here $\mathcal{O}$ is $\mathbb{Z}[\sqrt{-1}]$.

In the case where H is the identity matrix, this is given in Zeltinger (see [**10**]).

Lemma 2.5.1. *Let $G = SU(I_n)$; then we have*

$$\mathrm{vol}_{\omega_2}\left(G\left(\mathbb{Z}_{(2)}\right)\right) = 2^{-n+1}\prod_{r=1}^{[(n-1)/2]}(1-2^{-2r}).$$

For the two remaining cases, we first consider the unitary group $U(H, \mathcal{O}/2\mathcal{O})$, where we write $\mathcal{O}$ for the ring $\mathbb{Z}[\sqrt{-1}]$. Let $\mathcal{I}$ be the ideal of generated by $1+\sqrt{-1}$ and 2. There is an exact sequence

$$0 \to \mathcal{I}/2\mathcal{O} \to \mathcal{O}/2\mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0 \tag{2.5.2}$$

where $\mathcal{I}/2\mathcal{O}$ is cyclic of order 2 generated by $1+\sqrt{-1}$ and $\mathcal{O}/\mathcal{I}$ is isomorphic to the field of 2 elements, $\mathcal{O}/\mathcal{I} \approx \mathsf{F}_2$. It follows that $|\mathcal{O}/2\mathcal{O}| = 4$. This can also be seen from the fact that

$$\mathcal{O}/2\mathcal{O} = \{a + bT \mid T^2 = 1,\, a,\, b \in \mathsf{F}_2\}.$$

Denote by V the free $\mathcal{O}$-module $\mathcal{O}^n$ of rank $n = 2m$. An element x in V is said to be primitive if $\mathcal{O}x$ is a direct summand in V. An equivalent condition for x to be primitive is that $x \not\equiv 0 \bmod \mathcal{I}$. Let $P(V)$ be the set of primitive elements in V. From (2.5.2), we get an exact sequence

$$0 \to V \otimes \mathcal{I}/2\mathcal{O} \to V \otimes \mathcal{O}/2\mathcal{O} \to V \otimes \mathcal{O}/\mathcal{I} \to 0;$$

since $|V \otimes \mathcal{I}/2\mathcal{O}| = |V \otimes \mathcal{O}/\mathcal{I}| = 2^n$, it follows that

$$|P(V)| = 2^n(2^n - 1). \tag{2.5.3.}$$

Lemma 2.5.4. *Let H be the hermitian matrix*

$$H = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}.$$

Then the unitary group $U(H_{2m}, \mathcal{O}/2\mathcal{O})$ acts transitively on $S(V)$.

Proof: Given any x in $S(V)$, there exists a y in $S(V)$ such that $H(x, y) = 1$. For otherwise, $H(\overline{x}, \overline{y}) = 0$ for all $\overline{y}$ in $\mathcal{O}/\mathcal{I}$ and this would contradict x in $S(V)$.

Let V_0 be the subspace generated by x and y and let $V_0^{\perp}$ be its orthogonal complement, $V = V_0 \oplus V_0^{\perp}$. There exists a basis $\{x_1, y_1, \dots x_{m-1}, y_{m-1}\}$ in V such that with respect to the combined basis $\{x, y, x_1, y_1, \dots, x_{m-1}, y_{m-1}\}$, the hermitian matrix H takes the following form

$$\begin{bmatrix} & I_m \\ I_m & \end{bmatrix}.$$

It follows that there exists an isometry in $U(H_{2m}, \mathcal{O}/2\mathcal{O})$ which brings the element $e_1 = (1, 0, \dots, 0)$ to x.

Let $U(e_1)$ denote the isotropy subgroup in $U(H_{2m}, \mathcal{O}/2\mathcal{O})$ which keeps the element $e_1 = (1, 0, \dots, 0)$ in $P(V)$ fixed. Comparing this with the definition of a maximal parabolic subgroup, it is not difficult to see that every element in $U(e_1)$ has a unique product decomposition

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ x_2 & & & \\ \vdots & & & \\ x_m & I_{m-1} & 0 & 0 \\ z & y_2 \dots y_m & 1 & -\overline{x}_2 \dots -\overline{x}_m \\ y_2 & & & \\ \vdots & 0 & 0 & I_{m-1} \\ y_m & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & & & \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ & & & \\ 0 & C & 0 & D \\ & & & \end{bmatrix}$$

(Langlands' decomposition). Since in the first unipotent matrix, the entries $x_2, \dots, x_m$, $y_2, \dots, y_m$ can be any abitrary elements in $(\mathcal{O}/2\mathcal{O})^{2m-2}$, and $z = \overline{z}$, skew-symmetric elements in $\mathcal{O}/2\mathcal{O}$, it follows that

$$\begin{aligned} |U(e_1)| &= 2^2.2^{2(m-1)}2^{2(m-1)}.\left|U\left(H_{2(m-1)}, \mathcal{O}/2\mathcal{O}\right)\right|, \\ &= 2^{4m-2}.\left|U\left(H_{2(m-1)}, \mathcal{O}/2\mathcal{O}\right)\right|. \end{aligned}$$

Hence we obtain

$$\begin{aligned} |U(H_{2m}, \mathcal{O}/2\mathcal{O})| &= |S(V)|.|U(e_1)|, \\ &= 2^{8m-2}.(1-2^{-2m}).\left|U\left(H_{2(m-1)}, \mathcal{O}/2\mathcal{O}\right)\right|, \\ &= 2^{n^2+n}\prod_{r=1}^{[n/2]}(1-2^{-2r}). \end{aligned}$$

Let U_{2^j} denote the subgroup of level 2^j in $U(H, \mathcal{O})$:

$$U_2j = \{g \in U(H, \mathcal{O}) \mid g \equiv I \bmod 2^j\mathcal{O}\}.$$

Let $\mathfrak{u}(H, \mathcal{O}/2\mathcal{O})$ be the Lie "algebra" (strictly speaking, this is a Lie ring over $\mathcal{O}/2\mathcal{O}$ but a Lie algebra over F_2) of $2m \times 2m$ matrices X over $\mathcal{O}/2\mathcal{O}$ such that

$$X.H + H.\overline{X}^t = 0.$$

Then there is an exact sequence

$$1 \to U_{2^j} \hookrightarrow U_{2^{j-1}} \xrightarrow{\phi} \mathfrak{n}(H, \mathcal{O}/2\mathcal{O}) \to 1$$

where the map ϕ is defined by the formula $\phi(g) = (g-I)/2^{j-1}$.

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A straightforward computation shows that

$$[U_{2j-1} : U_{2j}] = |\mathrm{n}(H, \mathcal{O}/2\mathcal{O})| = 2^{n^2}$$

and so

$$[U_2 : U_{2j}] = 2^{n^2(j-1)}.$$

□

Lemma 2.5.7. *Let $U(H_{2m}, \mathcal{O})$ and U_{2j} be defined as above. Then*

$$[U(H_{2m}, \mathcal{O}) : U_{2j}] = 2^{n^2 j+n} \prod_{r=1}^{[n/2]} (1 - 2^{-2r}).$$

Proof: Use (2.5.5) and (2.5.6). □

The above formula also works for the unitary group $U(H'_{2m}, \mathcal{O})$ where

$$H'_{2m} = \begin{bmatrix} 0 & \sqrt{-1} & & & & & \\ \sqrt{-1} & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

This is because H_{2m} and H'_{2m} are GL-equivalant to each other, and so the corresponding unitary groups are conjugate to each other.

(2.5.8)
$$[U(H'_{2m}, \mathcal{O}) : U_{2j}] = 2^{n^2 j+n} \prod_{r=1}^{[n/2]} (1 - 2^{-2r}).$$

As for the special unitary group SU, we consider the exact sequence

(2.5.9)
$$1 \to SU(H, \mathcal{O}/2^j\mathcal{O}) \to U(H, \mathcal{O}/2^j\mathcal{O}) \overset{\det}{\to} \mathcal{U} \to 1$$

where H can be either H_{2m} or H'_{2m}, and $\mathcal{U}$ is the norm group, $\mathcal{U} = \{\xi\bar{\xi} \mid \in \mathcal{O}/2^j\mathcal{O}\}$. □

Proposition 2.5.10. *Let G be the special unitary $SU(H)$ where H is one of the following hermitian matrices:*

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{-1} & & & & & \\ \sqrt{-1} & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

Write G_{2^j} for the subgroup of level 2^j in $G(\mathbb{Z}_2)$. Then

$$[G(\mathbb{Z}_2) : G(2^j)] = 2^{j(n^2-1)+n} \prod_{r=1}^{[n/2]} (1 - 2^{-2r}), \quad j \geqslant 2$$

and
$$\mathrm{vol}_{\omega_2}(G(\mathbb{Z}_2)) = 2^n . \prod_{r=1}^{[n/2]} (1 - 2^{-2r}).$$

PROOF: There is a homomorphism

$$\begin{aligned} \mathcal{O}/2^j\mathcal{O}^\times &\to \mathcal{U} \\ \xi &\to \xi\bar{\xi} \end{aligned}$$

of the group of units onto $\mathcal{U}$, and the kernel of this homomorphism is the subgroup of norm 1 elements in $\mathcal{O}/2^j\mathcal{O}$. In other words, we have

$$\mathcal{U} = \frac{GL(1, \mathcal{O}/2^j\mathcal{O})}{U(1, \mathcal{O}/2^j\mathcal{O})}.$$

As computed before, we have

$$\begin{aligned} |GL(1, \mathcal{O}/2^j\mathcal{O})| &= 2^{2j-1}, \\ |U(1, \mathcal{O}/2^j\mathcal{O})| &= 2^{j+1}, \qquad j \geqslant 2 \end{aligned}$$

and so
$$|\mathcal{U}| = 2^{j-2}.$$

The first formula in (2.5.10) follows from our previous computation of $|U(H, \mathcal{O}/2^j\mathcal{O})|$, the exact sequence (2.5.9), and the above formula for $\mathcal{U}$.

As for the second formula, we have

$$\begin{aligned} \mathrm{vol}_{\omega_2}(G(\mathbb{Z}_2)) &= 2^{-j(n^2-1)}[G(\mathbb{Z}_2) : G(2^j\mathbb{Z})] \\ &= 2^{-j(n^2-1)} . 2^{j(n^2-1)+n} \prod_{r=1}^{[n/2]} (1 - 2^{-2r}) \\ &= 2^n \prod_{r=1}^{[n/2]} (1 - 2^{-2r}). \end{aligned}$$

$\square$

3. Sum over a class

In this section we use Rohlf's exact sequence (1.4) to sum up the $\chi(H)$ for those cohomology classes H which have the same image under the Hasse map h, that is, we first sum over a group class in C. We fix an hermitian matrix with coefficient in $\mathbb{Z}[\sqrt{-1}]$ and G in the special unitary group.

(3.1). Fix a maximal compact subgroup K of $G(\mathbf{R})$, let X_H be the symmetric space $K\setminus G(\mathbf{R})$, G_0 be the connected compact real form of G ([4], III Section 6), X_0 be the compact dual $K\setminus G_0(\mathbf{R})$ of X_H, $j\colon X_H \to X_0$ be the Borel embedding, and let Γ_H be the congruence subgroup of $G(\mathbf{Q})$ of level $\ell \neq 2$, ℓ prime.

Let ω (respectively ω_0) be the right-invariant volume form on X_H (respectively X_0) determined by the Riemannian metric. Then we can prove the following lemma.

Lemma 3.1.1. *If the hermitian form H has signature (p, q), $p+q=n$, then*

$$\chi(X_H/\Gamma_H) = \frac{(-1)^{pq}\binom{p+q}{q}}{\mathrm{vol}_{\omega_0} SU(n)} \int_{G(\mathbf{R})/\Gamma_H} \omega.$$

Proof: By the Gauss-Bonnet theorem according to Harder [3], there exists on X_H a G-right invariant differential form of degree $m = \dim_{\mathbf{R}} X_H$ such that

$$\chi(X_H/\Gamma_H) = \int_{X_H/\Gamma_H} \omega_\chi.$$

The same is true for X_0 with respect to $\omega_{0\chi}$.

Let $\mathcal{G}$ (respectively $\mathcal{G}_0$) be the Lie algebra of G (respectively G_0). Let $B(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ be the Killing form on $\mathcal{G}_{\mathbf{C}}$. Then B (respectively $-B$) defines a homogenous symmetric Riemannian metric on X_H (respectively X_0). Let R (respectively R_0) be the corresponding Riemannian curvature tensor. Then it follows from Cartan's formula [5] that

$$j^* R_0 = -R$$

at the "origin". As the Gauss-Bonnet form can be computed as the pfaffian of the curvature tensor [5], we see that, at the "origin",

$$j^*\omega_{0\chi} = (-1)^{pq}\omega_\chi$$

where $(p, q) =$ signature of H. Let ω (respectively ω_0) be the right-invariant volume form on X_H (respectively X_0) determined by the Riemannian metric above and suppose that

$$\omega_{0\chi} = c\omega_0.$$

Then

$$\omega_\chi = (-1)^{pq} c\omega.$$

By the classical Gauss-Bonnet theorem,

$$c = \chi(X_0)\,\mathrm{vol}_{\omega_0}(X_0)^{-1};$$

using

$$\int_{X_H/\Gamma_H} \omega = \mathrm{vol}_\omega(K)^{-1} \int_{G(\mathbf{R})/\Gamma_H} \omega$$

and the fact that G_0 and $SU(n)$ are conjugate in $SU(H, \mathbf{C})$ and so have the same volume, we get the lemma. ∎

(3.2). Write $\Gamma_\infty = SL(n,\mathbb{C})$, $\Gamma_v = SL(n,\mathbb{Z}_v[\sqrt{-1}])$ $(v\neq\ell)$, $\Gamma_\ell = \ker\left(SL(n,\mathbb{Z}_\ell[\sqrt{-1}])\to SL(n,(\mathbb{Z}/\ell\mathbb{Z})[\sqrt{-1}])\right)$, $\Gamma = \Gamma_\infty\cap SL(n,\mathbb{Q}(\sqrt{-1}))$, $\Gamma_H = \Gamma\cap G(\mathbb{Q})$, $\Gamma_{H,v} = \Gamma_v\cap G(\mathbb{Q}_v)$.

Lemma 3.2.1. *Let $\omega = \Pi_v\omega_v$ be the Tamagawa measure of G. Then there exists $g_i \in SL(n,\mathbb{Q}(\sqrt{-1}))$, $1\leqslant i\leqslant n(H)$ such that*

$$\sum_{i=1}^{n(H)} \mathrm{vol}_{\omega_\infty} G(\mathbb{R})/g_i\Gamma g_i^{-1}\cap G(\mathbb{Q}) = \prod_{v\neq\infty}(\mathrm{vol}_{\omega_v}\Gamma_{H,v})^{-1}.$$

Proof: There exists $y_i\in G(\mathbb{A})$ such that

$$G(\mathbb{A}) = \bigcup_{i=1}^{n(H)}\left(\prod_v \Gamma_{H,v}\right)y_i^{-1}G(\mathbb{Q}).$$

By strong approximation we can write $y_i = g_iu_i$ with g_i in $SL(n,\mathbb{Q}(\sqrt{-1}))$ and $u_i\in\prod_v\Gamma_v$. Then

$$y_i\left(\prod_v\Gamma_{H,v}y_i^{-1}\cap G(\mathbb{Q})\right) = g_i\Gamma g_i^{-1}\cap G(\mathbb{Q})$$

so we have

$$\left(\prod_v\Gamma_{H,v}\right).y_i^{-1}G(\mathbb{Q}) = y_i^{-1}\left(y_i\left(\prod_v\Gamma_{H,v}\right)y_i^{-1}/g_i\Gamma g_i^{-1}\cap G(\mathbb{Q})\right)\times G(\mathbb{Q}).$$

From
$$y_i\left(\prod_v\Gamma_{H,v}\right)y_i^{-1} = G(\mathbb{R})\times\prod_{v\neq\infty}y_{i,v}\Gamma_{H,v}y_{i,v}^{-1}$$

and the fibration

$$y_i\left(\prod_v\Gamma_{H,v}\right)y_i^{-1}/g_i\Gamma g_i^{-1}\cap G(\mathbb{Q})\to G(\mathbb{R})/g_i\Gamma g_i^{-1}\cap G(\mathbb{Q})$$

we get
$$G(\mathbb{A})/G(\mathbb{Q}) = \bigcup_{i=1}^{n(H)} y_i^{-1}\left((G(\mathbb{R})/g_i\Gamma g_i^{-1}\cap G(\mathbb{Q}))\times\prod_{v\neq\infty}y_{i,v}\Gamma_{H,v}y_{i,v}^{-1}\right).$$

If $\omega = \prod_v\omega_v$ is the Tamagawa measure for G then (by [9], pp.99, 72, 23)

$$\mathrm{vol}_\omega(G(\mathbb{A})/G(\mathbb{Q})) = 1.$$

So the lemma follows. ∎

(3.3). Write $H^1(\mathfrak{g}_H, \Gamma)$ for the nonabelian cohomology with the non-trivial element σ of $\mathfrak{g}$ acting as: $A \to H\overline{A}^{t-1}H^{-1}$. Then we get, by twisting Rohlf's exact sequence (1.4), an exact sequence

$$1 \to G(\mathbb{Q}) \setminus G(\mathbb{A}) / \prod_v \Gamma_{H,v} \to H^1(\mathfrak{g}_H, \Gamma) \overset{h_H}{\to} \coprod_v H^1(\mathfrak{g}_H, \Gamma_v).$$

The map h_H is prescribed in the following manner: given a class

$$G(\mathbb{Q})y^{-1} \prod_v \Gamma_{H,v}$$

use strong approximation to get $y = gx$ with $g \in SL(n, \mathbb{Q}(\sqrt{-1}))$ and $x \in \prod_v \Gamma_{H,v}$. Put $C = g^{-1}.H\overline{g}^{t-1}H^{-1}$. Then $h_H\left(G(\mathbb{Q})y^{-1} \prod_v \Gamma_{H,v}\right)$ is the cohomology class represented by $(1, C)$. If h is the Hasse map of (1.4), we can now consider the contribution to $\mathcal{L}$ by the $\chi(C)$ for C in the fibre $h^{-1}(h(H))$. Clearly $h(C) = h(H)$ if and only if $h_H(C) = 1$. The next lemma now follows immediately from the preceding lemmas of this section.

Lemma 3.3.1. *Given an integral hermitian form H of signature (p, q), $p+q = n$. Then*

$$\sum \chi(C) = \frac{(-1)^{pq}\binom{p+q}{q}}{\mathrm{vol}_{\omega_0}(SU(n))} \prod_{v \neq \infty} \mathrm{vol}_{\omega_v}(\Gamma_{H,v})^{-1}.$$

Here the sum is extended over those $C \in H^1(\mathfrak{g}, \Gamma)$ such that $h(C) = h(H)$.

4. The Lefschetz number

In this section we assemble together the computation when Γ is a congruence subgroup of level ℓ.

(4.1). The first cohomology $H^1(\mathfrak{g}, \Gamma_2)$ classifies equivalence classes of integral 2-adic hermitian forms. Using Gaussian elimination it is elementary to show that $H^1(\mathfrak{g}, \Gamma_2)$ is represented by three elements, namely, I, S, V as in (2.5). (See [6].)

Also $H^1(\mathfrak{g}, \Gamma_\infty)$ is classified by the set of integers (p, q) such that $p + q = n$ and $H^1(\mathfrak{g}, \Gamma_p) = 1$ for $p \neq 2, \infty$.

(4.2). Let $h_2 \colon H^1(\mathfrak{g}, \Gamma_2) \to H^1(\mathfrak{g}, SL(n, \mathbb{Z}_2[\sqrt{-1}]))$ be the cohomology map induced by the inclusion $\Gamma_2 \to SL(n, \mathbb{Z}_2[\sqrt{-1}])$. Using Rohlf's exact sequence (1.4) and the remarks in (4.1), we can write

$$\mathcal{L} = \sum_{h_2(\gamma)=E} \chi(\gamma) + \sum_{h_2(\gamma)=S} \chi(\gamma) + \sum_{h_2(\gamma)=V} \chi(\gamma).$$

Now we can use the results on summing over a class (Section 3) and the local volume computations (Section 2) to get the following formulae immediately. We write $|T|$ for the cardinality of a set T. Let ψ be the quadratic character $(-4/\cdot)$ attached to $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$, that is

$$\psi(p)\begin{cases} -1 & \text{if } \; p \equiv 3 \bmod 4 \\ 0 & \text{if } \; p = 2 \\ 1 & \text{if } \; p \equiv 1 \bmod 4. \end{cases}$$

Let $a(n)$ denote the following product.

$$\ell^{n^2-1}\prod_{r=2}^{n}\left(1-\frac{1}{(\psi(\ell)\ell)^r}\right)\frac{\Gamma(r)}{2\pi^r}\prod_{p\equiv 3(4)}\left(1-\frac{1}{(-p)^r}\right)^{-1}\prod_{p\equiv 1(4)}\left(1-\frac{1}{p^r}\right)^{-1}.$$

(4.2.1). The sum over $h_2(\gamma) = E$ is

$$a(n)2^{n+1}\prod_{r=1}^{(n-1/2)]}\left(1-2^{-2r}\right)^{-1}\left|h_2^{-1}(E)\right|.$$

(4.2.2). The sum over $h_2(\gamma) = S$ is

$$a(n)2^{-n}\prod_{r=1}^{[n/2]}\left(1-2^{-2r}\right)^{-1}\left|h_2^{-1}(V)\right|.$$

(4.2.3). The sum over $h_2(\gamma) = V$ is

$$a(n)2^{-n}\prod_{r=1}^{[n/2]}\left(1-2^{-2r}\right)^{-1}\left|h_2^{-1}(V)\right|.$$

(4.3). We now apply the functional equation of the Riemann ζ-function and the Dirichlet L-function:

$$\frac{\Gamma(2r)}{\pi^{2r}}\zeta(2r) = (-1)^r \,.\, 2^{2r-1} \,.\, \zeta(1-2r)$$
$$\frac{(2r+1)}{2r+1}L(2r+1,\psi) = (-1)^r 2^{-(2r+1)} L(-2r,\psi)$$

and we get our Theorem 1.6.

5. Remarks

The number $\mathcal{L}$ computed in this note is indeed the Lefschetz number of an involution on a symmetric space.

The symplectic group $Sp(2n)$ is the group of $2n \times 2n$ invertible matrices A such that

$$A\,J\,A^t = J$$

where
$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let Γ be the congruence subgroup of level ℓ inside the symplectic group $Sp(2n)$ of $2n$ varibles, that is

$$\Gamma = \operatorname{Ker}(Sp(2n, \mathbb{Z}) \to Sp(2n, \mathbb{Z}/\ell\mathbb{Z})).$$

Then Γ acts on the Siegel upper half space $\mathfrak{S}_n$ of degree n, that is,

$$\mathfrak{S}_n = \{Z \in M_n(\mathbb{C}) \mid Z = Z^t, \operatorname{Im} Z > 0\}.$$

The Cartan involution τ which takes a matrix A to the inverse of its transpose induces maps on the singular cohomology with rational coefficients:

$$\tau^i \colon H^i(\mathfrak{S}_n/\Gamma, \mathbb{Q}) \to H^i(\mathfrak{S}_n/\Gamma, \mathbb{Q}).$$

The Lefschetz number of τ is

$$\mathcal{L}(\tau) = \sum_{i=0}^{\infty} (-1)^i \operatorname{trace} \tau^i.$$

Write $(\mathfrak{S}_n/\Gamma)^\tau$ for the fixpoint set of the action of τ on the locally symmetric space $\mathfrak{S}_n/\Gamma$. The Lefschetz formula gives

$$\mathcal{L}(\tau) = \chi((\mathfrak{S}_n/\Gamma)^\tau).$$

Now observe that for a symmetric matrix $B \in \Gamma$, BJ is of order 4. This allows us to change the underlying ring from the rational integers to the Gaussian integers and replace the symplectic form J by a hermitian form. The fixpoint components then become locally symmetric spaces attached to special unitary groups. The number $\mathcal{L}$ computed in Theorem 1.6 is in fact the number $\mathcal{L}(\tau)$ above. This will be discussed in a paper written jointly with R. Lee.

References

[1] N. Bourbaki, *Groups et algebrès de Lie* (Hermann, Paris, 1972).

[2] R.W. Carter, *Simple groups of Lie type* (Wiley, New York, 1972).

[3] G. Harder, 'A Gauss-Bonnet formula for discrete arithmetically defined groups', *Ann. Sci. École Norm. Sup.* 4 (1971), 409–455.

[4] S. Helgason, *Differential geometry, Lie groups and symmetric spaces* (Academic Press, New York, 1978).

[5] F. Hirzebruch, 'Automorphe Formen und der Sat von Riemann Roch', in *Symposium International Topology and Algebra*, pp. 129–144 (Univ. de Mexico, 1958).

[6] R. Lee, 'Computation of Wall groups', *Topology* **10** (1971), 149–176.

[7] J. Rohlfs, 'Arithmetisch definierte Gruppen mit Galoisoperation', *Invent. Math.* **48** (1978), 185–205.

[8] J. Rohlfs, 'Lefschetz number of an involution on the space of classes of positive definite quadratic forms', *Comment. Math. Helv.* **56** (1981), 272–296.

[9] A. Weil, *Adeles and algebraic groups* (Birkhauser, Boston, 1982).

[10] H. Zeltinger, *Spitzenanzahlen und Volumina Picardscher Modulvanetaten* 136 (Bonner Math. Schriften, Bonn, 1981).

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