9

Matter models

This chapter provides a discussion of various matter models amenable to a treatment by means of conformal techniques. These matter models can be used as matter sources for the conformal Einstein field equations discussed in Chapter 8. The matter models to be considered are the electromagnetic field, radiation perfect fluids and the conformally invariant scalar field. These matter models share the property of having an energy-momentum tensor which is *trace free*. This property leads to simple transformation laws for the equations satisfied by the matter models. Moreover, the *unphysical equations* obtained by means of these transformations are regular at points where the conformal factor vanishes.

9.1 General properties of the conformal treatment of matter models

The fundamental object in the description of a matter model in general relativity is its ***energy-momentum tensor*** $\tilde{T}_{ab}$. The equations describing the model are then given by

$$\tilde{\nabla}^a \tilde{T}_{ab} = 0. \tag{9.1}$$

The energy-momentum tensor is related to the curvature of the spacetime via the Einstein field equations; see Equation (8.4). Despite this connection, a conformal transformation $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$ does not directly imply a transformation rule for the ***physical energy-momentum tensor*** $\tilde{T}_{ab}$. Nevertheless, it is convenient to define an ***unphysical energy-momentum tensor*** T_{ab} when rewriting Equation (9.1) in terms of geometric quantities derived from the rescaled metric $\boldsymbol{g}$.

9.1.1 The unphysical energy-momentum tensor

There is considerable freedom in a possible definition of T_{ab}. Guiding principles are simplicity in both the definition and the resulting form of the unphysical

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version of Equation (9.1). Arguably, the simplest definition of the unphysical energy-momentum tensor is one which is homogeneous with respect to the conformal factor Ξ. Accordingly, set

$$T_{ab} = \Xi^{-2}\tilde{T}_{ab}. \tag{9.2}$$

It follows then that

$$g^{ab}\nabla_a T_{bc} = \Xi^{-4}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{T}_{bc} - \Xi^{-5}\tilde{\nabla}_c\Xi\tilde{g}^{ab}\tilde{T}_{ab}. \tag{9.3}$$

Hence, Equation (9.1) *implies the equation* $\nabla^a T_{ab} = 0$ *only if*

$$\tilde{T} = 0, \qquad \tilde{T} \equiv \tilde{g}^{ab}\tilde{T}_{ab}.$$

This observation justifies definition (9.2) and the importance given in this chapter to trace-free matter models. As a result of the homogeneous nature of the transformation law in Equation (9.2), T_{ab} is trace free if $\tilde{T}_{ab}$ is trace free.

In the case of matter models with $\tilde{T}_{ab} \neq 0$, define $T \equiv g^{ab}T_{ab}$, so that $T = \Xi^{-4}\tilde{T}$. It follows from Equations (9.1) and (9.3) that

$$\nabla^a T_{ab} = \Xi^{-1}\nabla_b\Xi\, T.$$

This is an equation which is formally singular at the points where $\Xi = 0$. Dealing with this singularity is the essential problem faced in the analysis of general matter models by means of conformal methods.

9.1.2 The rescaled Cotton tensor

As discussed in Chapter 8, the matter fields couple to the conformal Einstein field equations through the ***rescaled Cotton tensor*** T_{abc}; compare Equations (8.22) and (8.23). Recall that the physical Cotton tensor is given by $T_{abc} = \Xi^{-1}\tilde{Y}_{abc}$ where

$$\tilde{Y}_{abc} = \tilde{\nabla}_a\tilde{L}_{bc} - \tilde{\nabla}_b\tilde{L}_{ac};$$

compare Equation (8.21). One readily finds that

$$T_{abc} = \frac{1}{2}\Xi^{-1}\left(\tilde{\nabla}_a\tilde{T}_{bc} - \tilde{\nabla}_b\tilde{T}_{ac}\right) - \frac{1}{6}\Xi^{-1}\left(\tilde{\nabla}_a\tilde{T}\tilde{g}_{bc} - \tilde{\nabla}_b\tilde{T}\tilde{g}_{ac}\right),$$

where it has been used that the physical Schouten tensor $\tilde{L}_{ab}$ is related to the physical energy-momentum tensor via Equation (8.5b). In what follows, attention will be restricted to the trace-free matter case so that

$$T_{abc} = \frac{1}{2}\Xi^{-1}\left(\tilde{\nabla}_a\tilde{T}_{bc} - \tilde{\nabla}_b\tilde{T}_{ac}\right).$$

The latter can be reexpressed in terms of the unphysical connection $\boldsymbol{\nabla}$ and the unphysical energy-momentum tensor T_{ab}. A computation using the methods of Chapter 5 yields

$$T_{abc} = \Xi\nabla_{[a}T_{b]c} + \nabla_{[a}\Xi T_{b]c} + g_{c[a}T_{b]e}\nabla^e\Xi. \tag{9.4}$$

From the above expression it follows that T_{abc} is regular whenever $\Xi = 0$ if T_{ab} is smooth at the conformal boundary.

Equation (9.4) can be expressed in terms of derivatives of the (conformal) matter fields. This feature complicates the construction of suitable conformal evolution equations as it introduces further derivatives of the fields into the principal part of the equations. This difficulty can be overcome by introducing evolution equations for the *derivatives of the matter fields* which cannot be eliminated with the equation $\nabla^a T_{ab} = 0$. This analysis depends on the specific properties of the matter model under consideration.

9.2 The Maxwell field

The ***electromagnetic*** or ***Maxwell field*** is the prototype of a relativistic matter model that can be treated by means of conformal methods. The Maxwell field is described by an antisymmetric tensor $\tilde{F}_{ab}$ – the ***Faraday tensor***. In terms of the latter, the source-free Maxwell equations are given by

$$\tilde{\nabla}^a \tilde{F}_{ab} = 0, \tag{9.5a}$$

$$\tilde{\nabla}_{[a} \tilde{F}_{bc]} = 0. \tag{9.5b}$$

Multiplying Equation (9.5b) by the volume form $\tilde{\epsilon}^{dabc}$, one obtains the alternative expression

$$\tilde{\nabla}^a \tilde{F}^*_{ab} = 0, \tag{9.6}$$

where $\tilde{F}^*_{ab} \equiv -\frac{1}{2}\tilde{\epsilon}_{ab}{}^{cd}\tilde{F}_{cd}$ denotes the ***dual Faraday tensor***. Now, introducing the ***self-dual Faraday tensor***

$$\tilde{\mathcal{F}}_{ab} \equiv \tilde{F}_{ab} + \mathrm{i}\tilde{F}^*_{ab},$$

it follows from (9.5a) and (9.6) that

$$\tilde{\nabla}^a \tilde{\mathcal{F}}_{ab} = 0. \tag{9.7}$$

This last equation contains the same information as Equations (9.5a) and (9.5b). The energy-momentum tensor of the electromagnetic field is quadratic in the Faraday tensor. It is given by

$$\tilde{T}_{ab} = \tilde{F}_{ac}\tilde{F}_b{}^c - \frac{1}{4}\tilde{g}_{ab}\tilde{F}_{cd}\tilde{F}^{cd}.$$

It can be readily verified that $\tilde{T} = 0$. Making use of the dual $\tilde{F}^*_{ab}$ one obtains the alternative expressions

$$\tilde{T}_{ab} = \frac{1}{2}\left(\tilde{F}_{ac}\tilde{F}_b{}^c + \tilde{F}^*_{ac}\tilde{F}^{*}_b{}^c\right) \tag{9.8a}$$

$$= \frac{1}{2}\tilde{\mathcal{F}}_{ac}\bar{\tilde{\mathcal{F}}}_b{}^c. \tag{9.8b}$$

It can be readily verified that the Maxwell Equations (9.5a) and (9.5b) imply that $\tilde{\nabla}^a\tilde{T}_{ab} = 0$.

Conformal transformation properties

The source-free Maxwell Equations (9.5a), (9.5b) and (9.6) are *conformally invariant*. In order to see this, assume that $(\mathcal{M}, \boldsymbol{g})$ is a conformal extension of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ with $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$, and define the ***conformal (unphysical) Faraday tensor*** via

$$F_{ab} \equiv \tilde{F}_{ab}. \tag{9.9}$$

It follows from this definition that $F^*_{ab} = \tilde{F}^*_{ab}$. Moreover, using the transformation laws between the connections $\tilde{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ one finds that Equations (9.5a), (9.5b) and (9.6) imply

$$\nabla^a F_{ab} = 0, \qquad \nabla_{[a} F_{bc]} = 0, \qquad \nabla^a F^*_{ab} = 0, \tag{9.10}$$

which shows the conformal invariance of the equations. Let $\hat{\boldsymbol{\nabla}}$ be a Weyl connection defined via $\hat{\boldsymbol{\nabla}} - \boldsymbol{\nabla} = \boldsymbol{S}(\boldsymbol{f})$ with $\boldsymbol{f}$ a covector. A further computation yields

$$\hat{\nabla}^a F_{ab} = 0, \qquad \hat{\nabla}_{[a} F_{bc]} = 0, \qquad \hat{\nabla}^a F^*_{ab} = 0.$$

Consistent with the transformation law (9.2) for the energy-momentum tensor one finds that

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} = \frac{1}{2} \left(F_{ac} F_b{}^c + F^*_{ac} F^{*c}_b \right).$$

Substituting the last expression in Equation (9.4) for the rescaled Cotton tensor one obtains

$$\begin{aligned} 2T_{abc} = {} & \nabla_{[a} F_{b]d} F_c{}^d + F_{d[a} \nabla_{b]} F_c{}^d + \nabla_{[a} F^*_{b]d} F^{*d}_c + F^*_{d[a} \nabla_{b]} F^{*d}_c \\ & + \nabla_{[a} \Xi F_{b]d} F_c{}^d + \nabla_{[a} \Xi F^*_{b]d} F^{*d}_c \\ & + g_{c[a} F_{b]e} F_d{}^e \nabla^d \Xi + g_{c[a} F^*_{b]e} F^{*e}_d \nabla^d \Xi. \end{aligned}$$

A direct inspection shows that the first four terms of the right-hand side contain derivatives of the Faraday tensor which cannot be eliminated using the (conformal) Maxwell Equations (9.10). Thus, it is necessary to consider equations for the derivatives of F_{ab}. A suitable equation can be obtained from the commutator of covariant derivatives applied to F_{ab}. More precisely, one has that

$$\nabla_a \nabla_b F_{cd} - \nabla_b \nabla_a F_{cd} = -R^e{}_{cab} F_{ed} - R^e{}_{dab} F_{ce}.$$

In view of this equation one introduces the ***auxiliary field*** $F_{abc} \equiv \nabla_a F_{bc}$ so that

$$\nabla_a F_{bcd} - \nabla_b F_{acd} = -R^e{}_{cab} F_{ed} - R^e{}_{dab} F_{ce}. \tag{9.11}$$

By construction one has that

$$F_{abc} = F_{a[bc]}, \qquad F_{[abc]} = 0, \qquad F^a{}_{ac} = 0.$$

9.2.1 The spinorial form of the Maxwell equations

The spinorial treatment of the Maxwell field is a direct consequence of the decomposition of spinors in irreducible components; see Section 3.1.6. The spinorial formulation of the Maxwell equations offers a number of computational advantages and makes more evident the similarities between the gravitational and electromagnetic fields.

In what follows, let $\tilde{F}_{AA'BB'}$ denote the ***spinorial counterpart of the Faraday tensor*** $\tilde{F}_{ab}$. By exploiting the antisymmetry of $\tilde{F}_{ab}$, it follows from Equation (3.13) that there exists a symmetric spinor $\tilde{\phi}_{AB}$, the ***Maxwell spinor***, such that

$$\tilde{F}_{AA'BB'} = \tilde{\phi}_{AB}\tilde{\epsilon}_{A'B'} + \bar{\tilde{\phi}}_{A'B'}\tilde{\epsilon}_{AB}, \qquad \tilde{\phi}_{AB} = \frac{1}{2}\tilde{F}_{AQ'B}{}^{Q'}. \tag{9.12}$$

Using the decomposition (9.12) it follows that

$$\tilde{\mathcal{F}}_{AA'BB'} = 2\tilde{\phi}_{AB}\tilde{\epsilon}_{A'B'},$$

where $\tilde{\mathcal{F}}_{AA'BB'}$ is the spinorial counterpart of the self-dual Faraday tensor $\tilde{\mathcal{F}}_{ab}$. Taking into account Equation (9.7) one obtains

$$\tilde{\nabla}^{A}{}_{A'}\tilde{\phi}_{AB} = 0. \tag{9.13}$$

This last equation is known as the ***spinorial Maxwell equation***. A further computation using Equation (9.8b) shows that the spinorial counterpart of the energy-momentum tensor takes the simple form

$$\tilde{T}_{AA'BB'} = \tilde{\phi}_{AB}\bar{\tilde{\phi}}_{A'B'}.$$

Behaviour under conformal rescalings

The definition of the unphysical (conformal) Faraday tensor given in Equation (9.9) suggests introducing the ***unphysical Maxwell spinor*** ϕ_{AB} as

$$\phi_{AB} \equiv \Xi^{-1}\tilde{\phi}_{AB}. \tag{9.14}$$

The factor Ξ^{-1} in the above definition is necessary to compensate for the factor Ξ picked up by the spinor $\tilde{\epsilon}_{AB}$ in Equation (9.12). It follows from Equation (9.13) and the transformation law of the connection upon conformal rescalings $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ given in Section 5.4 that

$$\nabla^{Q}{}_{A'}\phi_{BQ} = 0. \tag{9.15}$$

That is, the transformation rule (9.14) makes the spinorial Maxwell Equation (9.13) conformally invariant – this result was to be expected in view of the equations in (9.10). One readily sees the similarities between Equation (9.15) and the spinorial Bianchi identity $\nabla^{AA'}\phi_{ABCD} = 0$. Consistent with Equation (9.2) one finds that the spinorial counterpart of the unphysical energy-momentum tensor is given by

$$T_{AA'BB'} = \phi_{AB}\bar{\phi}_{A'B'}.$$

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Equation (9.15) can be expressed in terms of a Weyl connection $\hat{\boldsymbol{\nabla}} = \boldsymbol{\nabla} + \boldsymbol{S}(\boldsymbol{f})$ as

$$\hat{\nabla}^Q{}_{A'}\phi_{BQ} = f^Q{}_{A'}\phi_{BQ}.$$

In order to write a spinorial version of Equation (9.11) it is observed that the action of the commutator of covariant derivatives on the spinor ϕ_{AB} is given by

$$\nabla_{AA'}\nabla_{BB'}\phi_{CD} - \nabla_{BB'}\nabla_{AA'}\phi_{CD} = -\phi_{QD}R^Q{}_{CAA'BB'} - \phi_{CQ}R^Q{}_{DAA'BB'}.$$

Letting $\psi_{AA'BC} \equiv \nabla_{AA'}\phi_{BC}$, one obtains

$$\nabla_{AA'}\psi_{BB'CD} - \nabla_{BB'}\psi_{AA'CD} = -2\phi_{Q(C}R^Q{}_{D)AA'BB'}.$$

By construction, the auxiliary spinor $\psi_{AA'BC}$ possesses the symmetries

$$\psi_{AA'BC} = \psi_{AA'CB}, \qquad \psi^Q{}_{A'BQ} = 0.$$

9.3 The scalar field

A scalar field $\tilde{\phi}$ satisfying the wave equation

$$\tilde{\nabla}^a\tilde{\nabla}_a\tilde{\phi} = 0 \tag{9.16}$$

is a convenient matter model to couple to the Einstein field equations. It provides a way of incorporating dynamical degrees of freedom in spherically symmetric configurations; see, for example, Choptuik (1993) and Gundlach and Martín-García (2007). This idea has been exploited in a number of analyses of ***cosmic censorship*** and the formation of black holes through gravitational collapse; see, for example, Christodoulou (1986) and Dafermos (2003, 2005).

Unfortunately, as a direct computation shows, Equation (9.16) does not have good conformal transformation properties. This difficulty can be fixed by considering a modified version – the so-called ***conformally invariant scalar field equation***

$$\tilde{\nabla}^a\tilde{\nabla}_a\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi} = 0, \tag{9.17}$$

where $\tilde{R}$ denotes the Ricci scalar of the physical spacetime metric $\tilde{\boldsymbol{g}}$. Letting, as usual, $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ and defining the ***unphysical (conformal) scalar*** ϕ as

$$\phi \equiv \Xi^{-1}\tilde{\phi},$$

one finds, after a calculation using the transformation rule for the Ricci scalar Equation (5.6c), that

$$\nabla^a\nabla_a\phi - \frac{1}{6}R\phi = 0, \tag{9.18}$$

where R denotes the Ricci scalar of $\boldsymbol{g}$. An energy-momentum tensor for Equation (9.17) is given by

$$\tilde{T}_{ab} = \tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{4}\tilde{g}_{ab}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}^c\tilde{\phi} - \frac{1}{2}\tilde{\phi}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + \frac{1}{2}\tilde{\phi}^2\tilde{L}_{ab}. \tag{9.19}$$

A peculiarity of the above expression is the presence of the curvature terms $\tilde{L}_{ab}$ in the right-hand side of the energy-momentum tensor. Using the Einstein field Equations (8.4), the Schouten tensor can be reexpressed in terms of the energy-momentum tensor, so that Equation (9.19) takes the form

$$\tilde{T}_{ab} = \left(1 - \frac{1}{4}\tilde{\phi}^2\right)^{-1}\left(\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \frac{1}{4}\tilde{g}_{ab}\tilde{\nabla}_c\tilde{\phi}\tilde{\nabla}^c\tilde{\phi} - \frac{1}{2}\tilde{\phi}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + \frac{1}{12}(\lambda - \tilde{T})\tilde{\phi}^2\tilde{g}_{ab}\right).$$

Taking the trace of Equation (9.19) one finds that

$$\tilde{T} \equiv \tilde{g}^{ab}\tilde{T}_{ab} = -\frac{1}{2}\tilde{\phi}\left(\tilde{\nabla}^a\tilde{\nabla}_a\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi}^2\right).$$

Thus, the energy-momentum tensor of Equation (9.19) is trace free if and only if the conformally invariant wave Equation (9.17) is satisfied. A lengthier computation using the commutator of covariant derivatives and the Bianchi identity in the form $\tilde{\nabla}^a\tilde{L}_{ab} = \frac{1}{6}\tilde{\nabla}_b\tilde{R}$ shows that

$$\tilde{\nabla}^a\tilde{T}_{ab} = \tilde{\nabla}_b\tilde{\phi}\left(\tilde{\nabla}^c\tilde{\nabla}_c\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi}\right) - \frac{1}{2}\tilde{\phi}\tilde{\nabla}_b\left(\tilde{\nabla}^c\tilde{\nabla}_c\tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi}\right).$$

One concludes that $\tilde{T}_{ab}$ is divergence free if and only if Equation (9.17) holds. Finally, using the transformation law for the Schouten tensor under conformal rescalings, Equation (5.6b), one finds that

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{4}g_{ab}\nabla_c\phi\nabla^c\phi - \frac{1}{2}\phi\nabla_a\nabla_b\phi + \frac{1}{2}\phi^2 L_{ab},$$

so that $T_{ab} = \Xi^{-2}\tilde{T}_{ab}$. It follows from the previous discussion that

$$\nabla^a T_{ab} = 0, \qquad g^{ab}T_{ab} = 0.$$

Spinorial description

The straightforward spinorial counterpart of Equation (9.19) is given by

$$\begin{aligned}\tilde{T}_{AA'BB'} = \tilde{\nabla}_{AA'}\tilde{\phi}\tilde{\nabla}_{BB'}\tilde{\phi} - \frac{1}{4}\tilde{\epsilon}_{AB}\tilde{\epsilon}_{A'B'}\tilde{\nabla}_{PP'}\tilde{\phi}\tilde{\nabla}^{PP'}\tilde{\phi}\\ - \frac{1}{2}\tilde{\phi}\tilde{\nabla}_{AA'}\tilde{\nabla}_{BB'}\tilde{\phi} + \frac{1}{2}\tilde{\phi}^2\tilde{L}_{AA'BB'}.\end{aligned}$$

Applying the decomposition formula (3.12) to $\tilde{\nabla}_{AA'}\tilde{\phi}\tilde{\nabla}_{BB'}\tilde{\phi}$ and $\tilde{\nabla}_{AA'}\tilde{\nabla}_{BB'}\tilde{\phi}$ one finds that

$$\tilde{\nabla}_{AA'}\tilde{\phi}\tilde{\nabla}_{BB'}\tilde{\phi} = \tilde{\nabla}_{A(A'}\tilde{\phi}\tilde{\nabla}_{B')B}\tilde{\phi} + \frac{1}{4}\tilde{\epsilon}_{AB}\tilde{\epsilon}_{A'B'}\tilde{\nabla}_{PP'}\tilde{\phi}\tilde{\nabla}^{PP'}\tilde{\phi},$$

$$\tilde{\nabla}_{AA'}\tilde{\nabla}_{BB'}\tilde{\phi} = \tilde{\nabla}_{A(A'}\tilde{\nabla}_{B')B}\tilde{\phi} + \frac{1}{4}\tilde{\epsilon}_{AB}\tilde{\epsilon}_{A'B'}\tilde{\nabla}_{PP'}\tilde{\nabla}^{PP'}\tilde{\phi},$$

where it has been used that

$$\tilde{\nabla}_{P(A'}\tilde{\phi}\tilde{\nabla}^P{}_{B')}\tilde{\phi} = 0, \qquad \tilde{\nabla}_{P(A'}\tilde{\nabla}^P{}_{B')}\tilde{\phi} = 0.$$

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The above formulae, together with the wave equation (9.17), imply the following alternative spinorial expression for the energy-momentum tensor:

$$\tilde{T}_{AA'BB'} = \tilde{\nabla}_{A(A'}\tilde{\phi}\tilde{\nabla}_{B')B}\tilde{\phi} - \frac{1}{2}\tilde{\phi}\tilde{\nabla}_{A(A'}\tilde{\nabla}_{B')B}\tilde{\phi} + \frac{1}{2}\tilde{\phi}^2\tilde{\Phi}_{AA'BB'}$$

where $\tilde{\Phi}_{AA'BB'}$ is the (physical) trace-free Ricci spinor. The unphysical spacetime version of the above equation follows directly by removing the $\tilde{}$ of the various fields.

9.3.1 Equations for the derivatives of the scalar field

As in the case of the electromagnetic field, the coupling of the conformally invariant scalar field to the conformal field equations through the rescaled Cotton tensor T_{abc} involves derivatives of ϕ. Indeed, a calculation exploiting the fact that $\nabla_{[a}\nabla_{b]}\phi = 0$ shows that

$$\begin{aligned}\nabla_{[a}T_{b]c} = \frac{3}{2}\nabla_{[b}\phi\nabla_{a]}\nabla_c\phi - \frac{1}{2}g_{c[b}\nabla_{a]}\nabla_e\phi\nabla^e\phi + \phi\nabla_{[a}\phi L_{b]c}\\ - \frac{1}{2}\phi\nabla_{[a}\nabla_{b]}\nabla_c\phi + \frac{1}{2}\phi^2\nabla_{[a}L_{b]c}.\end{aligned}$$

The terms in the second line of the preceding equation can be rewritten using the commutator

$$\nabla_{[a}\nabla_{b]}\nabla_c\phi = -\frac{1}{2}R^e{}_{cab}\nabla_e\phi$$

and the Cotton Equation (8.23). Putting everything together in Equation (9.4) and rearranging one obtains

$$\begin{aligned}\left(1 - \frac{1}{4}\phi^2\Xi^2\right)T_{abc} = \frac{3}{2}\Xi\nabla_{[b}\phi\nabla_{a]}\nabla_c\phi - \frac{1}{2}\Xi g_{c[b}\nabla_{a]}\nabla_e\phi\nabla^e\phi + \phi\nabla_{[a}\phi L_{b]c}\\ + \frac{1}{4}\Xi\phi R^e{}_{cab}\nabla_e\phi + \frac{1}{4}\Xi\phi^2\nabla_e\Xi R^e{}_{cab} + \nabla_{[a}\Xi T_{b]c} + g_{c[a}T_{b]e}\nabla^e\Xi.\end{aligned}$$

The above expression contains first and second derivatives of ϕ which cannot be eliminated using the wave Equation (9.18). Accordingly, field equations for these derivatives need to be constructed.

In what follows, let $\phi_a \equiv \nabla_a\phi$, $\phi_{ab} \equiv \nabla_a\nabla_b\phi$. As $\boldsymbol{\nabla}$ is torsion free one has that $\phi_{[ab]} = 0$ and one can write

$$\phi_{ab} = \phi_{\{ab\}} + \frac{1}{4}g_{ab}\phi_e{}^e = \phi_{\{ab\}} + \frac{1}{24}g_{ab}R\phi, \tag{9.20}$$

where in the second equality one has used Equation (9.18) in the form $\phi_e{}^e = \frac{1}{6}R\phi$. Regarding ϕ_a and ϕ_{ab} as further field unknowns one obtains the *field equations*

$$\nabla_a\phi - \phi_a = 0, \qquad \nabla_a\phi_b - \phi_{\{ab\}} - \frac{1}{24}g_{ab}R\phi = 0, \qquad \nabla^e\phi_e - \frac{1}{6}R\phi = 0.$$

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To obtain equations for ϕ_{ab} one considers the commutator of covariant derivatives applied to $\nabla_c\phi$ in the form

$$\nabla_{[a}\phi_{b]c} = -\frac{1}{2}R^d{}_{cab}\phi_d.$$

Letting $\psi_{ab} \equiv \phi_{\{ab\}}$ and using the decomposition (9.20) one obtains

$$\nabla_{[a}\psi_{b]c} - \frac{1}{24}\left(Rg_{c[a}\phi_{b]} + \phi g_{c[b}\nabla_{a]}R\right) = -\frac{1}{2}R^d{}_{cab}\phi_d.$$

Finally, an equation for the trace term $\phi_e{}^e$ is obtained by differentiating Equation (9.18) so that

$$\nabla_a\phi_e{}^e - \frac{1}{6}\left(\phi_a R + \phi\nabla_a R\right) = 0.$$

9.3.2 Relation to other wave equations

Solutions to the conformally invariant wave Equation (9.17) on a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ are related to solutions of the standard wave equation on a conformally related spacetime $(\grave{\mathcal{M}}, \grave{\boldsymbol{g}})$ through a transformation first discussed in Bekenstein (1974): the scalar field $\tilde{\phi}$ can be used to define a metric $\hat{\boldsymbol{g}}$ conformally related to $\tilde{\boldsymbol{g}}$ via

$$\grave{\boldsymbol{g}} = \grave{\Xi}^2\tilde{\boldsymbol{g}}, \qquad \grave{\Xi} \equiv 1 - \frac{1}{4}\tilde{\phi}^2.$$

It follows from a direct computation that the scalar field

$$\grave{\phi} \equiv \sqrt{6}\arctan\frac{1}{2}\tilde{\phi}$$

is a solution of the equation

$$\grave{\nabla}_a\grave{\nabla}^a\grave{\phi} = 0.$$

As noticed in Bičák et al. (2010), this observation can be turned into a procedure to construct solutions to the Einstein-scalar field equations out of vacuum static solutions; see also Buchdahl (1959).

9.4 Perfect fluids

Perfect fluids constitute an important class of matter models for the Einstein field equations. In the cosmological context, perfect fluids are used to describe the matter content of the universe at a suitably large scale; see, for example, Ellis et al. (2012). Given a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, the ***energy-momentum tensor of a perfect fluid*** with ***4-velocity*** $\tilde{u}^a$, ***pressure*** $\tilde{p}$ and ***density*** $\tilde{\varrho}$ is given by

$$\tilde{T}_{ab} = (\tilde{\varrho} + \tilde{p})\tilde{u}_a\tilde{u}_b - \tilde{p}\tilde{g}_{ab}, \tag{9.21}$$

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with $\tilde{u}^a$ satisfying the normalisation condition $\tilde{u}_a\tilde{u}^a = 1$. The equations of motion for the fields $\tilde{u}^a$, $\tilde{\varrho}$ and $\tilde{p}$ are given by $\tilde{\nabla}^a\tilde{T}_{ab} = 0$. This last equation gives four equations for six unknowns. The normalisation of $\tilde{u}^a$ can be used to eliminate one of the components of the 4-velocity (usually the time component). To close the system a *phenomenological constitutive relation* linking the pressure and the density must be prescribed. A standard assumption made on perfect fluids is to have the density and the pressure related to each other by means of a ***barotropic equation of state*** $\tilde{p} = f(\tilde{\varrho})$ with f a smooth function of the density $\tilde{\varrho}$. From Equation (9.21) it follows that $\tilde{T} = \tilde{\varrho} - 3\tilde{p}$. Thus, the energy-momentum tensor of a perfect fluid is trace free if and only if

$$\tilde{p} = \frac{1}{3}\tilde{\varrho}. \tag{9.22}$$

This constitutive relation is known as the ***equation of state of radiation***. *In what follows, the discussion will be restricted to perfect fluids satisfying the equation of state* (9.22). To discuss the perfect fluid in the conformally rescaled spacetime $(\mathcal{M}, \boldsymbol{g})$ with $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ it is convenient to consider the following unphysical conformal fields

$$u_a \equiv \Xi\tilde{u}_a, \qquad \varrho \equiv \Xi^{-4}\tilde{\varrho}, \qquad p \equiv \Xi^{-4}\tilde{p}.$$

The above definitions are consistent with the transformation law for the energy-momentum tensor of Equation (9.2). Moreover, it follows that $p = \frac{1}{3}\varrho$, so that the unphysical energy-momentum tensor takes the form

$$T_{ab} = \frac{4}{3}\varrho u_a u_b - \frac{1}{3}\varrho g_{ab} \quad \text{with} \quad \nabla^a T_{ab} = 0. \tag{9.23}$$

Moreover, one has that $u_a u^a = 1$, so that differentiating along u^a one finds that

$$u^a\nabla_a(u_b u^b) = 0.$$

From this expression it follows that if $u_a u^a = 1$ at some point in a flow line, then $u_a u^a = 1$ everywhere along the flow line. From Equation (9.23) it readily follows that

$$\frac{4}{3}\left(u_a u^c\nabla_c\varrho + \varrho u_a\nabla_c u^c + \varrho u^c\nabla_c u_a\right) - \frac{1}{3}\nabla_a\varrho = 0.$$

Contracting this equation, respectively, with u^a and $g_{ab} - u_a u_b$ one obtains

$$u^a\nabla_a\varrho + \frac{4}{3}\varrho\nabla_a u^a = 0,$$
$$\frac{4}{3}\varrho u^c\nabla_c u_a + \frac{1}{3}u_a u^c\nabla_c\varrho - \frac{1}{3}\nabla_a\varrho = 0.$$

These equations are the conformal versions of the ***equation of energy conservation*** and the ***equations of motion***; see, for example, Choquet-Bruhat (2008). A discussion on how to use these equations to construct suitable evolution equations for the fields ϱ and the spatial components u^i of the fluid 4-velocity can be found in the same reference.

9.5 Further reading

A further matter model amenable to a treatment by means of conformal methods is the Yang-Mills field. The Yang-Mills equations can be regarded as a suitable generalisation of the Maxwell equations; see, for example, Frankel (2003) for a discussion. The conformal field equations with matter source given by a Yang-Mills field of arbitrary gauge group have been discussed in Friedrich (1991). The discussion of the Maxwell field presented in this chapter is adapted from that reference. A treatment of the conformal Einstein-Maxwell system by means of Weyl connections is given in Lübbe and Valiente Kroon (2012).

The discussion of the conformal field equations coupled to the conformally invariant wave equation was first given in Hübner (1995). An alternative approach to the analysis of the conformal Einstein field equations with a scalar field can be found in Bičák et al. (2010). In Friedrich (2015b) it has been shown that the Einstein–massive scalar field system has good conformal properties if the mass of the scalar field and the cosmological constant satisfy the relation $3m^2 = -2\lambda$.

Finally, the conformal Einstein-Euler equations have been analysed in Lübbe and Valiente Kroon (2013b) and used to prove the future non-linear stability of perturbations of Friedman-Lemaître-Robinson-Walker cosmological models with a radiation fluid. Analyses of the Einstein-Euler system not making use of conformal methods can be found in Rodnianski and Speck (2013) and Speck (2012).

The purpose of this chapter has been to present a discussion of matter models with properties which make them suitable sources for the conformal field equations. However, conformal methods have also been used for other types of constructions. As an example, one has Bičák and Krtouš (2001, 2002) where the conformal invariance of the Maxwell equations has been exploited to construct the analogue of the Born solution (describing the motion of a pair of uniformly accelerated charges) in the de Sitter spacetime.