42

Chiral perturbation theory

42.1 Introduction

In the general introduction of this book, we have discussed that, below the vector meson resonances region ($E \leq M_\rho$), the hadronic spectrum of light flavours only consists of an octet of quasi-Goldstone pseudoscalar mesons (π, K, η), whose interactions can be easily understood using the global symmetry of the QCD Lagrangian. In the limit of massless quarks, the QCD Lagrangian is invariant under the rotations of the left and right quark fields triplets:

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi\ , \qquad \psi_R \equiv \frac{1}{2}(1+\gamma_5)\psi\ , \qquad \psi \equiv u, d, s\ . \tag{42.1}$$

These rotations generate the chiral group $SU(3)_L \times SU(3)_R$, which at the level of hadronic spectrum is broken down to the diagonal flavour $SU(3)_V$ ($V \equiv L + R$) group of the eightfoldway [7]. The Goldstone bosons are associated to the spontaneous breakdown of chiral symmetry and obey low-energy theorems which are the basis of successful predictions of current algebra and pion PCAC [13]. Since there is a mass gap separating the Goldstone bosons from the rest of the hadronic spectrum, one can build an effective field theory including the symmetry of QCD where the Goldstone bosons are the only dynamic degrees of freedom [497]. This allows to a systematic analysis of the low-energy implications of the QCD symmetries which simplifies current algebra calculations and allows an investigation of higher-order corrections in the sense of perturbative field theory [498]. This approach is known as chiral perturbation theory (ChPT), which is a low-energy effective field theory of QCD, where many excellent reviews and lectures have been devoted to the subject [500–502]. Our presentation has been mainly inspired from the reviews in [500,501] and the works of Gasser–Leutwyler [499].

A well-known example of effective theories is the low-energy limit of QED ($E_\gamma \ll m_e$). In this limit the $\gamma\gamma$ scattering process can be described by the effective Euler–Heisenberg Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{A}{m_e^4}(F_{\mu\nu}(x)F^{\mu\nu}(x))^2 + \frac{B}{m_e^4}F_{\mu\nu}(x)F^{\nu\sigma}(x)F_{\sigma\rho}(x)F^{\rho\mu}(x) + \cdots \tag{42.2}$$

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which is only based on the gauge, Lorentz and parity invariance conditions. The coefficients A and B are known and can be computed by integrating out the electron field from the original QED generating functional, or equivalently by computing the corresponding $\gamma\gamma$ box diagram. They read [503]:

$$A = -\frac{\alpha^2}{36}, \qquad B = \frac{7}{90}\alpha^2 . \tag{42.3}$$

However, this QED example is academic since perturbation theory in terms of the QED coupling α is known to work at high accuracy. In QCD, due to confinement which induces that quark and gluon are not asymptotic states, the effective approach is more useful as we know the symmetry properties of QCD, from which we can write the effective theory in terms of hadronic asymptotic states, and parametrize the unknown dynamics of the theory in terms of some few couplings.

In the following discussions, we shall limit ourselves to the presentation of the main idea behind the method and illustrate its applications for the estimate of the light quark mass ratios.

42.2 PCAC relation from ChPT

One can also derive the previous PCAC relation ontained in Part I of this book using ChPT. In this approach, it is convenient to formulate the strong interactions of the pseudoscalar mesons in terms of an effective low-energy QCD Lagrangian described by the octet of Goldstone fields:

$$\phi(x) = \frac{1}{\sqrt{2}}\vec{\lambda}.\vec{\varphi}(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{42.4}$$

instead of in terms of the usual quark and gluon fields. The associated 3×3 unitary matrix:

$$U(\phi) = \exp(i\sqrt{2}\phi/f_\pi) , \tag{42.5}$$

transforms linearly under the global chiral rotations, although $\vec{\varphi}$ transforms non-linearly. The unique lowest order (in derivative) effective Lagrangian, satisfying chiral symmetry and generating non-trivial interaction is:

$$\mathcal{L} = \frac{f^2}{4}\mathbf{Tr}\{\partial_\mu U^\dagger \partial^\mu U\} , \tag{42.6}$$

where f is a constant which cannot be fixed by symmetry requirements alone. Expanding $U(\phi)$ in a power series of ϕ, the Lagrangian reads:

$$\mathcal{L} = \frac{1}{2}\mathbf{Tr}\{\partial_\mu\phi\partial^\mu\phi\} + \frac{1}{12f^2}\mathbf{Tr}\{(\phi\partial_\mu\phi)(\phi\partial^\mu\phi)\} + \mathcal{O}\left(\frac{\phi^6}{f^4}\right) , \tag{42.7}$$

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where one should note that the ϕ^4 interaction fixes the π–π scattering amplitude [504]:

$$T(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{t}{f^2} \tag{42.8}$$

where $t \equiv (p'_+ - p_+)^2$ is the usual kinematic variable. Now, one can go to a step further by introducing the couplings of external sources to the usual massless QCD Lagrangian:

$$\mathcal{L}_{\rm QCD}(x) = \mathcal{L}_{\rm QCD}^{\rm massless}(x) + \bar\psi\gamma^\mu(v_\mu + \gamma_5 a_\mu)\psi\bar\psi\gamma^\mu\,(s - i\gamma_5)\,\psi\ , \tag{42.9}$$

where $v_\mu,\ a_\mu,\ s$ and p are Hermitian 3×3 matrices in flavour and colour singlets. The Lagrangian $\mathcal{L}$ is now invariant under the *local* $SU(3)_L \times SU(3)_R$ gauge transformations. The generalized effective Lagrangian satisfying the local invariance reads to lowest order:

$$\mathcal{L}_{\rm eff}^{(2)} = \frac{f^2}{4}\mathbf{Tr}\{D_\mu U^\dagger D^\mu U\} + U^\dagger\chi + \chi^\dagger U\ , \tag{42.10}$$

where D_μ is the covariant derivative:

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu) \tag{42.11}$$

and:

$$\chi = 2B(s + ip)\ . \tag{42.12}$$

B is a constant which, like f, cannot be fixed by symmetry requirements alone. With the choice of directions:

$$\begin{aligned} s + ip &= \mathcal{M} + \cdots \\ r_\mu &= v_\mu + a_\mu = eQA_\mu + \cdots \\ l_\mu &= v_\mu - a_\mu = eQA_\mu + \frac{e}{\sqrt{2}\sin\theta_W}(W_\mu^\dagger T_+ + \text{h.c.}) + \cdots\ , \end{aligned} \tag{42.13}$$

where A_μ and W_μ are the photon and W^- bosons,

$$\mathcal{M} = \text{diag}(m_u, m_d, m_s), \qquad Q = \frac{1}{3}\text{diag}(2, -1, -1)\ , \tag{42.14}$$

and:

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ , \tag{42.15}$$

one can break chiral symmetry explicitly and select the electroweak standard model couplings. The Green functions are obtained as functional derivatives of the generating functional:

$$\exp\{iZ\} = \int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}A_\mu \exp\left\{i\int d^4x\ \mathcal{L}_{\rm QCD}\right\} = \int \mathcal{D}\exp\left\{i\int d^4x\ \mathcal{L}_{\rm eff}\right\}. \tag{42.16}$$

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At lowest order in momenta, the generating functional reduces to the classical action:

$$S_2 = \int d^4x \, \mathcal{L}^{(2)}_{\rm eff}(x) \, . \tag{42.17}$$

The Noether currents can be derived by taking appropriate derivatives with respect to the external fields:

$$\begin{aligned} J^\mu_L &= \frac{\delta S_2}{\delta l_\mu} = \frac{i}{2} f^2 D_\mu U^\dagger U = \frac{f}{\sqrt{2}} D_\mu \phi - \frac{i}{2}(\phi \vec{D}^\mu \phi) + \cdots \\ J^\mu_R &= \frac{\delta S_2}{\delta r_\mu} = \frac{i}{2} f^2 D_\mu U U^\dagger = -\frac{f}{\sqrt{2}} D_\mu \phi - \frac{i}{2}(\phi \vec{D}^\mu \phi) + \cdots \, , \end{aligned} \tag{42.18}$$

which shows the indentification of the coupling f with the decay constant $f_\pi = 92.4$ MeV to order p^2:

$$\langle 0 | J^\mu_A | \pi \rangle \equiv i\sqrt{2} f_\pi p^\mu \, . \tag{42.19}$$

In a similar way:

$$\begin{aligned} \bar{\psi}^i_L \psi^j_R &= -\frac{\delta S_2}{\delta (s - ip)^{ij}} = -\frac{f^2}{2} B \, U^{ij} \\ \bar{\psi}^i_R \psi^j_L &= -\frac{\delta S_2}{\delta (s + ip)^{ij}} = -\frac{f^2}{2} B (U^{ij})^\dagger \, , \end{aligned} \tag{42.20}$$

which implies:

$$\langle 0 | \bar{\psi}^j \psi^i | 0 \rangle = -f^2 B \delta^{ij} \, , \tag{42.21}$$

By taking $s = \mathcal{M}$ and $p = 0$, the χ term in Eq. (42.10) gives a quadratic pseudoscalar mass plus additional interactions proportional to the quark mass. Expanding in powers of ϕ, one obtains:

$$\frac{f^2}{4} 2B \mathbf{Tr}\{\mathcal{M}(U + U^\dagger)\} = B \left\{ -\mathbf{Tr}\,(\mathcal{M}\phi^2) + \frac{1}{6f^2} \mathbf{Tr}\,(\mathcal{M}\phi^4) + \cdots \right\} . \tag{42.22}$$

An explicit evaluation of the trace in the quadratic mass term provides:

$$\begin{aligned} M^2_{\pi^\pm} &= (m_u + m_d) B + \mathcal{O}\big(m_q^2\big) \, , \\ M^2_{\pi^0} &= (m_u + m_d) B - \epsilon + \mathcal{O}\big(\epsilon^2, m_q^2\big) \, , \\ M^2_{K^+} &= (m_u + m_s) B + \mathcal{O}\big(m_q^2\big) \, , \\ M^2_{K^0} &= (m_d + m_s) B + \mathcal{O}\big(m_q^2\big) \, , \\ M^2_{\eta_8} &= \frac{1}{3} (m_u + m_d + 4m_s)\, B + \epsilon + \mathcal{O}\big(\epsilon^2, m_q^2\big) \, , \end{aligned} \tag{42.23}$$

where:

$$\epsilon = \frac{B}{4} \frac{(m_u - m_d)^2}{(m_s - \hat{m})} \, , \quad \hat{m} = \frac{1}{2}(m_u + m_d), \tag{42.24}$$

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originates from the small mixing between the π^0 and η_8 fields. Previous relations explain why the masses of the multiplet break strongly explicitly the eightfoldway symmetry because $m_s \gg m_d > m_u$. Using also these results in Eqs. (42.21) and (42.23), one can deduce the pion PCAC relation given in Part I of this book, namely:

$$(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = -2 f_\pi^2 m_\pi^2 \ . \tag{42.25}$$

However, there is no rigorous evidence on the dominance of this linear quark mass term over the quadratic one in the previous relation in Eq. (42.23) leading to the previous PCAC relation where the quark mass is a quadratic function of the pseudoscalar mass. Some alternative scenario (*so-called Generalized ChPT*), where the value of the $\langle \bar{\psi}\psi \rangle$ condensate is smaller than the 'standard' value, is discussed in the literature [505]. We might expect that lattice calculations will clarify this issue in the near future, and at present, there are some lattice indications that M_P^2 behaves like m_q [506]. We shall see in the next section that direct extractions of the light quark masses from QCD spectral sum rules also favour the result that $m_q \sim M_P^2$.

42.3 Current algebra quark mass ratios

The ratios of the expressions in Eq. (42.23) imply the old current algebra mass ratios [21],[55–57]:

$$\frac{M_{\pi^\pm}^2}{(m_u + m_d)} = \frac{M_{K^+}^2}{(m_u + m_s)} = \frac{M_{K^0}^2}{(m_d + m_s)} \approx \frac{3M_{\eta_8}^2}{(m_u + m_d + 4m_s)} \ , \tag{42.26}$$

while the estimate of their absolute values needs more QCD theoretical inputs (renormalization and scale dependence). Neglecting the m^2 [1] and small $\mathcal{O}(\epsilon)$ corrections, one can deduce the mass ratios [55]:

$$\begin{aligned} \frac{m_u}{m_d} &\approx \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \approx 0.66 \\ \frac{m_s}{m_d} &\approx \frac{-M_{\pi^+}^2 + M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \approx 20 \ , \end{aligned} \tag{42.27}$$

where the electromagnetic part of the $K^+ - K^0$ squared mass-difference has been subtracted by using the fact that it is the same for the K^+ and π^+ [507]:

$$\left(M_{K^0}^2 - M_{K^+}^2\right)_{\text{QCD}} \simeq \left(M_{K^0}^2 - M_{K^+}^2\right) - \left(M_{\pi^0}^2 - M_{\pi^+}^2\right) . \tag{42.28}$$

Up to order $(m_d - m_u)$, one can also derive the quadratic Gell-Mann–Okubo mass relation [11]:[2]

$$3M_{\eta_8}^2 \approx 4M_K^2 - M_\pi^2 \ . \tag{42.29}$$

[1] This is not justified in the approach of [505].
[2] Analogous GMO mass formula for vector mesons might be affected by large perturbative m_s^2 corrections [32].

One should also note that the ϕ^4 interaction in Eq. (42.22) gives a mass correction to the π–π scattering amplitude given in Eq. (42.8):

$$T(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{t - M_\pi^2}{f_\pi^2}\,, \tag{42.30}$$

in good agreement with the current algebra result [504].

42.4 Chiral perturbation theory to order p^4

Improvements of this lowest order effective Lagrangian with the inclusion of p^4- and p^6-terms are actively discussed in the literature [502]. To order p^4, three different sources contribute to the generating functional:

- The most general effective Lagrangian $\mathcal{L}_{\rm eff}^{(4)}$ to order p^4 to be considered at the tree level.
- The one-loop graphs generated from the lowest order $\mathcal{L}_{\rm eff}^{(2)}$ Lagrangian.
- The Wess–Zumino–Witten functional [508,509] induced by the non-Abelian chiral anomaly [510].

42.4.1 The chiral Lagrangian to order (p^4)

The most general expression of the $\mathcal{O}(p^4)$ Lagrangian is:

$$\begin{aligned}
\mathcal{L}_{\rm eff}^{(4)} = {} & L_1\,\mathbf{Tr}\,(D_\mu U^\dagger D^\mu U)^2 + L_2\,\mathbf{Tr} D_\mu U^\dagger D_\nu U\,\mathbf{Tr} D^\mu U^\dagger D^\nu U \\
& + L_3\,\mathbf{Tr} D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \\
& + L_4\,\mathbf{Tr} D_\mu U^\dagger D^\mu U\,\mathbf{Tr}\,(\chi^\dagger U + U^\dagger\chi) + L_5\,\mathbf{Tr} D_\mu U^\dagger D^\mu U\,(\chi^\dagger U + U^\dagger\chi) \\
& + L_6\,[\mathbf{Tr}\,(\chi^\dagger \mathcal{U} + U^\dagger\chi)]^2 + L_7\,[\mathbf{Tr}\,(\chi^\dagger U - U^\dagger\chi)]^2 \\
& + L_8\,\mathbf{Tr}(U\chi^\dagger U\chi^\dagger + U^\dagger\chi U^\dagger\chi) \\
& + i L_9\,\mathbf{Tr}\left(F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\right) + L_{10}\,\mathbf{Tr} U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \\
& + H_1\,\mathbf{Tr}\left(F_R^{\mu\nu} F_{R\mu\nu} + F_L^{\mu\nu} F_{L\mu\nu}\right) + H_2\,\mathbf{Tr}\chi^\dagger\chi\,.
\end{aligned} \tag{42.31}$$

In this Lagrangian the parameters L_i, $i = 1, 2, 3, \ldots, 10$ are dimensionless coupling constants, which like f_π and B in the lowest order effective Lagrangian, are not fixed by chiral symmetry requirements alone. The terms proportional to the coupling constants H_1 and H_2 involve only the external fields. As a result these coupling constants cannot be fixed from low-energy observables alone. By contrast, most of the other couplings can be fixed from low-energy observables. The L_i constants, like f_π and B, are *in principle* calculable parameters in terms of the intrinsic $\Lambda_{\rm QCD}$ scale only.

42.4.2 Chiral loops

Here, we consider that ChPT is an effective field theory for low energies despite the fact that a simple power counting shows that loops generated by the lowest order Lagrangian

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are highly divergent as a consequence of the fact that the non-linear sigma model in four-dimensions is not renormalizable and then needs an infinite number of local counterterms. In order to define the loop integrals it is necessary to fix a regularization that preserves the symmetries of the Lagrangian, which can be done by using the well-known dimensional regularization technique. Since by construction, the $\mathcal{O}(p^4)$ Lagrangian $\mathcal{L}_{\rm eff}^{(4)}$ contains all possible terms which are allowed by chiral invariance, all the one-loop divergences from $\mathcal{L}_{\rm eff}^{(2)}$ can be absorbed by suitable renormalizations of the L_i and $H_{1,2}$ constants. This feature can be understood by power counting where one-loop divergences can only give rise to local $\mathcal{O}(p^4)$ terms. This program has been explicitly realized by Gasser and Leutwyler in [498], and leads to the renormalized low-energy couplings:

$$L_i = L_i^{\rm R}(\nu) + \gamma_i\,\lambda^{\rm loop}, \quad i = 1, 2, 3, \ldots 10; \qquad H_i = H_i^{\rm R}(\nu) + \tilde{\gamma}_j\,\lambda^{\rm loop}, \quad j = 1, 2, \tag{42.32}$$

where for $n = 4 - \epsilon$ space-time dimension:

$$\lambda^{\rm loop} = \frac{\nu^{-\epsilon}}{16\pi^2}\left\{-\frac{1}{\epsilon} - \frac{1}{2}[\log(4\pi) + \Gamma'(1) + 1]\right\}, \quad j = 1, 2; \tag{42.33}$$

and $\gamma_i, x\tilde{\gamma}_j$ have the following rational values:

$$\begin{aligned}
\gamma_1 &= \frac{3}{32}, \quad \gamma_2 = \frac{3}{16}, \quad \gamma_3 = 0, \quad \gamma_4 = \frac{1}{8}, \\
\gamma_5 &= \frac{3}{8}, \quad \gamma_6 = \frac{11}{144}, \quad \gamma_7 = 0, \quad \gamma_8 = \frac{5}{48}, \\
\gamma_9 &= \frac{1}{4}, \quad \gamma_{10} = -\frac{1}{4}, \quad \tilde{\gamma}_1 = -\frac{1}{8}, \quad \tilde{\gamma}_2 = \frac{5}{24}.
\end{aligned} \tag{42.34}$$

The renormalized coupling constants depend as usual on the scale ν introduced by the dimensional regularization. The running in ν is governed by the coefficients γ_i (and $\tilde{\gamma}_j$), which play the rôle of one–loop β–functions:

$$L_i^{\rm r}(\nu) = L_i^{\rm r}(\nu') + \frac{\gamma_i}{16\pi^2}\log\frac{\nu'}{\nu}. \tag{42.35}$$

The ν–scale dependence cancels however in the full $\mathcal{O}(p^4)$ calculation of a given physical observable. The non-polynomial contribution to a specific physical process will in general have a logarithmic ν–scale dependence (*the so called chiral logarithms*), which cancels with the ν–dependence of the tree level contribution modulated by the $L_i(\nu)$–constants. A typical $\mathcal{O}(p^4)$ amplitude will then consist of a non-polynomial part, coming from the loop computation, plus a polynomial in momenta and pseudoscalar masses, which depends on the unknown constants L_i. The non-polynomial part (the so-called chiral logarithms) is completely predicted as a function of the lowest-order coupling f and the Goldstone masses.

Finally, it is important to notice that ChPT is an expansion in powers of momenta over some typical hadronic scale, usually called the scale of chiral symmetry breaking Λ_χ.

The variation of the loop contribution under a rescaling of μ provides a natural order-of-magnitude estimate[3] of Λ_χ [498,512] :

$$\Lambda_\chi \sim 4\pi f_\pi \sim 1.2\,\mathrm{GeV} \approx M_\rho \approx M_p \,. \tag{42.36}$$

This result has been recovered from the analysis of the connection between the low- and high-energy behaviours of the pion form factor [281].

42.4.3 The non-Abelian chiral anomaly

Although the QCD Lagrangian with external sources is formally invariant under local chiral transformations, this is no longer true for the associated generating functional. The anomalies of the fermionic determinant break chiral symmetry at the quantum level. The anomalous change of the generating functional under an infinitesimal chiral transformation:

$$g_{L,R} = 1 + i\alpha \mp i\beta + \cdots \tag{42.37}$$

is given by [510]:

$$\delta Z[v,a,s,p] = -\frac{N_c}{16\pi^2}\int d^4x\, \mathrm{tr}\beta(x)\,\Omega(x), \tag{42.38}$$

where:

$$\Omega(x) = \varepsilon^{\mu\nu\sigma\rho}\left[v_{\mu\nu}v_{\sigma\rho} + \frac{4}{3}\nabla_\mu a_\nu \nabla_\sigma a_\rho + \frac{2}{3}i\,\{v_{\mu\nu}, a_\sigma a_\rho\} + \frac{8}{3}i\,a_\sigma v_{\mu\nu} a_\rho + \frac{4}{3}a_\mu a_\nu a_\sigma a_\rho\right], \qquad \varepsilon_{0123} = 1; \tag{42.39}$$

and:

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i\,[v_\mu, v_\nu], \qquad \nabla_\mu a_\nu = \partial_\mu a_\nu - i\,[v_\mu, a_\nu] \tag{42.40}$$

This anomalous variation of Z is an $\mathcal{O}(p^4)$ effect in the chiral counting. Chiral symmetry is the basic requirement to construct the effective χPT Lagrangian. Since chiral symmetry is explicitly violated by the anomaly at the fundamental QCD level, one is forced to add an effective functional with the property that its change under chiral gauge transformations reproduces Eq. (42.38). Such a functional was first constructed by Wess and Zumino [508]. An interesting topological interpretation was later found by Witten [509]. The functional in question, has the following explicit form:

$$\Gamma[U,\ell,r]_{WZW} = -\frac{iN_c}{240\pi^2}\int d\sigma^{ijklm}\,\mathbf{Tr}\left\{\Sigma_i^L\Sigma_j^L\Sigma_k^L\Sigma_l^L\Sigma_m^L\right\} - \frac{iN_c}{48\pi^2}\int d^4x\,\varepsilon_{\mu\nu\alpha\beta}(W(U,\ell,r)^{\mu\nu\alpha\beta} - W(\mathbf{1},\ell,r)^{\mu\nu\alpha\beta})\,, \tag{42.41}$$

[3] Since the loop amplitude increases with the number of possible Goldstone mesons in the internal lines, this estimate results in a slight dependence of Λ_χ on the number of light-quark flavours N_f [511]: $\Lambda_\chi \sim 4\pi f_\pi/\sqrt{N_f}$.

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with:

$$W(U,\ell,r)_{\mu\nu\alpha\beta} = \mathbf{Tr}\Big\{U\ell_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta + \frac{1}{4}U\ell_\mu U^\dagger r_\nu U\ell_\alpha U^\dagger r_\beta + iU\partial_\mu\ell_\nu\ell_\alpha U^\dagger r_\beta$$
$$+ i\partial_\mu r_\nu U\ell_\alpha U^\dagger r_\beta - i\Sigma^L_\mu \ell_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu U^\dagger \partial_\nu r_\alpha U\ell_\beta$$
$$- \Sigma^L_\mu\Sigma^L_\nu U^\dagger r_\alpha U\ell_\beta + \Sigma^L_\mu\ell_\nu\partial_\alpha\ell_\beta + \Sigma^L_\mu\partial_\nu\ell_\alpha\ell_\beta$$
$$- i\,\Sigma^L_\mu\ell_\nu\ell_\alpha\ell_\beta + \frac{1}{2}\Sigma^L_\mu\ell_\nu\Sigma^L_\alpha\ell_\beta - i\Sigma^L_\mu\Sigma^L_\nu\Sigma^L_\alpha\ell_\beta\Big\}$$
$$- (L \leftrightarrow R)\,, \tag{42.42}$$

where:

$$\Sigma^L_\mu = U^\dagger\partial_\mu U\,, \qquad \Sigma^R_\mu = U\partial_\mu U^\dagger\,, \tag{42.43}$$

and $(L \leftrightarrow R)$ stands for the interchanges $U \leftrightarrow U^\dagger$, $\ell_\mu \leftrightarrow r_\mu$ and $\Sigma^L_\mu \leftrightarrow \Sigma^R_\mu$. The integration in the first term of Eq. (42.41) is over a five-dimensional manifold whose boundary is four-dimensional Minkowski space. The integrand is a surface term; therefore both the first and the second terms of Γ_{WZW} are $\mathcal{O}(p^4)$ according to the chiral counting rules.

Since the effect of anomalies is perturbatively calculable, their translation from the fundamental quark-gluon level to the effective chiral level is unaffected by hadronization problems. Despite its apparent complexity, the anomalous action [Eq. (42.41)] has no free parameters. It is responsible for the $\pi^0 \to 2\gamma$, $\eta \to 2\gamma$ decays, and the $\gamma 3\pi$, $\gamma\pi^+\pi^-\eta$ interactions among others. The five-dimensional surface term generates interactions among five or more Goldstone bosons.

42.5 Some low-energy phenomenology to order p^4

At lowest order in momenta, the predictive power of the chiral Lagrangian was quite impressive; with only two low-energy couplings, it was possible to describe all Green functions associated with the pseudoscalar-meson interactions, and to reproduce all old current algebra results [13]. The symmetry constraints become less powerful at higher orders. Ten additional constants appear in the $\mathcal{L}_4$ Lagrangian, and many more would be needed at $\mathcal{O}(p^6)$.

Higher-order terms in the chiral expansion are much more sensitive to the non-trivial aspects of the underlying QCD dynamics. With $p \ll M_K$ (M_π), we expect $\mathcal{O}(p^4)$ corrections to the lowest-order amplitudes at the level of $p^2/\Lambda^2_\chi \leq 20\%$ (2%). We need to include those corrections if we aim to increase the accuracy of the ChPT predictions beyond this level. Although the number of free constants in $\mathcal{L}_4$ looks quite big, only a few of them contribute to a given observable. In the absence of external fields, for instance, the Lagrangian reduces to the first three terms; elastic $\pi\pi$ and πK scatterings are then sensitive to $L_{1,2,3}$. The two-derivative couplings $L_{4,5}$ generate mass corrections to the meson decay constants (and mass-dependent wave-function renormalizations). Pseudoscalar masses are affected by the non-derivative terms $L_{6,7,8}$; L_9 is mainly responsible for the charged-meson electromagnetic radius and L_{10}, finally, only contributes to amplitudes with at least two external vector or

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Table 42.1. *Phenomenological values of the renormalized couplings $L_i^r(M_\rho)$.*

i	$L_i^r(M_\rho)\times 10^3$	Source
1	0.7 ± 0.5	K_{e4}, $\pi\pi \to \pi\pi$
2	1.2 ± 0.4	K_{e4}, $\pi\pi \to \pi\pi$
3	-3.6 ± 1.3	K_{e4}, $\pi\pi \to \pi\pi$
4	-0.3 ± 0.5	Zweig rule
5	1.4 ± 0.5	$F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule
7	-0.4 ± 0.2	Gell-Mann–Okubo, L_5, L_8, sum rules
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m}) : (m_d - m_u)$
9	6.9 ± 0.7	$\langle r^2 \rangle^\pi_{\text{em}}$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$

axial-vector fields, like the radiative semi-leptonic decay $\pi \to e\nu\gamma$. Table 42.1 summarizes the present status of the phenomenological determination of the renormalized constants L_i [499,502], evaluated at a scale $\mu = M_\rho$. The values of these couplings at any other renormalization scale can be trivially obtained, through the logarithmic running given in Eq. (42.35).

42.5.1 Decay constants

In the isospin limit ($m_u = m_d = \hat{m}$), the $\mathcal{O}(p^4)$ calculation of the meson-decay constants gives [499]:

$$
\begin{aligned}
f_\pi &= f\left\{1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2}L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2}L_4^r(\mu)\right\}, \\
f_K &= f\left\{1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2}L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2}L_4^r(\mu)\right\}, \\
f_{\eta_8} &= f\left\{1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2}L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2}L_4^r(\mu)\right\}, \qquad (42.44)
\end{aligned}
$$

where:

$$
\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2}\,\log\left(\frac{M_P^2}{\mu^2}\right). \qquad (42.45)
$$

The result depends on two $\mathcal{O}(p^4)$ couplings, L_4 and L_5. The L_4 term generates a universal shift of all meson-decay constants, $\delta f^2 = 16 L_4 B \mathbf{Tr}\mathcal{M}$, which can be eliminated taking

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ratios. From the experimental value [513]:

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \,, \tag{42.46}$$

one can then fix $L_5(\mu)$; this gives the result quoted in Table 42.1. Moreover, one gets the absolute prediction [499]:

$$\frac{f_{\eta_8}}{f_\pi} = 1.3 \pm 0.05 \,. \tag{42.47}$$

Taking into account isospin violations, one can also predict [499] a tiny difference between $f_{K^\pm}$ and f_{K^0}, proportional to $m_d - m_u$.

42.5.2 Electromagnetic form factors

At $\mathcal{O}(p^2)$ the electromagnetic coupling of the Goldstone bosons is just the minimal one, obtained through the covariant derivative. The next-order corrections generate a momentum-dependent form factor:

$$F_V^{\phi^\pm}(p^2) = 1 + \frac{1}{6}\,\langle r^2\rangle_V^{\phi^\pm}\, p^2 + \cdots \quad ; \quad F_V^{\phi^0}(p^2) = \frac{1}{6}\,\langle r^2\rangle_V^{\phi^0}\, p^2 + \cdots \tag{42.48}$$

The pion electromagnetic radius $\langle r^2\rangle_V^{\phi}$ gets local contributions from the L_9 term, plus logarithmic loop corrections [499]:

$$\langle r^2\rangle_V^{\pi^\pm} = \frac{12L_9^r(\mu)}{f^2} - \frac{1}{32\pi^2 f^2}\left\{2\log\left(\frac{M_\pi^2}{\mu^2}\right) + \log\left(\frac{M_K^2}{\mu^2}\right) + 3\right\} \tag{42.49}$$

The measured electromagnetic pion radius, $\langle r^2\rangle_V^{\pi^\pm} = 0.439 \pm 0.008\,\mathrm{fm}^2$ [514], is used as input to estimate the coupling L_9.

- The factor $1/(16\pi^2 f^2)$ is a characterisitc factor of a loop-expansion, where chiral logs are expected to contribute as $p^2/(16\pi^2 f^2)$ log in physical processes.
- The form factor provides a good example of the importance of higher-order local terms in the chiral expansion [515]. If one tries to ignore the L_9 contribution, using instead some *physical* cut-off $p_{\max}$ to regularize the loops, one needs an unrealistic value $p_{\max} \sim 60\,\mathrm{GeV}$, in order to reproduce the experimental value. This fact shows that the pion charge radius is dominated by the $L_9^r(\mu)$ contribution, for any reasonable value of μ, which can be better understood from a $1/N_c$ (number of colour) counting rules, where for large N_c, L_9 and f_π^2 are order N_c, implying that the chiral loops are $1/N_c$ suppressed compared to the tree level contributions.
- The phenomenological value of dimensionless couplig L_9 might be understood as originating from the $f_\pi^2/4$ factor from $\mathcal{L}_{\text{eff}}^{(4)}$ divided by the chiral symmetry breaking scale Λ_χ^2, which leads to the order of magnitude value of about 10^{-3}; an expected value for all other L_i couplings as found experimentally in Table 42.1.

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The kaon electromagnetic radius reads:

$$\langle r^2\rangle_V^{K^0} = -\frac{1}{16\pi^2 f^2}\,\log\left(\frac{M_K}{M_\pi}\right), \tag{42.50}$$

$$\langle r^2\rangle_V^{K^\pm} = \langle r^2\rangle_V^{\pi^\pm} + \langle r^2\rangle_V^{K^0}\,. \tag{42.51}$$

Since neutral bosons do not couple to the photon at tree level, $\langle r^2\rangle_V^{K^0}$ only gets a loop contribution, which is moreover finite (there cannot be any divergence because there exists no counterterm to renormalize it). The predicted value:

$$\langle r^2\rangle_V^{K^0} = -0.04 \pm 0.03\,\text{fm}^2\,, \tag{42.52}$$

is in perfect agreement with the experimental determination [516]

$$\langle r^2\rangle_V^{K^0} = -0.054 \pm 0.026\,\text{fm}^2\,. \tag{42.53}$$

The measured K^+ charge radius [517]:

$$\langle r^2\rangle_V^{K^\pm} = 0.28 \pm 0.07\,\text{fm}^2\,, \tag{42.54}$$

has a larger experimental uncertainty. Within present errors, it is in agreement with the parameter-free relation in Eq. (42.51).

42.5.3 K_{l3} decays

The semi-leptonic decays $K^+ \to \pi^0 l^+ \nu_l$ and $K^0 \to \pi^- l^+ \nu_l$ are governed by the corresponding hadronic matrix elements of the vector current [$t \equiv (P_K - P_\pi)^2$]:

$$\langle \pi|\bar{s}\gamma^\mu u|K\rangle = C_{K\pi}\,[(P_K + P_\pi)^\mu\;f_+^{K\pi}(t) + (P_K - P_\pi)^\mu\;f_-^{K\pi}(t)]\,, \tag{42.55}$$

where $C_{K^+\pi^0} = 1/\sqrt{2}$, $C_{K^0\pi^-} = 1$. At lowest order, the two form factors reduce to trivial constants: $f_+^{K\pi}(t) = 1$ and $f_-^{K\pi}(t) = 0$. There is however a sizeable correction to $f_+^{K^+\pi^0}(t)$, due to $\pi^0\eta$ mixing, which is proportional to $(m_d - m_u)$:

$$f_+^{K^+\pi^0}(0) = 1 + \frac{3}{4}\,\frac{m_d - m_u}{m_s - \hat{m}} = 1.017\,. \tag{42.56}$$

This number should be compared with the experimental ratio:

$$\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1.028 \pm 0.010\,. \tag{42.57}$$

The $\mathcal{O}(p^4)$ corrections to $f_+^{K\pi}(0)$ can be expressed in a parameter-free manner in terms of the physical meson masses [499]. Including those contributions, one gets the more precise values:

$$f_+^{K^0\pi^-}(0) = 0.977\,, \qquad \frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1.022\,, \tag{42.58}$$

which are in perfect agreement with the experimental result of Eq. (42.57). The accurate ChPT calculation of these quantities allows us to extract [513] the most precise determination

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of the Cabibbo–Kobayashi–Maskawa matrix element V_{us}:

$$|V_{us}| = 0.2196 \pm 0.0023 \,. \tag{42.59}$$

At $\mathcal{O}(p^4)$, the form factors get momentum-dependent contributions. Since L_9 is the only unknown chiral coupling occurring in $f_+^{K\pi}(t)$ at this order, the slope λ_+ of this form factor can be fully predicted:

$$\lambda_+ \equiv \frac{1}{6}\,\langle r^2\rangle_V^{K\pi}\, M_\pi^2 = 0.031 \pm 0.003 \,. \tag{42.60}$$

This number is in excellent agreement with the experimental determinations [16], $\lambda_+ = 0.0300 \pm 0.0016$ (K_{e3}^0) and $\lambda_+ = 0.0286 \pm 0.0022$ ($K_{e3}^{\pm}$). Contrary to this case, the experimental determination of the slope of the form factor $f_0^{K\pi}$ is still controversial. It is predicted to be [499]:

$$\lambda_0 \equiv \frac{1}{6}\,\langle r^2\rangle_S^{K\pi}\, M_\pi^2 = 0.017 \pm 0.004 \,, \tag{42.61}$$

and is determined by the constant L_5.

42.5.4 Ratios of light quark masses to order p^4

Ratios of light quark masses to this order have been discussed in details in [57]. Here, we outline the different derivations of the results obtained there. The relations in Eq. (42.23) get modified at $\mathcal{O}(p^4)$. The additional contributions depend on the low-energy constants L_4, L_5, L_6, L_7 and L_8. It is possible, however, to obtain one relation between the quark and meson masses, which does not contain any of the $\mathcal{O}(p^4)$ couplings. The dimensionless ratios

$$Q_1 \equiv \frac{M_K^2}{M_\pi^2}\,, \qquad Q_2 \equiv \frac{\left(M_{K^0}^2 - M_{K^+}^2\right)_{\rm QCD}}{M_K^2 - M_\pi^2}\,, \tag{42.62}$$

get the same $\mathcal{O}(p^4)$ correction [499]:

$$Q_1 = \frac{m_s + \hat{m}}{2\hat{m}}\,\{1 + \Delta_m\}\,, \qquad Q_2 = \frac{m_d - m_u}{m_s - \hat{m}}\,\{1 + \Delta_m\}\,, \tag{42.63}$$

where

$$\Delta_m = -\mu_\pi + \mu_{\eta_8} + \frac{8}{f^2}\left(M_K^2 - M_\pi^2\right)\left[2L_8^r(\mu) - L_5^r(\mu)\right]\,. \tag{42.64}$$

Therefore, at this order, the ratio Q_1/Q_2 is just given by the corresponding ratio of quark masses,

$$Q^2 \equiv \frac{Q_1}{Q_2} = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}\,. \tag{42.65}$$

where $Q^2 = 22.7 \pm 0.8$ using the value of the $\eta \to \pi^+\pi^-\pi^0$ decay rate from the PDG average [16], though this value can well be in the range 22–26, to be compared with the Dashen's formula [507] value of 24.2 including next-to-leading chiral corrections [518];

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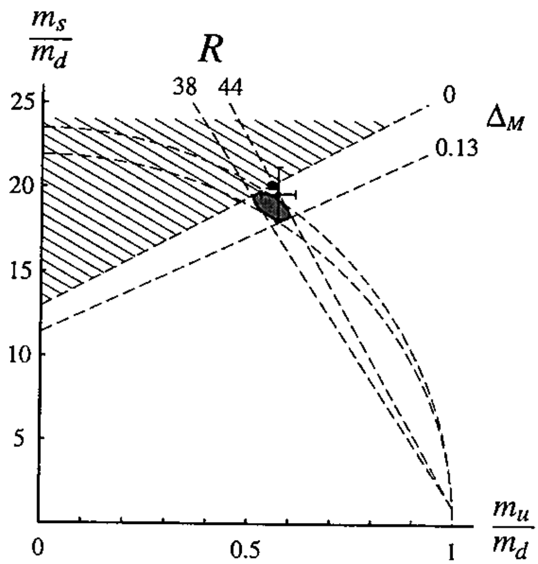

Fig. 42.1. m_s/m_d versus m_u/m_d from [57].

$\hat{m} \equiv (1/2)(m_u + m_d)$. To a good approximation, Eq. (42.65) constrains the quark-mass ratios to be on the ellipse,

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1\,, \tag{42.66}$$

In Fig. 42.1, one shows the range spanned by the corrections to the GMO mass formula:

$$\Delta_M :\ M^2_{\eta_8} = (1/3)\big(4M_K^2 - M_\pi^2\big)(1+\Delta_M)\,, \tag{42.67}$$

where to order p^4, one has:

$$\Delta_M \equiv \left(\frac{M_8^2 - M_\pi^2}{4M_K^2 - M_\pi^2}\right)\Delta_{\rm GMO}\,, \tag{42.68}$$

with:

$$\Delta_{\rm GMO} \equiv \frac{4M_K^2 - 3M_{\eta_8}^2 - M_\pi^2}{M_{\eta_8}^2 - M_\pi^2}\,. \tag{42.69}$$

Neglecting the mass difference $m_d - m_u$, one gets [499]

$$\Delta_{\rm GMO} = \frac{-2\left(4M_K^2\mu_K - 3M_{\eta_8}^2\mu_{\eta_8} - M_\pi^2\mu_\pi\right)}{M_{\eta_8}^2 - M_\pi^2} - \frac{6}{f^2}\left(M_{\eta_8}^2 - M_\pi^2\right)\left[12L_7^r(\mu) + 6L_8^r(\mu) - L_5^r(\mu)\right]\,. \tag{42.70}$$

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Experimentally, correcting the masses for electromagnetic effects, one obtains:

$$\Delta_{\rm GMO} = 0.21. \tag{42.71}$$

Since L_5 is already known, this allows the combination $2L_7 + L_8$ to be fixed. However, in order to determine the individual quark-mass ratios from Eqs. (42.63), we would need to fix the constant L_8. However, there is no way to find an observable that isolates this coupling. The reason is an accidental symmetry of the Lagrangian $\mathcal{L}_2 + \mathcal{L}_4$, which remains invariant under the following simultaneous change [519] of the quark-mass matrix and some of the chiral couplings:

$$\begin{aligned} \mathcal{M}' &= \alpha\,\mathcal{M} + \beta\,(\mathcal{M}^\dagger)^{-1}\,\det\mathcal{M}\,, \qquad B_0' = B_0/\alpha\,, \\ L_6' &= L_6 - \zeta\,, \qquad L_7' = L_7 - \zeta\,, \qquad L_8' = L_8 + 2\zeta\,, \end{aligned} \tag{42.72}$$

where α and β are arbitrary constants, and $\zeta = \beta f^2/(32\alpha B_0)$. The only information on the quark-mass matrix $\mathcal{M}$ that we used to construct the effective Lagrangian was that it transforms as $\mathcal{M} \to g_R \mathcal{M} g_L^\dagger$.

The matrix $\mathcal{M}'$ transforms in the same manner; therefore, symmetry alone does not allow us to distinguish between $\mathcal{M}$ and $\mathcal{M}'$. In order to resolve this ambiguity, additional information outside the framework of the pseudoscalar meson chiral Lagrangian has been used, by the introduction of the ratio:

$$R \equiv (m_s - \hat{m})/(m_d - m_u)\,. \tag{42.73}$$

Its value comes from the analysis of isospin breaking in the $\omega - \rho$ mixing and from the baryon spectrum [499]. At the intersection of different ranges, one deduces from Fig. 42.1:

$$\begin{aligned} \frac{m_u}{m_d} &= 0.553 \pm 0.043\,, \qquad \frac{m_s}{m_d} = 18.9 \pm 0.8\,, \\ \frac{2m_s}{(m_d + m_u)} &= 24.4 \pm 1.5\,. \end{aligned} \tag{42.74}$$

However, the possibility to have a $m_u = 0$ advocated in [519], where chiral symmetry can be still broken by, for example, instantons, appears to be unlikely as it implies too strong flavour symmetry breaking and is not supported by the QSSR results from two-point correlators of the divergences of the axial and vector currents, as will be shown in the following chapters.

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