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The Ergodic Hilbert Transform for Admissible Processes

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Abstract. It is shown that the ergodic Hilbert transform exists for a class of bounded symmetric admissible processes relative to invertible measure preserving transformations. This generalizes the well-known result on the existence of the ergodic Hilbert transform.

1 Introduction

Let (X, Σ, μ) be a probability space and $T\colon X \to X$ be an invertible measure preserving transformation. For $f \in L_p(X)$, define $\mathbf{H}_n f = {\sum'}_{i=-n}^{n} \frac{T^i f}{i}$, where ${\sum'}_{i=-n}^{n}$ stands for summation without the term with zero denominator. The operator $\mathbf{H} f = \lim_{n\to\infty} \mathbf{H}_n f = {\sum'}_{i\in\mathbb{Z}} \frac{T^i f}{i}$ is known as the ergodic Hilbert transform. In 1955 M. Cotlar proved that when T is induced by an invertible measure preserving transformation, the ergodic Hilbert transform exists a.e. for every $f \in L_1$ [C]. This result has since been revisited by various authors [CaP, DL, P1]. Among them, Petersen [P1] gave a direct proof of this result, and Derriennic and Lin [DL] investigated the conditions under which the a.e. convergence of the one-sided ergodic Hilbert transform holds. All the results above deal with additive processes in their respective settings. This naturally leads to the question of whether the analogous results can be obtained for not neccessarily additive processes. In this article, we will answer this question affirmatively for a class of bounded superadditive processes, namely, for *admissible* processes. This type includes additive processes $\{T^k f\}_{k\in\mathbb{Z}}$ as a special case. Hence, our result generalizes some of the theorems mentioned above.

A family $F = \{f_i\}_{i\in\mathbb{Z}} \subset L_p$ is called a *T-superadditive process on* $\mathbb{Z}$ if the sequence of its partial sums $\{F_k\}_{k\in\mathbb{Z}}$, where $F_k = \sum_{i=0}^{k-1} f_i$, if $k \geq 1$ and $F_k = \sum_{i=k+1}^{0} f_i$, if $k \leq -1$, satisfies that

$$F_{n+m} \geq F_n + T^n F_m \quad \text{and} \quad F_{-(n+m)} \geq F_{-n} + T^{-n} F_{-m}, \text{ for all } n, m \geq 0.$$

The process F is called *T-subadditive* when the reverse inequalities hold. If F is both superadditive and subadditive, it is called *T-additive* and is necessarily of the form $\{T^i f\}_{i\geq 1}$, for some f. For a T-superadditive process F we will adopt the notation $\mathbf{H}_n F = {\sum'}_{i=-n}^{n} \frac{1}{i} f_i$ and $\mathbf{H} F = \lim_n \mathbf{H}_n F$.

A process $F = \{f_i\}_{i\in\mathbb{Z}} \subset L_p$ is called a *T-admissible process on* $\mathbb{Z}$ if

$$T f_i \leq f_{i+1} \text{ for } i \geq 0, \quad \text{and} \quad T^{-1} f_i \leq f_{i-1} \text{ for } i \leq 0.$$

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Obviously, any T-admissible process is T-superadditive. An admissible family $F = \{f_i\} \subset L_p$, $1 \le p < \infty$, is called *strongly bounded* if $\sup_{n\in\mathbb{Z}} \|f_n\|_p < \infty$. When $p = 1$, a T-superadditive process $F \subset L_1$ on $\mathbb{Z}$ is called *bounded* if

$$\gamma_F := \sup_{n\in\mathbb{Z}} \frac{1}{|n|}\|F_n\|_1 < \infty.$$

In what follows, T will always be an invertible measure preserving transformation, and all families $F = \{f_i\}_{i\in\mathbb{Z}}$ defining a T-admissible process will be assumed to satisfy the *symmetry* condition:

$$T^{2i} f_{-i} = f_i \qquad \text{for all } i \in \mathbb{Z}.$$

When F is a T-admissible process on $\mathbb{Z}$, we have $f_i - T^i f_0 \ge 0$, and consequently, if a result holds for $\mathbf{H}_n f_0$, then the same is also valid for $\mathbf{H}_n F$ if and only if it is valid for $\mathbf{H}_n G$, where $g_i = f_i - T^i f_0$. Therefore, in such a case, we can assume that $f_i \ge 0$, for each $i \in \mathbb{Z}$. In particular, since we will consider sums without 0-th term, for convenience, we will always assume that $f_0 = 0$.

2 Preliminaries

By Kingman's decomposition theorem [AS, K], any bounded T-superadditive process $F = \{F_n\}_{n\ge 1} \subset L_1$ can be decomposed into a difference of two processes as $F_n = G_n - H_n$, where $G = \{G_n\}$ is an additive process and $H = \{H_n\}$ is a *purely subadditive* process, in the sense that H does not dominate any nonzero T-additive process and $\lim_n \frac{1}{n} H_n = 0$ a.e. (and hence in norm). The function δ for which $G_n = \sum_{k=0}^{n-1} T^k\delta$ is called the *exact dominant* of the process, and $\int \delta = \gamma_F$ holds. If $H = \{H_n\}$ is the purely subadditive part of a superadditive process, then necessarily $H_n = \sum_{i=0}^{n-1} h_i \ge 0$, for each $n \ge 0$. However, in general, this does not imply that each $h_i \ge 0$.

The following statement shows that positive symmetric strongly bounded admissible processes relative to invertible measure preserving transformations admit a simpler representation.

Proposition 2.1 *Let T be an invertible measure preserving transformation and $F = \{f_n\}$ be a positive symmetric strongly bounded T-admissible process. Then there exists a sequence of nonnegative functions $\{v_k\} \in L_p$ such that $f_n = T^n v_{|n|}$ for all $n \in \mathbb{Z}$. Furthermore, there exists $\delta \in L_p$ such that $f_n \le T^n\delta$ for all $n \in \mathbb{Z}$ and $\|\delta\|_p = \sup_{n\in\mathbb{Z}} \|f_n\|_p$.*

Proof Define $\{v_i\} \subset L_p^+$ by $v_i = T^{-i} f_i$ and $v_{-i} = T^i f_{-i}$, $i \ge 0$. From the symmetry condition $v_i = v_{-i}$ for all $i \in \mathbb{Z}$. Since F is T-admissible, we have $v_i \le v_{i+1}$ for all $i \ge 0$. By monotone convergence theorem and by (strong) boundedness of the process, there exists $\delta \in L_p^+$ such that $\|\delta\|_p = \lim_i \|v_i\|_p$. It is clear that for any $n \in \mathbb{Z}$, $v_n \le \delta$, and hence $f_n \le T^n\delta$ for all $n \in \mathbb{Z}$. ∎

Remarks (1) The function $\delta \in L_p^+$ is an exact dominant for F if $p = 1$. When $p > 1$, it is called a *dominant* for F.

(2) Proposition 2.1 also implies that if F is a strongly bounded admissible process relative to an invertible measure preserving transformation, and if $H = \{\sum_{i=0}^{n-1} h_i\}$ is the purely subadditive part of F, then each $h_i \geq 0$. Indeed, $h_i = T^i\delta - f_i = T^i(\delta - \nu_i)$.

The following simple statement shows that boundedness and strong boundedness are the same for admissible processes in L_1.

Proposition 2.2 *A positive T-admissible process $F \subset L_1$ is bounded if and only if it is strongly bounded.*

Proof Obviously, if $\sup_i \|f_i\|_1 < \infty$, then F is bounded. Conversely, assume that F is bounded. By the symmetry condition, it is enough to consider the case $i \geq 0$. Since T is measure preserving and F is T-admissible,

$$\begin{aligned}\|f_i\|_1 &= \frac{1}{m}\sum_{k=0}^{m-1}\int T^k f_i \leq \frac{1}{m}\sum_{i=k}^{m+k-1}\int f_i \leq \frac{1}{m}\int [F_{m+k} - F_k] \leq \frac{1}{m}\int F_{m+k}\\ &= \frac{m+k}{m}\Big[\frac{1}{m+k}\int F_{m+k}\Big] \leq \frac{m+k}{m}\gamma_F.\end{aligned}$$

Letting $m \to \infty$ we have $\sup_{i\geq 0}\|f_i\|_1 \leq \gamma_F$. ∎

Remark Proposition 2.2 is not valid if $p \neq 1$. Indeed, there are admissible processes in L_2 satisfying $\sup_n \frac{1}{|n|}\|F_n\|_2 < \infty$, which are not strongly bounded. We provide an example of such a process here.

Example 1 Let bounded positive functions $g_n, n = 0, 1, 2, \dots$ be given on some probability space, and suppose that a measure-preserving point transformation T is given such that the entire doubly-indexed family $\{T^n g_m\}_{n,m=0,1,2,\dots}$ is independent, considered as a family of random variables. For example, we could take T to be the product of countably many shifts on countably many infinite product spaces, and choose g_n as a function of the first coordinate of the n-th product space. Consider the case in which $g_0 = 0$ and $\int g_n = 1/2^n$, $\int g_n^2 = 1/\sqrt{n}$. Define f_n, $n = 0, 1, 2, \dots$ by

$$f_n = T^n g_0 + T^{n-1} g_1 + \cdots + T g_{n-1} + g_n.$$

As usual, define $F_n = f_0 + f_1 + \cdots + f_{n-1}$. It is easy to check that the sequence $\{f_n\}$ is T-admissible. Clearly $\int f_n \leq 1$ for all n. Also

$$\int f_n^2 \leq \Big(\int f_n\Big)^2 + \int g_0^2 + \int g_1^2 + \cdots + \int g_n^2,$$

and therefore,

$$\int f_n^2 \leq 1 + c_1\sqrt{n}.$$

The independence assumption implies that $\{f_i\}$ is an independent family. It follows that
$$\int F_n^2 \leq \Big(\int F_n\Big)^2 + \int f_0^2 + \cdots + \int f_{n-1}^2 \leq n^2 + c_2 n\sqrt{n},$$
and so $\sup_n \frac{1}{n}\|F_n\|_2 < \infty$. But
$$\int f_n^2 \geq \int g_0^2 + \int g_1^2 + \cdots + \int g_n^2 \geq c_3\sqrt{n},$$
and hence F is not strongly bounded.

It is well known that if $\{a_i\}$ is a sequence of nonnegative real numbers with mean zero (*i.e.*, $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} a_i = 0$), then there exists a set of zero density $K \subset \mathbb{N}$ such that $\lim_{n\in\mathbb{N}\setminus K, n\to\infty} a_n = 0$. So, for a fixed $x \in X$, since $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} h_i = 0$, there exists a set $K \subset \mathbb{N}$, of density 0 such that $\lim_{n\in\mathbb{N}\setminus K, n\to\infty} h_n(x) = 0$. On the other hand, since $0 \leq H_n$, we also have $0 \leq \int \frac{1}{n}H_n = \frac{1}{n}\int \sum_{k=0}^{n-1} h_i = \frac{1}{n}\sum_{k=0}^{n-1}\int h_i$. By subadditivity, $H_n \leq \sum_{k=0}^{n-1} T^i h_0$, which implies that $0 \leq \int \frac{1}{n}H_n \leq \frac{1}{n}\sum_{k=0}^{n-1}\int T^i h_0 \leq \|h_0\|_1$. Hence the sequence $\{\frac{1}{n}H_n\}$ is L_1-norm bounded. Now, if $g_n = \frac{1}{n}\sum_{k=0}^{n-1} T^i h_0$, then $\frac{1}{n}H_n \leq g_n$, and $g_n \to g^*$ a.e. for some $g^* \in L_1$. Since $\frac{1}{n}H_n \to 0$ a.e., by the generalized Lebesgue Convergence Theorem, $\lim \int g_n \to \int g^*$ implies that $0 = \lim_n \frac{1}{n}\int H_n = \lim_n \frac{1}{n}\sum_{k=0}^{n-1}\int h_i$. Consequently, there exists a set $K \subset \mathbb{N}$, of density 0 such that $\lim_{n\in\mathbb{N}\setminus K, n\to\infty} \int h_n(x) = 0$. Since, by admissibility, $h_m \leq T^k h_{m-k}$ for any $m \geq k \geq 0$, we must have that $\lim_{n\to\infty}\int h_n(x) = 0$ while n ranges through all positive integers. This fact, though, does not exclude the possibility that $\|H_n\|_1 \uparrow \infty$, indeed, this is the case that is most interesting. In the same vein, the convergence of averages of the form $\frac{1}{n^{1-\alpha}}\sum_{k=1}^{n}\|h_k\|_1$, for some $\alpha \in (0,1)$, is not guaranteed either.

Under some assumptions on the sequence $\{\|h_k\|_p\}$, however, one can say more about the convergence of averages like $\frac{1}{n^{1-\alpha}}\sum_{k=1}^{n}\|h_k\|_p$. To that end, first we state the following proposition (without proof), which can be referred to as *l'Hôpital's Rule for sequences*:

Proposition 2.3 *Let $\{a_n\}$ be any sequence of positive real numbers and let $\{b_n\}$ be a sequence with $b_n \uparrow \infty$. Then*
$$\liminf_n \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \leq \liminf_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_{n+1}-a_n}{b_{n+1}-b_n}.$$
Consequently, if $\lim \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ exists, then so does $\lim \frac{a_n}{b_n}$.

3 Existence Theorems for Admissible Processes

We begin with proving a lemma needed for the proof of the weak (1,1) maximal inequality. Following the observations made in the preceding section, we will assume that if $H = \{h_i\}$ is the purely subadditive part of an admissible process $F \subset L_1$, then

$$(*) \qquad n^\alpha \|h_n - h_{-n}\|_1 = O(1), \quad \text{for some } 0 < \alpha < 1.$$

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This condition, which is rather easy to check, seems necessary for the proof of maximal inequality, at least when the techniques utilized in this article are concerned. That is why, in what follows, we will assume that all the admissible processes satisfy this property.

Lemma 3.1 *Let $F \subset L_1$ be a bounded symmetric T-admissible process with exact dominant δ. Assume that the purely subadditive part $H = \{h_n\}_{n\in\mathbb{Z}}$ of F satisfies $(*)$. Then there exists a constant $A > 0$ such that*

$$\int \Big| \sum_{i=1}^{\infty} \frac{H_i - H_{-i}}{i(i+1)} \Big| \, d\mu \le A\|\delta\|_1.$$

Proof First, observe that for any $n \ge 1$,

$$\int \Big| \sum_{i=1}^{n} \frac{H_i - H_{-i}}{i(i+1)} \Big| \, d\mu \le \sum_{i=1}^{n} \int \frac{|H_i - H_{-i}|}{i(i+1)} \, d\mu$$
$$\le \lim_n \sum_{i=1}^{n} \Big(\frac{1}{i(i+1)} \sum_{j=1}^{i-1} \|h_j - h_{-j}\|_1 \Big).$$

Now,

$$\sum_{i=1}^{n} \Big(\frac{1}{i(i+1)} \sum_{j=1}^{i-1} \|h_j - h_{-j}\|_1 \Big) \le \sum_{i=1}^{n} \frac{1}{i^{1+\alpha}} \Big(\frac{1}{i^{1-\alpha}} \sum_{j=1}^{i-1} \|h_j - h_{-j}\|_1 \Big)$$

for any $\alpha \in (0,1)$. Let $a_n = \sum_{j=1}^{n-1} \|h_j - h_{-j}\|_1$, and $b_n = n^{1-\alpha}$. Since H satisfies $(*)$, without loss of generality we can assume that $\sup_n n^{\alpha}\|h_n - h_{-n}\|_1 \le \|\delta\|_1$. Then,

$$\limsup_n \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \limsup_n \frac{\|h_n - h_{-n}\|_1}{(n+1)^{1-\alpha} - n^{1-\alpha}} \le \lim_n \frac{n^{-\alpha}}{(n+1)^{1-\alpha} - n^{1-\alpha}} \|\delta\|_1$$
$$= \lim_n \Big[\frac{\frac{1}{n}}{(1+\frac{1}{n})^{1-\alpha} - 1} \Big] \|\delta\|_1 = \frac{1}{1-\alpha}\|\delta\|_1.$$

Thus, by Proposition 2.3,

$$\limsup_i \frac{1}{i^{1-\alpha}} \sum_{j=1}^{i-1} \|h_j - h_{-j}\|_1 \le \frac{1}{1-\alpha}\|\delta\|_1,$$

which implies that the sequence $\{\frac{1}{i^{1-\alpha}} \sum_{j=1}^{i-1} \|h_j - h_{-j}\|_1\}$ is bounded above by a constant multiple of $\|\delta\|_1$. Therefore,

$$\int \sum_{i=1}^{n} \frac{|H_i - H_{-i}|}{i(i+1)} \, d\mu \le A\|\delta\|_1, \quad \text{for some constant } A. \qquad \blacksquare$$

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Theorem 3.2 *Let $F \subset L_1$ be a bounded symmetric T-admissible process and let its purely subadditive part $H = \{h_n\}_{n\in\mathbb{Z}}$ satisfy $(*)$. If $\lambda > 0$, then there is a constant C (which may not be the same at each occurrence) such that*

$$\mu\Big(\Big\{x : \sup_{n\geq 1}\Big|{\sum_{i=-n}^{n}}' \frac{f_i(x)}{i}\Big| > \lambda\Big\}\Big) \leq \frac{C}{\lambda}\|\delta\|_1,$$

where δ is an exact dominant for F.

Proof Using the Kingman decomposition,

$${\sum_{i=-n}^{n}}' \frac{f_i}{i} = {\sum_{i=-n}^{n}}' \frac{T^i\delta}{i} - {\sum_{i=-n}^{n}}' \frac{h_i}{i},$$

where $\{h_i\}$ is the purely subadditive part of F. Therefore, by Abel's summation by parts formula,

$${\sum_{i=-n}^{n}}' \frac{f_i}{i} = {\sum_{i=-n}^{n}}' \frac{T^i\delta}{i} - \sum_{i=1}^{n} \frac{H_i - H_{-i}}{i(i+1)} - \frac{1}{n}H_n - \frac{1}{n}H_{-n}.$$

If $E = \{x : \sup_{n\geq 1} |\sum{'}_{i=-n}^{n} \frac{f_i(x)}{i}| > \lambda\}$, then $E \subset E_0 \cup E_1 \cup E_2 \cup E_3$, where

$$E_0 = \Big\{x : \sup_n\Big|{\sum_{i=-n}^{n}}' \frac{T^i\delta(x)}{i}\Big| > \frac{\lambda}{4}\Big\},$$

$$E_1 = \Big\{x : \sup_n\Big|\sum_{i=1}^{n} \frac{H_i(x) - H_{-i}(x)}{i(i+1)}\Big| > \frac{\lambda}{4}\Big\},$$

$$E_2 = \Big\{x : \sup_n \frac{1}{n}H_n(x) > \frac{\lambda}{4}\Big\},$$

$$E_3 = \Big\{x : \sup_n \frac{1}{n}H_{-n}(x) > \frac{\lambda}{4}\Big\}.$$

Clearly, since $\{\sum{'}_{i=-n}^{n} \frac{T^i\delta}{i}\}$ admits weak (1,1) maximal inequality [P$_1$], $\mu(E_0) \leq \frac{C_1}{\lambda}\|\delta\|_1$, for some constant C_1. On the other hand, by subadditivity,

$$\frac{1}{n}H_n \leq \frac{1}{n}\sum_{i=0}^{n-1} T^i h_0 \quad \text{and} \quad \frac{1}{n}H_{-n} \leq \frac{1}{n}\sum_{i=0}^{n-1} T^{-i} h_0,$$

where $h_0 = \delta - f_0$. Hence, we have $\mu(E_2) \leq \frac{C_2}{\lambda}\|\delta\|_1$ and $\mu(E_3) \leq \frac{C_3}{\lambda}\|\delta\|_1$, for some constants C_2 and C_3.

Now, observe that the sequence $\{\sum_{i=1}^{n}\frac{|H_i-H_{-i}|}{i(i+1)}\}_{n\geq 1}$ is monotone increasing. Hence, if we define $\hat{h}(x) := \lim_n \sum_{i=1}^{n}\frac{|H_i(x)-H_{-i}(x)|}{i(i+1)}$, then $\sup_n|\sum_{i=1}^{n}\frac{H_i(x)-H_{-i}(x)}{i(i+1)}| \leq \hat{h}(x)$ a.e. By the monotone convergence theorem and Lemma 3.1,

$$\int \hat{h}\,d\mu = \lim_n \int \sum_{i=1}^{n}\frac{|H_i - H_{-i}|}{i(i+1)}\,d\mu \leq A\|\delta\|_1.$$

Hence, by Chebyshev's inequality,

$$\begin{aligned}\mu\Big\{x : \sup_n\Big|\sum_{i=1}^{n}\frac{H_i(x)-H_{-i}(x)}{i(i+1)}\Big| > \frac{\lambda}{4}\Big\} &\leq \mu\Big\{x : \hat{h}(x) > \frac{\lambda}{4}\Big\}\\ &\leq \frac{4}{\lambda}\|\hat{h}\|_1 \leq \frac{4A}{\lambda}\|\delta\|_1.\end{aligned}$$

Therefore, we obtain that $\mu(E) \leq \frac{C}{\lambda}\|\delta\|_1$, where $C = C_1 + 4A + C_2 + C_3$, proving the assertion. ■

Let $F = \{f_n\}_{n\in\mathbb{Z}} \subset L_1$ be a positive bounded symmetric T-admissible process. For $k \geq 0$, define $g_i^k(x) = f_i(x)$ for $0 \leq |i| \leq k$ and

$$g_i^k(x) = \begin{cases} T^{i-k}f_k(x) & \text{for } i > k,\\ T^{-i+k}f_{-k}(x) & \text{for } -i > k.\end{cases}$$

Thus, $g_i^k(x) \leq f_i(x)$ for every $i \in \mathbb{Z}$ and for each $k \geq 0$. Also, by Proposition 2.1,

$$0 \leq f_i(x) - g_i^k(x) \leq T^i(\delta - \nu_k)(x) \quad \text{if } |i| > k,$$

and

$$0 = f_i(x) - g_i^k(x) \quad \text{if } |i| \leq k.$$

Observe that, $\|\delta - \nu_k\|_1 \downarrow 0$ as $k \to \infty$.

The following theorem generalizes existence of the ergodic Hilbert transform [C, P1] to the setting of bounded symmetric admissible processes.

Theorem 3.3 *Let T be an invertible measure preserving transformation and $F \subset L_1$ be a bounded symmetric T-admissible process whose purely subadditive part satisfies the condition $(*)$. Then*

$$\mathbf{H}F(x) = \lim_n {\sum_{i=-n}^{n}}' \frac{f_i(x)}{i} \quad \textit{exists a.e.}$$

Proof By the existence of the ergodic Hilbert transform for additive processes [P1], we can assume without loss of generality that $f_i \geq 0$ for each $i \in \mathbb{Z}$. Fix $k \geq 1$, and let $G^{(k)} = \{g_i^k\}_{i\in\mathbb{Z}}$ be the process where each g_i^k is defined as above. Since $g_i^k =$

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$T^i(T^{-k}f_k) = T^i\nu_k$ for $|i| > k$, actually, $\sum_{-n}'^{n} \frac{g_i^k}{i} = \sum_{-n}'^{n} \frac{T^i\nu_k}{i} + \sum_{-k}'^{k} \frac{f_i - T^i\nu_k}{i}$, therefore $\lim_n \sum_{-n}'^{n} \frac{g_i^k}{i}$ exists a.e. Now, $\sum_{-n}'^{n} \frac{f_i(x)}{i} - \sum_{-n}'^{n} \frac{g_i^k}{i} = \sum_{-n}'^{n} \frac{s_i}{i}$, where

$$s_i(x) = \begin{cases} 0 & \text{for } 1 \le |i| \le k, \\ f_i(x) - T^{i-k}f_k(x) & \text{for } i > k, \\ f_{-i}(x) - T^{-i+k}f_{-k}(x) & \text{for } -i > k. \end{cases}$$

Since

$$Ts_i = T(f_i - T^{i-k}f_k) \le f_{i+1} - T^{i+1-k}f_k = s_{i+1} \quad \text{when } i > k,$$

and

$$T^{-1}s_{-i} = T^{-1}(f_{-i} - T^{-i+k}f_{-k}) \le f_{-i-1} - T^{-i-1+k}f_{-k} = s_{-i-1} \quad \text{when } -i > k,$$

the process $S = \{s_i\}$ is T-admissible. From the construction, S is bounded with exact dominant $\delta - \nu_k$. Letting $f^* = \limsup_n \sum_{-n}'^{n} \frac{f_i}{i}$ and $f_* = \liminf_n \sum_{-n}'^{n} \frac{f_i}{i}$, we observe that $0 \le f^* - f_* \le 2|f^* - g_k^*|$, where $g_k^* = \lim_n \sum_{-n}'^{n} \frac{g_i^k}{i}$. Therefore, if $E = \{x : f^*(x) - f_*(x) > \lambda\}$, then

$$E \subset \Big\{ x : \limsup_n \Big| \sum_{-n}^{n}{}' \frac{s_i(x)}{i} \Big| > \frac{\lambda}{2} \Big\}.$$

From Theorem 3.2 it follows that, for some constant $C > 0$,

$$\mu(E) \le \frac{C}{\lambda} \|\delta - \nu_k\|_1.$$

By letting $k \to \infty$, we obtain that $\mu(E) = 0$. Thus $\lim_n \mathbf{H}_n F(x)$ exists a.e. ■

Remarks (1) Since $L_p \subset L_1$, the strong boundedness in L_p implies strong boundedness in L_1 (we deal with a probability space). Consequently, the assertion of Theorem 3.3 is also valid for strongly bounded symmetric admissible processes $F \subset L_p, 1 < p < \infty$, whose purely subadditive part satisfies the condition

$$(**) \qquad n^\alpha \|h_n - h_{-n}\|_p = O(1) \quad \text{for some } 0 < \alpha < 1.$$

(2) It should be stressed here that without the symmetry condition, which is needed for cancellation, Theorem 3.3 is false. In fact, there are functions $f \in L_\infty$ with integral zero for which the one-sided ergodic Hilbert transform (*i.e.*, $\sum_{k=1}^{\infty} \frac{T^k f}{k}$) need not exist (see [P2, pp. 94–99]. Take the above f and define $\{f_k\}$ by $f_k = T^k f$ for $k > 0$ and $f_k = 0$ for $k \le 0$. Then $\{f_k\}$ is T-admissible, and the desired convergence fails. Also, it should be observed that given any invertible ergodic measure preserving transformation T, the one-sided ergodic Hilbert transform diverges a.e. for any nonnegative f (not identically zero), since the convergence of one-sided ergodic Hilbert transform implies convergence of the ergodic averages to zero (Kronecker's Lemma), so $\int f = 0$.

Further Comments

In [DL], among others, norm and a.e. convergence of the one-sided Hilbert transform was studied. An equivalent formulation of the a.e. result of [DL] is: if

$$\sup \Big\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} T^k f \Big\|_1 < \infty$$

for some $0 < \beta < 1$, where T is a linear contraction on L_1 with mean ergodic modulus, then $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converges a.e. and in L_1-norm. Naturally, this prompts the question whether under similar conditions one would hope to have the a.e. (or norm) convergence of the one-sided ergodic Hilbert transform for admissible processes. The answer to this question is, as the following argument shows, affirmative for all (not necessarily admissible) bounded superadditive processes.

Let T be a measure preserving transformation and $F = \{f_i\}_{i\geq 0} \subset L_1$ be a bounded T-superadditive process with $\sup \|\frac{F_n}{n^{1-\beta}}\|_1 < \infty$ for some $0 < \beta < 1$. In [DL, (3.1)] replace $T^k f$ by f_k and S_k by F_k. Then, since $\frac{F_n}{n}$ converges a.e. by the superadditive ergodic theorem [K], we obtain as in [DL], that $\sum_{k=1}^{\infty} \frac{F_k}{k}$ converges a.e. Note that in Theorem 3.3, for the two-sided Hilbert transform, we use a condition *only* on the purely subadditive part (which trivially holds in the additive case).

Recently, in [CoL] pointwise ergodic theorems with rate are obtained as well as convergence of the series of the form $\sum_{k=1}^{\infty} \frac{f_k}{k^{1-\gamma}}$, $\gamma \in [0,1)$, where $\{f_k\}$ is a norm bounded sequence in L_p, $1 < p < \infty$. There, the sequence $\{f_k\}$ is also assumed to satisfy a growth condition

$$(\dagger) \qquad \sup_n \Big\| \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} f_k \Big\|_p < \infty,$$

for some $\beta \in (0,1]$. Hence, when one-sided Hilbert transform is considered, if $F = \{f_i\} \subset L_p$, $1 < p < \infty$, is a purely subadditive process (*i.e.*, with $\delta = 0$ a.e.), then the condition $(**)$ takes the form

$$\sup_{n>1} n^{\alpha} \|f_n\|_p < \infty.$$

Clearly, this condition is stronger than $(\dagger)$. Consequently, for $1 < p < \infty$, if $F \subset L_p$ is a strongly bounded admissible process satisfying $\sup_n n^{\alpha}\|f_n\|_p < \infty$, for some $0 < \alpha < 1$, then [CoL, Theorem 1] implies that $\sum_{k=1}^{n} \frac{f_k}{k}$ converge a.e. This fact, in turn, implies that if $F \subset L_p$, $1 < p < \infty$ is a strongly bounded symmetric admissible process satisfying $\sup_{n\in\mathbb{Z}} n^{\alpha}\|f_n\|_p < \infty$, then $\mathbf{H}F(x)$ exists a.e. As noted in [CoL, p. 1001], their result fails for $p = 1$, hence Theorem 3.3 does not follow from their results. It should also be noted here that the family of functions $\{f_k\}$ considered in [CoL] is more general than the family of admissible processes, and their method of proof is different from the method used in this article.

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