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Equivariant Formality for Actions of Torus Groups

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Abstract. This paper contains a comparison of several definitions of equivariant formality for actions of torus groups. We develop and prove some relations between the definitions. Focusing on the case of the circle group, we use S^1-equivariant minimal models to give a number of examples of S^1-spaces illustrating the properties of the various definitions.

1 Introduction

The idea of a formal space is a space whose rational homotopy type is determined by its rational cohomology ring, indicating relatively simple geometric properties and also making calculations much easier. Many classes of interesting spaces turn out to be formal, including Lie groups, classifying spaces and compact 1-connected Kähler manifolds; see for example the discussion in [6].

Various authors have extended the concept of formality to the equivariant setting. One definition was used in the influential paper of Goresky, Kottwitz and MacPherson [8] for actions of torus groups. For finite group actions, Triantafillou and Fine have an alternate approach in [7], and a definition following this spirit was given for actions of circle groups by the author in [12]. Lillywhite [9] has recently presented yet another approach for actions of general Lie groups.

This paper looks at actions of torus groups, denoted $\mathbb{T}$. In this context, we compare the various definitions, and show some relations between them. Specializing to the circle $\mathbb{T}^1$, we give a number of examples of $\mathbb{T}^1$-spaces which are formal in one sense or another, illustrating the difference between the definitions. The main technique for producing examples will be to use the $\mathbb{T}^1$-minimal models developed by the author in [11], and the assorted concepts of formality will also be interpreted via these models.

Throughout this paper, we will assume that all cohomology has rational coefficients, and all homotopy groups are rational homotopy groups. The organization of the paper is as follows. Section 2 contains an outline of the main results of [11], with the definition of $\mathbb{T}^1$-minimal models and their properties; and also discusses the generalization of these ideas to other Abelian compact Lie groups. Section 3 contains the assorted definitions of $\mathbb{T}$-equivariant formality: for circle actions, the definition of the author from [12] using $\mathbb{T}^1$-minimal models, and some of its properties, extending this to actions of higher tori and the other definitions mentioned in the second

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paragraph. Some comparison results relating different definitions of $\mathbb{T}$-equivariant formality are given in Section 4. Section 5 focuses on the case of the circle group, and contains a number of examples of $\mathbb{T}^1$-spaces exhibiting different types of formality and illustrating the relations between the various types of formality. Finally, Section 6 contains the proofs of a couple of technical results from Section 3.

2 Equivariant Algebraic Models

This section contains a brief summary of the major results of the author [11] on $\mathbb{T}^1$-minimal models, and the results of Mandell and the author in extending these to Abelian compact Lie groups. These models are used directly in one of the definitions of $\mathbb{T}$-equivariant formality. In addition, the $\mathbb{T}^1$-minimal models are used in this paper to compare the various types of formality and to produce examples.

Throughout this paper, all $\mathbb{T}$-spaces are assumed to be $\mathbb{T}$-CW complexes; note that this ensures that all orbit spaces and related constructions, such as Borel spaces, are also CW complexes. In addition, all $\mathbb{T}$-spaces are assumed to have finitely many orbit types. We also assume that all $\mathbb{T}$-spaces are $\mathbb{T}$-simply connected in the sense that the fixed point subspaces $X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}$ are all connected and simply connected (and also non-empty). Lastly, we assume that the rational cohomology of each fixed point subspace X^H is of finite type. We refer to spaces satisfying all of these conditions as $\mathbb{Q}$-good, and this will be a standing assumption in what follows.

In general, the equivariant homotopy type of a G-space X depends not only on the homotopy type of the space itself but also on the homotopy type of all the fixed point subspaces X^H for closed subgroups $H \leq G$. Together with the natural inclusions and maps induced by the action of G, these form a diagram of spaces. Studying this diagram is a standard equivariant technique, going back to a theorem by Bredon [3] which states that a map which induces isomorphisms on all homotopy groups of the diagram of fixed point subsets is an equivariant homotopy equivalence. The shape of the diagram is described by the orbit category $\mathcal{O}_G$. This category has as objects the canonical orbits G/H, and morphisms given by the equivariant maps between them. The fixed point subspaces X^H thus form a functor from $\mathcal{O}_G$ to spaces, and this functor completely determines the equivariant homotopy type, as shown by Elmendorf in [5]. When considering discrete algebraic invariants, we look at functors from the discrete homotopy category $h\mathcal{O}_G$ instead, which has the same objects G/H with homotopy classes of maps between them. Note that the objects of $\mathcal{O}_G$ will be abbreviated from G/H to H for simplicity of notation.

Observe that if X is a G-space and H is *not* an isotropy type of X, then the fixed set X^H consists of a union of the fixed sets $\{X^K\}$ contained in it for $H \supset K$. Since the value of an algebraic functor at a subgroup H reflects H-fixed information, a space X corresponds to an algebraic functor with the following property: for all closed subgroups H which are not isotropy subgroups of X, the value of the functor $A(H)$ is determined by the value at the subgroups $K \supset H$ by the equation $A(H) = \lim_{K \supset H}(K)$. If the space has finitely many orbit types, the value of $A(H)$ will be determined in this way for all but a finite number of subgroups; such a functor will be said to have finitely many orbit types.

In the case of the circle group $G = \mathbb{T}^1$, this indexing category can be described quite simply. Objects are canonical orbits $\mathbb{T}^1/H$ for subgroups $H = \mathbb{Z}/n$ or $H = \mathbb{T}^1$. Group theory tells us that any equivariant map between orbits $\mathbb{T}^1/H \to \mathbb{T}^1/K$ is of the form $\hat{a}\colon gH \to gaK$ for some $a \in \mathbb{T}^1$ for which $a^{-1}Ha \subseteq K$; since $\mathbb{T}^1$ is Abelian this is equivalent to $H \subseteq K$. Two such maps $\hat{a}$ and $\hat{b}$ are the same if and only if $aK = bK$, that is, $ab^{-1} \in K$. Thus the orbit category $\mathcal{O}_{\mathbb{T}^1}$ has morphisms

$$\mathrm{Hom}\,(H,K) = \begin{cases} \mathbb{T}^1/K & \text{if } H \subseteq K \\ \varnothing & \text{otherwise.} \end{cases}$$

All the equivariant maps from $\mathbb{T}^1/H$ to $\mathbb{T}^1/K$ are homotopic, since $\mathbb{T}^1$ is connected; so the homotopy orbit category $h\mathcal{O}_{\mathbb{T}^1}$ has exactly one morphism from H to K if $H \subseteq K$ and no other morphisms. This gives the shape of the diagram category we use to study $\mathbb{T}^1$-spaces.

For the circle group, the algebraic category used to model the rational homotopy of $\mathbb{T}^1$-spaces is a category of functors from $h\mathcal{O}_{\mathbb{T}^1}$ to commutative differential graded algebras (CDGAs). In order to algebraically encode the $\mathbb{T}^1$-action, we work with CDGAs which are modules over $H^*(B\mathbb{T}^1) = \mathbb{Q}[c]$, the polynomial ring with a single generator of degree 2. Precisely, the category is given by the following.

Definition 2.1 ([11, Defn. 5.18]) A $\mathbb{T}$-system consists of

(1) A covariant functor $\mathcal{A}$ from $hO_{\mathbb{T}^1}$ to the category of finitely generated CDGAs under $\mathbb{Q}[c]$such that $\mathcal{A}$ has finitely many orbit types, and such that the functor is an *injective* object when regarded by neglect of structure as a functor to rational vector spaces.
(2) A distinguished sub-CDGA $\mathcal{A}_{\mathbb{T}}$ of $\mathcal{A}(\mathbb{T})$ such that the map $\mathcal{A}_{\mathbb{T}} \otimes \mathbb{Q}[c] \to \mathcal{A}(\mathbb{T})$ is a quasi-isomorphism.

A morphism between $\mathbb{T}$-systems $\mathcal{A}$ and $\mathcal{B}$ is a natural transformation such that $\mathcal{A}_{\mathbb{T}}$ lands in $\mathcal{B}_{\mathbb{T}}$.

The restriction to the injective objects of the category makes sense geometrically, and it is needed for the existence of minimal models. To establish the equivariant analogue of minimality, we use the idea of an "elementary extension" (defined in [11], Section 11), which builds systems of CDGAs out of diagrams of vector spaces.

Definition 2.2 ([11, Defn. 3.4]) A system of CDGAs $\mathcal{M}$ is *minimal* if $\mathcal{M} = \bigcup_n \mathcal{M}(n)$ where $\mathcal{M}(0) = \mathcal{M}(1) = \mathbb{Q}$ and $\mathcal{M}(n) = \mathcal{M}(n-1)(\underline{V}_n)$ is an elementary extension of degree n, for some diagram of vector spaces $\underline{V}_n$.

These minimal $\mathbb{T}$-systems are particularly easy to understand. Some important properties of the minimal $\mathbb{T}$-systems are given by the following results.

Proposition 2.3 ([11, Prop. 5.26]) *If $\mathcal{A}$ is a $\mathbb{T}$-system, then there is a minimal $\mathbb{T}$-system $\mathcal{M}$ with a quasi-isomorphism $\mathcal{M} \to \mathcal{A}$.*

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Proposition 2.4 ([11, Prop. 5.23]) *If $f\colon \mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism between two minimal $\mathbb{T}$-systems, then $f \simeq g$ where g is an isomorphism.*

Corollary 2.5 ([11, Cor. 5.24]) *If $\mathcal{M}$ and $\mathcal{M}'$ are two minimal $\mathbb{T}$-systems and $\rho\colon \mathcal{M} \to \mathcal{A}$ and $\rho'\colon \mathcal{M}' \to \mathcal{A}$ are quasi-isomorphisms, then there is an isomorphism $f\colon \mathcal{M} \cong \mathcal{M}'$ such that $\rho' f \simeq \rho$.*

These results show that the minimal $\mathbb{T}$-systems can serve as preferred representatives for quasi-isomorphism classes, since there is a unique minimal $\mathbb{T}$-system in each class; this allows us to make the following definition.

Definition 2.6 ([11, Defn. 5.25]) Let A be a $\mathbb{T}$-system. If $\mathcal{M}$ is a minimal $\mathbb{T}$-system and $\rho\colon \mathcal{M} \to \mathcal{A}$ is a quasi-isomorphism, we say that $\mathcal{M}$ is the *minimal model* of $\mathcal{A}$. Note that Corollary 2.5 implies that the minimal model is unique.

To get a $\mathbb{T}$-system associated to a $\mathbb{T}^1$-space X we use the Borel bundle construction combined with a suitable version of the functor of de Rham differential forms [11, Defn. 4.12] denoted here by Ω.

Definition 2.7 ([11, Defn. 6.27]) Let X be a $\mathbb{Q}$-good $\mathbb{T}^1$-space, and consider the Borel construction $X \times_{\mathbb{T}^1} E\mathbb{T}^1$. Let $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$ be the $\mathbb{T}$-system defined by

$$\underline{\mathcal{E}}_{\mathbb{T}^1}(X)(H) = \Omega(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1),$$

with special sub-CDGA

$$\mathcal{E}_{\mathbb{T}^1} = \Omega(X^{\mathbb{T}^1}) \subset \Omega(X^{\mathbb{T}^1} \times B\mathbb{T}^1) = \underline{\mathcal{E}}_{\mathbb{T}^1}(X)(\mathbb{T}^1),$$

where the inclusion $\Omega(X^{\mathbb{T}^1}) \subset \Omega(X^{\mathbb{T}^1} \times B\mathbb{T}^1)$ is induced by the projection $p_1\colon X^{\mathbb{T}^1} \times B\mathbb{T}^1 \to X^{\mathbb{T}^1}$.

As defined here, $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$ is a $\mathbb{T}$-system, since it is injective as a functor to vector spaces. Thus by Proposition 2.3 it has a minimal model $\mathcal{M}_X$, which by Corollary 2.5 is unique. We say that this minimal model $\mathcal{M}_X$ is the equivariant minimal model of the $\mathbb{T}^1$-space X. There is a quasi-isomorphism $\mathcal{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X)$, and $\mathcal{M}_X$ encodes rational homotopy information. The main theorem of [11] is the following.

Theorem 2.8 ([11, Thm. 6.28]) *Let X be a $\mathbb{Q}$-good $\mathbb{T}^1$-space, and $\mathcal{M}_X$ be the minimal model of $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$. Then the correspondence $X \to \mathcal{M}_X$ induces a bijection between rational homotopy types of $\mathbb{Q}$-good spaces and isomorphism classes of minimal $\mathbb{T}$-systems.*

Moreover, the minimal model $\mathcal{M}_X$ computes geometric information in certain specific ways, as described below [11, Sec. 6, 14–18].

- The cohomology of $\mathcal{M}_X(H)$ is equal to the rational cohomology of the Borel construction of the fixed set X^H, as a module over $H^*(B\mathbb{T}^1) = \mathbb{Q}[c]$. In particular, the Borel cohomology of the space X can be recovered by taking the cohomology of $\mathcal{M}_X(e)$.

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- $\mathcal{M}_X$ is generated by free $\mathbb{Q}[c]$-CDGAs from the diagram of vector spaces given by $\underline{\pi}^*$, the duals of the rational homotopy groups $\underline{\pi}_n(X^H)$.
- The Grivel-Halperin-Thomas theorem [6] implies that the non-equivariant minimal model of $X \times_{\mathbb{T}^1} E\mathbb{T}^1$ is given by $\mathcal{N} \otimes \mathbb{Q}[c]$, where $\mathcal{N}$ is the Sullivan minimal model of X. Therefore for any fixed set X^H, we can recover the Sullivan model $\mathcal{N}_{X^H}$ by taking a minimal model for $\mathcal{M}_X(H)/(c)$ where (c) is the ideal generated by c; and the cohomology of $\mathcal{M}_X(H)/(c)$ is the rational cohomology of X^H.

Thus the quasi-isomorphism class of the $\mathbb{T}$-system determines the rational homotopy type of the $\mathbb{T}^1$-space, and the minimal model provides a concrete way of calculating many rational geometric invariants.

Extending these models to more general Abelian compact Lie groups is done in [10], unfortunately giving algebraic models which are more complicated and less explicit. Instead of the orbit category described above, we need to use a larger category $\mathcal{D}$ which reflects pairs of subgroups $H[K]$ for $K \subseteq H$. The algebraic category used for the algebraic models are functors from $\mathcal{D}$ to CDGA's with a map from a certain initial object $\underline{P}$. The main result for rational homotopy is the following.

Theorem 2.9 ([10, Thm. A]) *Let G be an Abelian compact Lie group. Then there is a functor $\underline{A}$ from the equivariant rational homotopy category of G-spaces to the homotopy category of $\mathcal{D}-$CDGAs under $\underline{P}$. On the full subcategory of $\mathbb{Q}$-good spaces, this functor is full and faithful.*

The basic intuition behind the functor $\underline{A}$ is similar to the case of the circle; we look at a diagram given by taking the Borel construction of various fixed sets $\{X^K \times_{\mathbb{T}/H} E(\mathbb{T}/H)\}$, and the initial object $\underline{P}$ is reflecting a diagram of classifying spaces $B(\mathbb{T}/H)$. If we consider these more complicated objects in the case of the circle $\mathbb{T}^1$, we can simplify the diagram back to the orbit category almost to $hO_{\mathbb{T}^1}$ and take $\underline{P}$ to be the constant functor $\mathbb{Q}[c]$ with zero differential. One additional object is required, which encodes the distinguished sub-CDGA of Definition 2.1, (2). Thus we recover the category of $\mathbb{T}$-systems discussed above.

3 $\mathbb{T}$-Equivariant Formality

Non-equivariantly, formality is defined using commutative differential graded algebras. A CDGA is formal if it is quasi-isomorphic to its cohomology ring, regarded as a CDGA with differential $d = 0$. This means that $\mathcal{A}$ is formal if there is a chain of quasi-isomorphisms $\mathcal{A} \to \mathcal{A}_1 \leftarrow \mathcal{A}_2 \to \cdots \to H^*(\mathcal{A})$. For a minimal CDGA $\mathcal{M}$ this is equivalent to requiring the existence of a single quasi-isomorphism $\mathcal{M} \to H^*(\mathcal{M})$. To define formality for spaces, we can use the minimal model $\mathcal{M}_X$ of a space X defined by Sullivan in [4, 13], which is a CDGA: a space X is formal if its minimal model $\mathcal{M}_X$ is formal. The CDGA $\mathcal{M}_X$ encodes all rational homotopy information, including homotopy and homology groups, and for a formal space there is a quasi-isomorphism $\mathcal{M}_X \to H^*(\mathcal{M}_X) = H^*(X)$; so $\mathcal{M}_X$ may be computed as the minimal model for the rational cohomology ring $H^*(X)$. Therefore we see that the rational homotopy type of a formal space can be recovered from the cohomology ring.

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We can use the $\mathbb{T}^1$-minimal models of Section 2 to mimic the non-equivariant definition of formality. This is the approach used in [7] for finite group actions. Recall that in dealing with equivariant minimal models, it is necessary to restrict to diagrams of CDGAs which are injective. However, the cohomology of an injective object may not be injective, and thus the diagram given by taking the cohomology of a $\mathbb{T}$-system may not be a $\mathbb{T}$-system, so it may not be possible to produce a minimal model as described in Definition 2.6. To remedy this we need the following result from [12]. Note that taking the cohomology of a $\mathbb{T}$-system does give a functor to $\mathbb{Q}[c]$-CDGAs, where the differential is taken to be identically zero; all morphisms in the following theorem are thus in this category.

Theorem 3.1 ([12, Thm. 3.6 and Lemma 3.9]) *Let $\mathcal{A}$ be a $\mathbb{T}$-system, and $\mathcal{H}$ be the diagram given by taking its cohomology. Then there is a $\mathbb{T}$-system $\mathcal{I}$ and an inclusion $\phi\colon \mathcal{H} \hookrightarrow \mathcal{I}$ which is a quasi-isomorphism, such that (I, ϕ) satisfies the following: For any $\mathbb{T}$-system $\mathcal{B}$ with a morphism $\mathcal{H} \to \mathcal{B}$, there exists a morphism $\mathcal{I} \to \mathcal{B}$ making the following diagram commute.*

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{B} \\ {\scriptstyle \phi}\downarrow & \nearrow & \\ \mathcal{I} & & \end{array}$$

The $\mathbb{T}$-system $\mathcal{I}$ of the previous theorem is called the injective envelope of $\mathcal{H}$; it is unique up to quasi-isomorphism. We use the injective envelope to define a notion of formality for $\mathbb{T}$-systems, analogous to the definition for CDGAs.

Definition 3.2 Let $\mathcal{M}$ be a minimal $\mathbb{T}$-system. Then $\mathcal{M}$ is formal if there is a quasi-isomorphism of $\mathbb{T}$-systems $\mathcal{M} \to \mathcal{I}_H$, where $\mathcal{I}_H$ is an injective envelope of the cohomology $\underline{H}^*(\mathcal{M})$.

As in the non-equivariant case, we use $\mathbb{T}^1$-minimal models to define formality for $\mathbb{T}^1$-spaces.

Definition 3.3 A $\mathbb{T}^1$-space X is model formal if its minimal model $\mathcal{M}_X$ is formal.

Recall that $\mathcal{M}_X$ is a model for the de Rham differential forms of the Borel construction $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$, and the cohomology of the minimal model gives the Borel cohomology of the various fixed sets $\underline{H}^*(\mathcal{M}_X) = \underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$. Therefore model formality is equivalent to requiring that the $\mathbb{T}$-systems $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$ and $\mathcal{I}_X$ are quasi-isomorphic, where $\mathcal{I}_X$ is the injective envelope of $\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1) = \underline{H}^*(\mathcal{M}_X)$. Because the $\mathbb{T}^1$-minimal model encodes the rational $\mathbb{T}^1$-homotopy type, this can be interpreted as saying that the equivariant rational homotopy type of the space is determined by its equivariant cohomology, with a suitable interpretation of equivariant cohomology—in this case, we mean the diagram obtained by taking the Borel cohomology (as $\mathbb{Q}[c]$-modules) of the fixed point sets.

In extending the definition of model formal to higher tori, we need to adapt to the fact that there are at present no "minimal models" to work with. However, we can use the algebraic category of $\mathcal{D}-CDGAs$ under $\underline{P}$ of Theorem 2.9. An object $\mathcal{A}$ in this category is defined to be formal if there is a chain of quasi-isomorphisms $\mathcal{A} \to \mathcal{A}_1 \leftarrow \mathcal{A}_2 \to \cdots \to H^*(\mathcal{A})$ between the diagram and the diagram obtained by taking its cohomology. Then we define a torus version of "model formality" as follows.

Definition 3.4 A space X is model formal if $\underline{\mathrm{A}}(X)$ is formal as a $\mathcal{D}-CDGA$ under $\underline{P}$.

Notice that we have avoided the problem of injective envelopes by working in this larger category. However, it is much harder to detect this condition since actually writing down one of these more general models is rather difficult, and they contain less concrete geometric information. Because the models of Theorem 2.9 simplify to the $\mathbb{T}$-systems of Theorem 2.8, the two definitions are equivalent in the case of the circle; wherever possible, we work with the minimal models.

This definition satisfies the following properties, analogous to the properties of non-equivariant formality.

Proposition 3.5 *If X, Y are model formal then $X \vee Y$ is also.*

In the case of the circle, we can also prove the following.

Proposition 3.6 *Let $\mathbb{T} = \mathbb{T}^1$. Suppose X, Y are model formal, and one of*

$$\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1) \text{ or } \underline{H}^*(Y^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$$

is a free $\mathbb{Q}[c]$-module for all $H \leq \mathbb{T}$. Then $X \times Y$ is formal.

Note that if we remove the free $\mathbb{Q}[c]$-module condition, this may fail; see Example 5.6. Thus even though we have stayed as close to the original definition as possible, the properties are different in this more complicated equivariant context.

In the case of the torus, the models are sufficiently complicated that it is quite difficult to analyze the situation; it is probable that a result similar to Proposition 3.6 holds, but no proof is attempted here.

Proofs of Propositions 3.5 and 3.6 are rather technical, and so are given at the end of the paper in Section 6.

There are several alternate approaches to defining equivariantly formal $\mathbb{T}$-spaces, seeking to capture the flavour of the non-equivariant definition while taking into account the richer structure of the $\mathbb{T}$-action. One might ask about the individual formality of the fixed point sets. Taking another approach, one can consider the Borel bundle $X \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$, classically the tool used to study equivariant structures. This is the approach taken by Lillywhite in [9] (which also applies to more general compact Lie groups).

Definition 3.7 A $\mathbb{T}$-space X is *bundle formal* if $\Omega(X \times_{\mathbb{T}} E\mathbb{T})$ is formal in the category of augmented CDGAs under $H^*(B\mathbb{T})$.

Observe that a bundle formal space requires a quasi-isomorphism of $H^*(B\mathbb{T})$-CDGAs between the (non-equivariant) minimal model of the Borel construction $\mathcal{M}_{X\times_{\mathbb{T}}E\mathbb{T}}$ and the Borel cohomology $H^*(X \times_{\mathbb{T}} E\mathbb{T})$. Because of the action of $H^*(B\mathbb{T})$, this is stronger than simply requiring that the space $X \times_{\mathbb{T}} E\mathbb{T}$ be non-equivariantly formal. If a space is bundle formal, this implies that the homotopy type of the Borel bundle $X \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$ is determined by the cohomology of the bundle. However the Borel bundle does not determine the $\mathbb{T}$-equivariant homotopy type of the space X; we need to involve diagrams of fixed points in some way to do so.

The Borel bundle is also used by Goresky-Kottwitz-MacPherson in [8], where they interpret the collapse of the spectral sequence associated to the Borel construction of the space as a type of equivariant formality. The collapse of this spectral sequence and its implications were first discussed by Borel in [1], although it was not called formality there. In our case this condition is also equivalent to another condition, classically referred to as *X totally non-homologous to zero in* $X \times_{\mathbb{T}} E\mathbb{T}$, which is defined to mean that that the map $i^*\colon H^*(X\times_{\mathbb{T}}E\mathbb{T}) \to H^*(X)$ induced by the inclusion of the fibre is surjective. For this reason we refer to this condition here as TNHZ formality.

Definition 3.8 A $\mathbb{T}$-space X is *TNHZ formal* if the spectral sequence for the Borel fibration

$$H^p(B\mathbb{T}; H^q(X)) \Rightarrow H^{p+q}(X \times_{\mathbb{T}} E\mathbb{T})$$

collapses.

At first glance, this definition seems to have little to do with the usual notion of formality. But it is shown in [8] that this condition is related to a more usual notion of formality of a certain equivariant chain complex, working in the derived category of differential graded $H^*(B\mathbb{T})$-modules.

For actions of the circle, we can use $\mathbb{T}^1$-minimal models concretely to recognize various formality conditions. As observed at the end of Section 2, if we look at the minimal model of a space X, then the value of the minimal model at H, $\mathcal{M}_X(H)$ is a model (not necessarily minimal) for the Borel space $X^H \times_{\mathbb{T}^1} E\mathbb{T}^1$; and $\mathcal{M}_X(H)/(c)$ is a model for the fixed set X^H. Thus to see if a given fixed set is non-equivariantly formal, we can simply determine whether the CDGA $\mathcal{M}_X(H)/(c)$ is formal.

When considering the other definitions of $\mathbb{T}^1$-equivariant formality, we note that the bundle $X \times_{\mathbb{T}^1} E\mathbb{T}^1 \to B\mathbb{T}^1$ is described by $\mathcal{M}_X(e)$, the value of the $\mathbb{T}^1$-minimal model $\mathcal{M}_X$ at the identity subgroup e, and the Borel cohomology $H^*(X \times_{\mathbb{T}^1} E\mathbb{T}^1)$ is computed by taking the cohomology of $\mathcal{M}_X(e)$. Thus a $\mathbb{T}^1$-space X is bundle formal if $\mathcal{M}_X(e)$ is formal as a $\mathbb{Q}[c]$-CDGA, and X is TNHZ formal if the cohomology of $H^*(\mathcal{M}_X(e))$ is given by $\mathcal{H} \otimes \mathbb{Q}[c]$, where $\mathcal{H}$ is the cohomology of the CDGA $\mathcal{M}_X(e)/(c)$ which models X. We will use these interpretations via the $\mathbb{T}^1$-minimal models when discussing the examples of Section 5.

4 Comparing Definitions

In this section, we examine the relations that exist between the various definitions of $\mathbb{T}$-equivariant formality given in Section 3. Overall, these concepts are fairly independent, as illustrated by the examples in Section 5. However, we do have the following implications.

Theorem 4.1 *If X is model formal, then each fixed set X^H is bundle formal.*

Proof We first consider the case of the circle. Observe that a map of $\mathbb{T}^1$-systems $\mathcal{A} \to \mathcal{B}$ can be evaluated at a subgroup H to give a map $\mathcal{A}(H) \to \mathcal{B}(H)$ in the category of augmented CDGAs under $H^*(B\mathbb{T}^1) = \mathbb{Q}[c]$, the category used in defining bundle formal. By the definition of minimal model, there is a quasi-isomorphism of $\mathbb{T}$-systems $\mathcal{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X)$; evaluating at H gives a quasi-isomorphism of $\mathbb{Q}[c]$-CDGAs $\mathcal{M}_X(H) \to \Omega(X^H \times_{\mathbb{T}} E\mathbb{T})$. Now $\mathcal{M}_X(H)$ is not necessarily minimal as a $\mathbb{Q}[c]$-CDGA, so let $\mathcal{N}_H$ be a minimal model for $\mathcal{M}_X(H)$. Then we have quasi-isomorphisms $\mathcal{N}_H \to \mathcal{M}_X(H) \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X^H)$, and so $\mathcal{N}_H$ is a $\mathbb{Q}[c]$-CDGA minimal model for $X^H \times_{\mathbb{T}^1} E\mathbb{T}^1$.

Because X is model formal, there is also a quasi-isomorphism $\mathcal{M}_X(H) \to \mathcal{I}(H)$, where $\mathcal{I}$ is the injective envelope of $\underline{H}^*(X \times_{\mathbb{T}^1} E\mathbb{T}^1)$. So $\mathcal{N}_H$ is also a minimal model for $\mathcal{I}(H)$. The construction of the injective envelope comes with a quasi-isomorphism of $\mathbb{T}$-systems $\underline{H}^* \to \mathcal{I}$, and consequently a quasi-isomorphism of $\mathbb{Q}[c]$-CDGAs $\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1) \to \mathcal{I}(H)$, so $\mathcal{N}_H$ is quasi-isomorphic to $\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$ as well. Thus $\Omega(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$ and $\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$ are quasi-isomorphic as $\mathbb{Q}[c]$-CDGAs and the space X^H is bundle formal.

For the general torus case, the argument is similar, and we do not have the injective envelopes to worry about. Evaluating a map of $\mathcal{D}-CDGA$s under $\underline{P}$ at the pair $H[e]$ gives a map of $\underline{P}(H[e])$-CDGA's; the initial object $\underline{P}$ is designed so that $\underline{P}(H[e])$ is quasi-isomorphic to the de Rham forms on the classifying space $B\mathbb{T}$, and the value of the equivariant model $\underline{A}(X)$ at $H[e]$, $\underline{A}(X)(H[e])$, is given by the de Rham forms on the Borel space $X^H \times_{\mathbb{T}} E\mathbb{T}$. The chain of quasi-isomorphisms assumed by the model formality condition therefore will in particular give a chain of quasi-isomorphisms in the category of $H^*(B\mathbb{T})$-CDGAs used in the definition of bundle formal. ■

The converse of this theorem fails; Example 5.3 shows that it is possible to have a space which is not model formal but whose fixed sets are all bundle formal.

The next results provide links between equivariant and non-equivariant formality.

Theorem 4.2 *Suppose X is model formal and each fixed point set X^H is TNHZ-formal. Then each fixed set X^H is non-equivariantly formal.*

Proof Theorem 4.1 gives that X^H is bundle formal for each H. In [9, Corr. 4.5] it is shown that a space which is bundle formal and TNHZ formal must be non-equivariantly formal; applying this to each fixed set yields the result. ■

Proposition 4.3 *If X is model formal, then the fixed set $X^{\mathbb{T}}$ is non-equivariantly formal.*

Proof Again, we begin with the case of the circle. If X is model formal, then by definition there is a quasi-isomorphism of $\mathbb{T}$-systems $\mathcal{M}_X \to \mathcal{I}$, where $\mathcal{I}$ is the injective envelope of the cohomology $\underline{H}^*(\mathcal{M}_X)$. The construction of the injective envelope $\mathcal{I}$ given in [12] consists of enlargements of $\underline{H}^*(\mathcal{M}_X)$, and does not change its value at $\mathbb{T}^1$, so in fact $\mathcal{I}(\mathbb{T}^1) = H^*(\mathcal{M}_X(\mathbb{T}^1))$; consequently the map of $\mathbb{T}$-systems restricts to a quasi-isomorphism of $\mathbb{Q}[c]$-CDGAs $\mathcal{M}(\mathbb{T}^1) \to H^*(\mathcal{M}_X(\mathbb{T}^1))$. The value of the minimal model $\mathcal{M}(\mathbb{T}^1)$ is given by $\mathcal{N} \otimes \mathbb{Q}[c]$ where $\mathcal{N}$ is a minimal model for the space $X^{\mathbb{T}^1}$; similarly $H^*(\mathcal{M}_X(\mathbb{T}^1)) = H^*(X^{\mathbb{T}^1}) \otimes \mathbb{Q}[c] = H^*(\mathcal{N}) \otimes \mathbb{Q}[c]$. So the quasi-isomorphism of $\mathbb{Q}[c]$-CDGAs at $\mathbb{T}^1$ comes from a quasi-isomorphism of CDGAs $\mathcal{N} \to H^*(\mathcal{N})$ and consequently $X^{\mathbb{T}^1}$ is formal.

A similar argument will give the result for the general torus, by examining the value of the $\mathcal{D}-CDGA$ model at the subgroup pair $\mathbb{T}[e]$ to give a model for the space $X^{\mathbb{T}} \times_{\mathbb{T}} E\mathbb{T} = X^{\mathbb{T}} \times B\mathbb{T}$. ■

There are no other direct general relations between the various definitions of equivariant formality. However, we now look at certain special types of $\mathbb{T}$-actions where the situation is simpler and more implications can be drawn.

Theorem 4.4 *Suppose X has trivial $\mathbb{T}$-action. Then X is TNHZ formal and the following are equivalent:*

(1) *X is model formal,*
(2) *X is bundle formal,*
(3) *X is non-equivariantly formal.*

Proof If X has trivial $\mathbb{T}$-action, then the Borel bundle $X \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$ is the trivial bundle $X \times B\mathbb{T}$. Therefore X is TNHZ formal.

(2) ⇒ (1): Each fixed set X^H of X is equal to X itself, and so the algebra model functors $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$, $\mathcal{M}_X$, $\underline{A}(X)$, and $\underline{H}^*(X^H \times_{\mathbb{T}} E\mathbb{T})$ are all constant, taking the same value at each subgroup H or pair $H[K]$, with structure maps given by the identity map. A constant functor is injective, and so $\mathcal{I} = \underline{H}^*(X^H \times_{\mathbb{T}} E\mathbb{T})$. Therefore a quasi-isomorphism of $H^*(B\mathbb{T})$-CDGAs between $\Omega(X \times_{\mathbb{T}} E\mathbb{T})$ and $H^*(X \times_{\mathbb{T}} E\mathbb{T})$ as given for a bundle formal space induces a quasi-isomorphism between the constant $\mathbb{T}$-systems $\underline{\mathcal{E}}_{\mathbb{T}^1}(X)$ and $\underline{H}^*(X^H \times_{\mathbb{T}} E\mathbb{T})$, and hence between $\mathcal{M}_X$ and $\mathcal{I}$ (for the circle) or, for the general torus, between the functors $\underline{A}(X)$ and $\underline{H}^*(\underline{A}(X))$.

(1) ⇒ (3): Since X is TNHZ formal, we can apply Proposition 4.3.

(3) ⇒ (2): Since the Borel bundle of X is trivial, the $H^*(B\mathbb{T})$-CDGA of de Rham forms $\Omega(X \times_{\mathbb{T}} E\mathbb{T})$ is quasi-isomorphic to $\Omega(X) \otimes H^*(B\mathbb{T})$, and $H^*(X \times_{\mathbb{T}} E\mathbb{T})$ is quasi-isomorphic to $H^*(X) \otimes H^*(B\mathbb{T})$. If X is a non-equivariantly formal space, then $\Omega(X)$ is quasi-isomorphic to $H^*(X)$ and so $\Omega(X \times_{\mathbb{T}} E\mathbb{T})$ is quasi-isomorphic to $H^*(X \times_{\mathbb{T}} E\mathbb{T})$ as a $H^*(B\mathbb{T})$-CDGA. ■

Proposition 4.5 *Let $\mathbb{T} = \mathbb{T}^1$. If X has a single isotropy type (apart from the fixed basepoint) then X is model formal if and only if X is bundle formal.*

Proof If X is model formal, then X is automatically bundle formal by Theorem 4.1, since $X = X^e$ is one fixed point set of X. Conversely, suppose X is bundle formal. The $\mathbb{T}^1$-minimal model of a space with a single isotropy type is created out of systems V_H generated at the given isotropy subgroup H; to produce $\mathcal{M}_X$ we create a minimal $\mathbb{Q}[c]$-CDGA model for $X^H \times_{\mathbb{T}^1} E\mathbb{T}^1 = X \times_{\mathbb{T}^1} E\mathbb{T}^1$, and let $\mathcal{M}_X$ be an injective functor generated by this at the isotropy subgroup H. Similarly the cohomology of the minimal model is generated by its value at the subroup H. So the cohomology is already injective and so is equal to its injective envelope; and a $\mathbb{Q}[c]$-CDGA quasi-isomorphism $\mathcal{M}_X(H) \to H^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$ will extend to a quasi-isomorphism of $\mathbb{T}$-systems $\mathcal{M}_X \to H^*(\mathcal{M}_X)$. ■

For actions of general torus groups, this argument is complicated by the fact that the indexing category $\mathcal{D}$ is larger than the orbit category, and so a given fixed set X^H contributes in several places and in several forms in the more complicated diagram $\underline{A}(X)$. Thus the resulting models are not "generated" by a single vector space and a single indexing object. The models are difficult to analyze even in the particular case of a single isotropy type.

5 Examples of $\mathbb{T}^1$-Spaces

In this section we examine a number of examples of spaces with circle actions to illustrate the various definitions of equivariant formality and how they relate to each other. The tool we use to compute information about these spaces is the $\mathbb{T}^1$-minimal model. Throughout Section 5, therefore, the group $\mathbb{T}$ will always be $\mathbb{T}^1$.

A class of spaces which are formal in all senses are the $\mathbb{T}^1$-equivariant Eilenberg-MacLane spaces.

Theorem 5.1 *If $K = K(\underline{\pi}, n)$ is a $\mathbb{T}^1$-equivariant Eilenberg-MacLane space, then K is model formal, and all fixed sets are bundle formal, TNHZ formal and non-equivariantly formal.*

Proof In [12, Thm. 4.3], we saw that the minimal model of a $\mathbb{T}^1$-equivariant Eilenberg-MacLane space is given by taking an injective resolution of $\underline{H}^n = \underline{\pi}^*$

$$\underline{\pi}^* \to \underline{V}_0 \to \underline{V}_1 \to \cdots,$$

and defining the $\mathbb{T}$-system by

$$\mathcal{M}_X = \otimes_i \mathbb{Q}(\underline{V}_i) \otimes \mathbb{Q}[c]$$

with differential induced by the resolution, and distinguished sub-CDGA given by $\mathcal{M}_{\mathbb{T}} = \otimes_i \mathbb{Q}(\underline{V}_i(\mathbb{T}^1)) \otimes \mathbb{Q}$. It is shown in [12] that this $\mathbb{T}$-system is formal, and so X is model formal and consequently all fixed sets are bundle formal by Theorem 4.1. Examining the cohomology

$$\underline{H}^*(\mathcal{M}_X) = \underline{H}^*(K^H \times_{\mathbb{T}^1} E\mathbb{T}^1) = \mathbb{Q}(\underline{\pi}^*) \otimes \mathbb{Q}[c]$$

we see that each fixed set is also TNHZ-formal and hence by Theorem 4.2 is also non-equivariantly formal. ■

A couple of concrete examples of $\mathbb{T}^1$-equivariant Eilenberg-MacLane spaces are examined in detail in [12, Section 4]. One is given by a free (off the fixed basepoint) $\mathbb{T}^1$-action on a space which is homotopy equivalent to S^3. The other is the space S^∞, given complex coordinates as the units in $\mathbb{C}^\infty$, with $\mathbb{T}^1$-action given by $\lambda[z_0, z_1, z_2, z_3, z_4, \dots] = [z_0, z_1, \lambda z_2, \lambda z_3, \lambda z_4, \dots]$.

The rest of our examples demonstrate the independence of the various definitions, showing that a wide range of possibilities exist for spaces being formal in some senses but not others. For simplicity, all of our examples will be $\mathbb{T}^1$-spaces with semifree $\mathbb{T}^1$-action, and so $X^H = X^{\mathbb{T}^1}$ for all $H \neq e$. Consequently all functors will also have $\mathcal{A}(H) = \mathcal{A}(\mathbb{T}^1)$ for all $H \neq e$, and so to simplify notation the functors will be written $\mathcal{A}(e) \to \mathcal{A}(\mathbb{T}^1)$ and all other subgroups will be suppressed.

Example 5.2 The following example was discussed in [3] as one of the simplest manifolds with a $\mathbb{T}^1$-action which is *not* TNHZ formal. It is bundle formal but not formal in any other sense.

Let τ be the tangent bundle of S^8, with trivial $\mathbb{T}^1$-action, and ϵ the trivial 2-plane bundle on S^8 with non-trivial $\mathbb{T}^1$-action. Let η be the pullback bundle of $\tau \oplus \epsilon$ over $S^3 \times S^5$ along a map of degree 1. We let X be the total space of $S(\eta)$, the unit sphere bundle of η. Observe that the fixed set $X^{\mathbb{T}^1}$ is the sphere bundle of the pullback of τ over $S^3 \times S^5$; all other fixed sets X^H are equal to $X^{\mathbb{T}^1}$ except for $X = X^e$, which is homotopic to $S^3 \times S^5 \times S^9$ since the bundle $\tau \oplus \epsilon$ is trivial. Therefore the only non-trivial homotopy groups are in dimensions $3, 5, 7$ and 9, and the diagrams of the dual homotopy groups $\underline{\pi}_n^* = \pi_n^*(X^e) \to \pi_n^*(X^{\mathbb{T}^1})$ are given by the diagrams $\underline{\pi}_3^* = \underline{\pi}_5^* = \mathbb{Q} \to \mathbb{Q}$, $\underline{\pi}_7^* = 0 \to \mathbb{Q}$ and $\underline{\pi}_9^* = \mathbb{Q} \to 0$. These generate the minimal model $\mathcal{M}_X$; note that $\underline{\pi}_7^*$ is not injective, and so an injective resolution will be used.

The $\mathbb{T}^1$-minimal model for this space is given by the following functor, where the subscripts on the generators indicate their degrees.

$$\begin{array}{c} \mathcal{M}_X(e) = \mathbb{Q}(x_3, y_5, w_7, w_8', z_9) \otimes \mathbb{Q}[c], dw = xy - w', dz = xyc \\ \Big\downarrow \\ \mathcal{M}_X(H) = \mathcal{M}_X(\mathbb{T}^1) = \mathbb{Q}(x_3, y_5, w_7) \otimes \mathbb{Q}[c], dw = xy \end{array}$$

The generator w' in $\mathcal{M}_X(e)$ comes from the injective resolution of $\underline{\pi}_7^*$. In $X^{\mathbb{T}^1}$, the twisting of the non-trivial sphere bundle is reflected by the differential $dw = xy$.

The cohomology of $H^*(X)$ is a free CDGA generated by classes $[x], [y], [z]$, whereas the Borel cohomology $H^*(X \times_{\mathbb{T}^1} E\mathbb{T}^1)$ is calculated by $H^*(\mathcal{M}_X(e))$, and is generated by classes $[c^n], [xc^n], [yc^n]$ and $[xyc^n]$; in particular, z is not a cocycle since $dz = xyc$. Therefore $H^*(X \times_{\mathbb{T}^1} E\mathbb{T}^1)$ is not given by $H^*(X) \otimes \mathbb{Q}[c]$, and X is not TNHZ formal.

Note that although the space itself is non-equivariantly formal, the fixed set $X^{\mathbb{T}^1}$ is not; in fact, this space is given in [6] as an example of a non-formal space. Thus by Theorem 4.4, X cannot be model formal. However, the Borel construction of X is modeled by $\mathcal{M}_X(e)$, and this is formal as a $\mathbb{Q}[c]$-CDGA, since there is an obvious map of $\mathbb{Q}[c]$-CDGAs $\mathcal{M}_X(e) \to H^*(\mathcal{M}_X(e))$ which is a quasi-isomorphism. Thus X is bundle formal.

Example 5.2 does not provide a counterexample to the converse of Proposition 4.3, since although X is bundle formal the fixed set $X^{\mathbb{T}^1}$ is not. Instead, it demonstrates the way that different fixed sets of a space may exhibit very different behaviour. The promised counterexample is given by the following.

Example 5.3 Let X be the sphere S^5, given coordinates (z_0, z_1, z_2) as the unit sphere in $\mathbb{C}^3$. Give X an action of $\mathbb{T}^1$ by $\lambda[z_0, z_1, z_2] = [z_0, z_1, \lambda z_2]$. Then the fixed set $X^{\mathbb{T}^1} \cong S^3$ is given by $(z_1, z_2, 0)$; and $X^H = X^{\mathbb{T}^1}$ for all $H \neq e$.

The space X has one non-trivial homotopy group π_5, and $X^{\mathbb{T}^1}$ has one non-trivial homotopy group π_3; so the diagrams of the dual homotopy groups of the fixed point sets $\pi_n^*(X^e) \to \pi_n^*(X^{\mathbb{T}^1})$ are given by $\underline{\pi}_3^* = 0 \to \mathbb{Q}$ and $\underline{\pi}_5^* = \mathbb{Q} \to 0$, and X has a two-stage equivariant Postnikov tower with a non-trivial k-invariant. Note that $\underline{\pi}_3^*$ is not injective, and so when we form the minimal model we use an injective resolution.

The minimal model of X is given by the following, where the subscripts indicate degrees of the generators.

$$\begin{array}{c} \mathcal{M}_X(e) = \mathbb{Q}(x_3, x_4', y_5) \otimes \mathbb{Q}[c],\ dx = x', dy = x'c \\ \downarrow \\ \mathcal{M}_X(\mathbb{T}^1) = \mathbb{Q}(x_3) \otimes \mathbb{Q}[c],\ d = 0 \end{array}$$

The generator x' in $\mathcal{M}_X(e)$ comes from the injective resolution of $\underline{\pi}_3^*$; the non-trivial k-invariant is expressed by $dy = x'c$. A calculation shows that the cohomology of this system $\underline{H}^*(\mathcal{M}_X) - \mathcal{H}_X$ is given by

$$\begin{array}{c} \mathcal{H}_X(e) = \mathbb{Q}(b_5) \otimes \mathbb{Q}[c] \\ \downarrow \\ \mathcal{H}_X(\mathbb{T}^1) = \mathbb{Q}(a_3) \otimes \mathbb{Q}[c] \end{array}$$

where the cohomology class $a = [x]$ and $b = [y - xc]$.

Observe that this space is bundle formal, since there is a $\mathbb{Q}[c]$-CDGA map $\mathcal{M}_X(e) \to \mathcal{H}_X(e)$ given by taking $y \to b$ and $x, x' \to 0$ which is a quasi-isomorphism. Moreover $\mathcal{H}_X(e) = H^*(X) \otimes \mathbb{Q}[c]$ and the space is TNHZ formal. The fixed set $X^{\mathbb{T}^1}$ is easily seen to be both bundle and TNHZ formal also.

However, this space fails to be model formal. The cohomology $\mathcal{H}_X$ is not injective, and so we form the injective envelope as described in [12]. In this case, the map $\mathcal{H}_X(e) \to \lim_{K \supset e} \mathcal{H}_X(K)$ is not surjective, and *coker*$\{\mathcal{H}_X(\mathbb{T}^1) \to \lim_{K \supset \mathbb{T}^1} \mathcal{H}_X(K)\}$ is generated by the elements ac^m. To create the injective envelope we choose a to be a $\mathbb{Q}[c]$-generator of this cokernel and define the enlargement $\mathcal{H}_X \otimes_{\mathbb{Q}[c]} \Lambda_{\mathbb{T}^1}(a)$ where

$\Lambda_{\mathbb{T}^1}(a)(H) = \mathbb{Q}[c]$ and $\Lambda_{\mathbb{T}^1}(a)(\mathbb{T}^1)$ is the free acyclic $\mathbb{Q}[c]$-CDGA generated by a and a', with $d(a) = a'$. We see that the injective envelope of $\mathcal{H}_X$ has generators of the same degrees as the minimal model $\mathcal{M}_X$, but with $db = 0$. There is no quasi-isomorphic map between these two $\mathbb{T}$-systems: the generator x must be mapped to (some non-zero multiple of) a, and then to respect the differential x' must map to a'. But to get a quasi-isomorphism we must also have $y - xc$ map to b, so y maps to $b - ac$. It is impossible to get a map which does this and also commutes with structure maps, since y maps to 0 in $\mathcal{M}_X(\mathbb{T}^1)$ whereas $b - ac$ maps to ac in $\mathcal{H}_X(\mathbb{T}^1)$. Thus X is not model formal.

It is interesting to compare this space to the space Y which has the same homotopy groups of all fixed sets, but with a trivial equivariant k-invariant. The space Y is therefore a product of equivariant Eilenberg-MacLane spaces. The minimal model in this case has generators of the same degrees as $\mathcal{M}_X$ above, but with $dy = 0$. The cohomology is the same as the cohomology functor $\mathcal{H}$ described above and so the injective envelope is also the same; therefore the minimal model for Y is isomorphic to the injective envelope, and Y is model formal.

Example 5.4 Any space with a trivial group action is automatically TNHZ formal by Theorem 4.4. Therefore if we take a space X which is not formal and consider it as a $\mathbb{T}^1$-space with trivial $\mathbb{T}^1$ action we produce an example of a space which is TNHZ formal but not model formal, bundle formal or non-equivariantly formal (these are equivalent in this case, by Theorem 4.4). For example, we can take the fixed set of Example 5.2, with non-equivariant minimal model given by $\mathcal{N} = \mathbb{Q}(x, y, z)$ with $dx = dy = 0$ and $dz = xy$, $\deg x = 3$, $\deg y = 5$. Considering this as a $\mathbb{T}^1$-space with a trivial $\mathbb{T}^1$-action, we see that every fixed point set is the same, and so it has a constant $\mathbb{T}^1$-minimal model given by $\mathcal{M}_X(H) = \mathcal{N} \otimes \mathbb{Q}[c]$ for every $H \leq \mathbb{T}^1$. We can see directly that this is not bundle or model formal, since $H^*(\mathcal{M}_X) = H^*(\mathcal{N}) \otimes \mathbb{Q}[c]$ and there is no quasi-isomorphism $\mathcal{N} \to H^*(\mathcal{N})$, and hence no $\mathbb{Q}[c]$-CDGA quasi-isomorphism $\mathcal{M}_X \to H^*(\mathcal{M}_X)$.

Example 5.5 We describe a space X which is free off the fixed basepoint, so that $\mathcal{M}_X(H) = \mathbb{Q}[c]$ for every $H \neq e$. We build X using a two-stage Postnikov tower with non-zero homotopy groups π_4 and π_5, and with a non-trivial equivariant k invariant; so that the minimal model $\mathcal{M}_X$ is a free $\mathbb{Q}[c]$-CDGA generated by x of degree 4, y of degree 5 with $d(y) = xc$. Then the cohomology of $H^*(\mathcal{M}_X(e))$ is generated by classes $[x^n]$ and $[c^n]$, where $[x][c] = 0$.

If we ignore the $\mathbb{T}^1$-action, the non-equivariant model for X is given by $\mathcal{M}(e)/(c) = \mathbb{Q}(x, y)$ with $dx = dy = 0$, and we see that the space X is the product of Eilenberg-MacLane spaces $K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 5)$. Thus the Borel cohomology $H^*(\mathcal{M}(e))$ is not freely generated by $H^*(X)$ and this space is not TNHZ formal. However, we have an obvious map $\mathcal{M}_X \to \underline{H}^*(\mathcal{M}_X)$ induced by $x \to [x]$ which is a quasi-isomorphism. (Note that in this case $H^*(\mathcal{M}_X)$ is already injective, and so equal to its injective envelope.) Thus X is model formal and consequently bundle formal.

We can use Example 5.5 to show that the product of two model formal spaces may not be model formal if the spaces are not TNHZ formal and their Borel cohomologies are not $\mathbb{Q}[c]$-free.

Example 5.6 Let X be the space from Example 5.5 and consider $Y = X \times X$. Then Y is also free off the fixed basepoint, and the $\mathbb{T}^1$-minimal model of Y is given by $\mathcal{M}_Y(H) = \mathbb{Q}[c]$ for every $H \neq e$. The model for $Y = Y^e$ is given by $\mathcal{M}_Y(e) = \mathbb{Q}(x_1, x_2, y_1, y_2) \otimes \mathbb{Q}[c]$ with x_i of degree 4, y_i of degree 5 and $d(y_i) = x_i c$. A calculation shows that the cohomology $H^*(\mathcal{M}_Y(e))$ is generated by classes $[x_1^m x_2^n]$, $[c^n]$, and $[x_1^m x_2^{n+1} y_1 - x_1^{m+1} x_2^n y_2]$. It is the existence of these last-named cohomology classes which show that this space is not model formal; a map $\mathcal{M} \to \underline{H}^*(\mathcal{M})$ must take $y_1 \to 0$ and $y_2 \to 0$ and so cannot be a quasi-isomorphism. (Note that once again the cohomology functor $\underline{H}^*(\mathcal{M}_X)$ is injective.) Thus we have a space which is not $\mathbb{T}^1$-equivariantly formal in any sense, even though its underlying space is non-equivariantly formal since Y, like X, is a product of Eilenberg-MacLane spaces.

Our final example demonstrates that it is possible to have a $\mathbb{T}^1$-space which is model formal, but whose underlying space is not formal as an ordinary non-equivariant space. Hence this space is in some sense the simplest $\mathbb{T}^1$-homotopy type which has the given diagram of Borel cohomologies, the $\mathbb{Q}[c]$-module diagram $\underline{H}^*(X^H)$; but this equivariant homotopy type is not realized by a formal underlying space. Thus we see the separation between equivariant and non-equivariant phenomena, and the fact that spaces which are relatively easy to understand equivariantly may still be quite complicated.

Example 5.7 Let X be a $\mathbb{T}^1$-space which is free off the basepoint, so that as in the previous two examples $\mathcal{M}_X(H) = \mathbb{Q}[c]$ for $H \neq e$. We consider the space whose minimal model at e is given as follows, where again the subscripts indicate the degrees of the generators: $\mathcal{M}_X(e) = \mathbb{Q}(x_3, y_3, w_{,4}\, z_5) \otimes \mathbb{Q}[c]$ with $d(x) = d(y) = d(w) = 0$ and $d(z) = -xy + wc$. Observe that the minimal model for the underlying space is given by $\mathcal{N} = \mathcal{M}_X(e)/(c) = \mathbb{Q}(x, y, w, z)$ with $d(z) = -xy$, and so has cohomology generated by $[1], [x], [y], [xz], [yz], [xyz]$ as well as any of the above multiplied by $[w^n]$; we see immediately that the presence of elements $[xz]$ mean that this is not a formal space, since any map $\mathcal{N} \to H^*(\mathcal{N})$ must take z to 0.

Nevertheless this space is model formal, since we find that the cohomology of $H^*(\mathcal{M}_X(e))$ consists of free $\mathbb{Q}[c]$-generators $[x], [y], [w]$, and generators $[xw^n]$, $[yw^n]$ and $[xyw^n]$ which become 0 when multiplied by $[c]$; note that in this algebra, we also have the relation $[xy] = [wc]$. So there is an obvious map of $\mathbb{T}$-systems $\mathcal{M}_X \to \underline{H}^*(\mathcal{M}_X)$ induced by taking $x \to [x]$, $y \to [y]$, $w \to [w]$ and $z \to 0$, which is a quasi-isomorphism.

Note that this space is not TNHZ formal, and in fact the Borel cohomology $H^*(X \times_{\mathbb{T}^1} E\mathbb{T}^1) = H^*(\mathcal{M}_X(e))$ is far from being $\mathbb{Q}[c]$-free. We expect this, since otherwise we would have a counter-example to Theorem 4.5.

6 Proofs from Section 3

We now give the deferred proofs of Propositions 3.5 and 3.6 about model formality.

Proof of Proposition 3.5 Once again, we begin with the case $\mathbb{T} = \mathbb{T}^1$. For simplicity of notation we assume that direct sum and tensor product are always taken over the base ring $\mathbb{Q}[c]$. Observe that $(X \vee Y) \times_{\mathbb{T}^1} E\mathbb{T}^1 = X \times_{\mathbb{T}^1} E\mathbb{T}^1 \vee_{B\mathbb{T}^1} Y \times_{\mathbb{T}^1} E\mathbb{T}^1$ and $(X \vee Y)^H = X^H \vee Y^H$, so $\underline{\mathcal{E}}_{\mathbb{T}^1}(X \vee Y) = \underline{\mathcal{E}}_{\mathbb{T}^1}(X) \oplus \underline{\mathcal{E}}_{\mathbb{T}^1}(Y)$ and $\underline{H}^*((X \vee Y)) \times_{\mathbb{T}^1} E\mathbb{T}^1 = \underline{H}^*(X \times_{\mathbb{T}^1} E\mathbb{T}^1) \oplus \underline{H}^*(Y \times_{\mathbb{T}^1} E\mathbb{T}^1) = \underline{H}^*(\mathcal{M}_X) \oplus \underline{H}^*(\mathcal{M}_Y)$. Now if $\underline{H}^*(\mathcal{M}_X) \hookrightarrow \mathcal{I}_X$ and $\underline{H}^*(\mathcal{M}_Y) \hookrightarrow \mathcal{I}_Y$ are the quasi-isomorphic inclusions to the injective envelopes, then the induced map $\underline{H}^*(\mathcal{M}_{X \vee Y}) \to \mathcal{I}_X \oplus \mathcal{I}_Y$ is also a quasi-isomorphism. Then by Lemma 3.1 we have a map from the injective envelope $\mathcal{I}_{X \vee Y}$ of $\underline{H}^*(\mathcal{M}_X) \oplus \underline{H}^*(\mathcal{M}_Y)$ to $\mathcal{I}_X \oplus \mathcal{I}_Y$, and by the commutativity of the diagram of Lemma 3.1 this map will also be a quasi-isomorphism. Uniqueness of minimal models ensures that both $\mathcal{I}_{X \vee Y}$ and $\mathcal{I}_X \oplus \mathcal{I}_Y$ have the same minimal model. Therefore it will suffice to show that there is a quasi-isomorphism $\mathcal{M}_{X \vee Y} \to \mathcal{I}_X \oplus \mathcal{I}_Y$, and so $\mathcal{M}_{X \vee Y}$ is a minimal model for both $\mathbb{T}$-systems.

The quasi-isomorphisms $\mathcal{M}_X \to \underline{H}^*(\mathcal{M}_X)$ and $\mathcal{M}_Y \to \underline{H}^*(\mathcal{M}_Y)$ induce a quasi-isomorphism $\mathcal{M}_X \oplus \mathcal{M}_Y \to \mathcal{I}_X \oplus \mathcal{I}_Y$. Now $\mathcal{M}_X \oplus \mathcal{M}_Y$ may not be the minimal model for $X \vee Y$, but the quasi-isomorphisms $\mathcal{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X)$ and $\mathcal{M}_Y \to \underline{\mathcal{E}}_{\mathbb{T}^1}(Y)$ induce a quasi-isomorphism $\mathcal{M}_X \oplus \mathcal{M}_Y \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X) \oplus \underline{\mathcal{E}}_{\mathbb{T}^1}(Y)$, and so again using the uniqueness of minimal models, the minimal models of $\underline{\mathcal{E}}_{\mathbb{T}^1}(X) \oplus \underline{\mathcal{E}}_{\mathbb{T}^1}(Y)$ and $\mathcal{M}_X \oplus \mathcal{M}_Y$ are isomorphic. But by definition, $\mathcal{M}_{X \vee Y}$ is a model for $\underline{\mathcal{E}}_{\mathbb{T}^1}(X \vee Y) = \underline{\mathcal{E}}_{\mathbb{T}^1}(X) \oplus \underline{\mathcal{E}}_{\mathbb{T}^1}(Y)$. Therefore we have quasi-isomorphisms $\mathcal{M}_{X \vee Y} \to \mathcal{M}_X \oplus \mathcal{M}_Y \to \mathcal{I}_X \oplus \mathcal{I}_Y$ and so $X \vee Y$ is model formal.

The same reasoning (minus the injective envelopes) holds for the general torus case, since examining the algebraic model of Theorem 2.9 in detail shows that it also satisfies the basic equality $\underline{A}(X \vee Y) = \underline{A}(X) \oplus_{H^*(B\mathbb{T})} \underline{A}(Y)$, and the rest follows. ■

From here on, we restrict to the case $\mathbb{T} = \mathbb{T}^1$; again, the base ring for direct sums and tensor products is always $\mathbb{Q}[c]$.

Lemma 6.1 *The $\mathbb{T}^1$-minimal model $\mathcal{M}_{X \times Y}$ of $X \times Y$ is isomorphic to the tensor product $\mathcal{M}_X \otimes \mathcal{M}_Y$.*

To prove this we use the following construction.

Definition 6.2 A commutative square of CDGAs

$$\begin{array}{ccc} A & \longrightarrow & A'' \\ \big\downarrow & & \big\downarrow \\ A' & \longrightarrow & C \end{array}$$

is an *EM* square if the induced map $\mathrm{Tor}^*_A(A', A'') \to H^*C$ is an isomorphism (see [2, p. 13]).

EM squares satisfy the following algebraic property, proved in [11]

Lemma 6.3 *If a map between EM squares*

$$\begin{array}{ccc} A & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & C \end{array} \quad \longrightarrow \quad \begin{array}{ccc} B & \longrightarrow & B'' \\ \downarrow & & \downarrow \\ B' & \longrightarrow & D \end{array}$$

induces isomorphisms $H^*(A) \cong H^*(B)$, $H^*(A') \cong H^*(B')$ and $H^*(A'') \cong H^*(B'')$ then it also induces an isomorphism $H^*(C) \cong H^*(D)$.

Proof of 6.1 It is shown in [11] that any $\mathbb{T}^1$-equivariant pullback of spaces produces an EM square of $\mathbb{T}$-systems when the differential form functor $\underline{\mathcal{E}}_{\mathbb{T}^1}(-)$ is applied. Therefore we have an EM square

$$\begin{array}{ccc} \underline{\mathcal{E}}_{\mathbb{T}^1}(B\mathbb{T}^1) & \longrightarrow & \underline{\mathcal{E}}_{\mathbb{T}^1}(X) \\ \downarrow & & \downarrow \\ \underline{\mathcal{E}}_{\mathbb{T}^1}(Y) & \longrightarrow & \underline{\mathcal{E}}_{\mathbb{T}^1}(X \times Y) \end{array}$$

Now any minimal $\mathbb{T}$-system is a free $\mathbb{Q}[c]$-module, and so $\mathrm{Tor}^*_{\mathbb{Q}[c]}(\mathcal{M}_X, \mathcal{M}_Y) = \mathcal{M}_X \otimes \mathcal{M}_Y$. Therefore

$$\begin{array}{ccc} \mathbb{Q}[c] & \longrightarrow & \mathcal{M}_X \\ \downarrow & & \downarrow \\ \mathcal{M}_Y & \longrightarrow & \mathcal{M}_X \otimes \mathcal{M}_Y \end{array}$$

is also an EM square. Then the quasi-isomorphisms $\mathcal{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X)$, $\mathcal{M}_Y \to \underline{\mathcal{E}}_{\mathbb{T}^1}(Y)$, and $\mathbb{Q}[c] \to \underline{\mathcal{E}}_{\mathbb{T}^1}(B\mathbb{T}^1)$ ensure that the induced map $\mathcal{M}_X \otimes \mathcal{M}_Y \to \underline{\mathcal{E}}_{\mathbb{T}^1}(X \times Y)$ is also a quasi-isomorphism by Lemma 6.3. Since $\mathcal{M}_X \otimes \mathcal{M}_Y$ is also minimal, it is the minimal model of $X \times Y$. ■

Proof of 3.6 We know that there are quasi-isomorphisms $\mathcal{M}_X \to \mathcal{I}_X$ and $\mathcal{M}_Y \to \mathcal{I}_Y$, where $\mathcal{I}_-$ is the injective envelope of $H^*(-)$. We wish to show that there is a quasi-isomorphism $\mathcal{M}_{X\times Y} \to \mathcal{I}_{X\times Y}$. We know that $\mathcal{M}_{X\times Y} = \mathcal{M}_X \otimes \mathcal{M}_Y$ by Lemma 6.1. Moreover, since the minimal models $\mathcal{M}_-$ are free, the Künneth theorem gives an isomorphism $H^*(\mathcal{M}_X \otimes \mathcal{M}_Y) \cong H^*(\mathcal{M}_X) \otimes H^*(\mathcal{M}_Y)$.

Again because of freeness of minimal models, $\mathrm{Tor}^*_{\mathbb{Q}[c]}(\mathcal{M}_X, \mathcal{M}_Y) = \mathcal{M}_X \otimes \mathcal{M}_Y$. By assumption, one of $\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$ and $\underline{H}^*(Y^H \times_{\mathbb{T}^1} E\mathbb{T}^1)$ is a free $\mathbb{Q}[c]$-modules for all $H \leq \mathbb{T}^1$; assume it is $\underline{H}^*(X^H \times_{\mathbb{T}^1} E\mathbb{T}^1) = \underline{H}^*(\mathcal{M}_X)$. Then its injective envelope $\mathcal{I}_X$ is also a free $\mathbb{Q}[c]$-module, since it is created by adding free generators to $\underline{H}^*(\mathcal{M}_X)$.

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Therefore $\mathrm{Tor}^*_{\mathbb{Q}[c]}(\mathcal{I}_X, \mathcal{I}_Y) = \mathcal{I}_X \otimes \mathcal{I}_Y$, and the quasi-isomorphisms $\mathcal{M}_X \to \mathcal{I}_X$ and $\mathcal{M}_Y \to \mathcal{I}_Y$ induce a map of the EM squares

$$\begin{array}{ccccccc} \mathbb{Q}[c] & \longrightarrow & \mathcal{M}_X & & \mathbb{Q}[c] & \longrightarrow & \mathcal{I}_X \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ \mathcal{M}_Y & \longrightarrow & \mathcal{M}_X \otimes \mathcal{M}_Y & & \mathcal{I}_Y & \longrightarrow & \mathcal{I}_X \otimes \mathcal{I}_Y \end{array}$$

and so the map $\mathcal{M}_X \otimes \mathcal{M}_Y \to \mathcal{I}_X \otimes \mathcal{I}_Y$ is also a quasi-isomorphism.

The lifting property of injective envelopes of Lemma 3.1 gives a map from the injective envelope of $H^*(\mathcal{M}_X) \otimes H^*(\mathcal{M}_Y)$, denoted $\mathcal{I}_{X\times Y}$, to $\mathcal{I}_X \otimes \mathcal{I}_Y$; and commutativity ensures that this is also a quasi-isomorphism. Then since $\mathcal{M}_X \otimes \mathcal{M}_Y$ is the minimal model for $\mathcal{I}_X \otimes \mathcal{I}_Y$, it is also the minimal model for $\mathcal{I}_{X\times Y}$ and so the required quasi-isomorphism $\mathcal{M}_X \otimes \mathcal{M}_Y \to \mathcal{I}_{X\times Y}$ exists. Thus $X \times Y$ is model formal. ■

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