M. Enomoto and K. Tamaki
Nagoya Math. J.
Vol. 56 (1974), 7–11

FREELY ACTING AUTOMORPHISMS OF ABELIAN C^*-ALGEBRAS

MASATOSHI ENOMOTO AND KAZUHIRO TAMAKI

1. Introduction.

Very recently, M. Choda, I. Kasahara and R. Nakamoto [3] extend the concept of free action of automorphisms for C^*-algebras and prove several theorems which are hither to known for von Neumann algebras. In the present note, we shall concern with freely acting automorphisms on abelian C^*-algebras. In § 2, several equivalent conditions for the free action are obtained. In § 3, we shall apply them to an automorphism which has a transversal group.

2. Equivalent conditions.

Let A be a unital abelian C^*-algebra and X be the character space of A, i.e. the compact space of all characters (multiplicative states) of A equipped with the weak* topology.

Following after [2], [3] for an automorphism α on A, an element $a \in A$ is called a dependent element of α if

(1) $$ax = x^\alpha a$$

is satisfied for every $x \in A$; if every dependent element of α is automatically 0, then we say that α is freely acting.

An automorphism α of A naturally induces a homeomorphism of X onto itself by

(2) $$\chi^\alpha(x) = \chi(x^\alpha)$$

for every $\chi \in X$ and $x \in A$. Therefore, we shall consider α as an automorphism of A and a homeomorphism of X onto itself. For a set $U \subset X$ (resp. $K \subset A$) we shall denote $U^\alpha = \{\chi^\alpha ; \chi \in U\}$ (resp. $K^\alpha = \{x^\alpha ; x \in K\}$).

Received February 5, 1973.

THEOREM 1. *The following conditions on an automorphism of a unital abelian C^*-algebra A are equivalent:*

(A) *α is freely acting,*

(B) *the set of all fixed points of α in X is nondense,*

(C) *for any nonempty open set $U \subset X$, there is a nonempty open set $V \subset U$ such that $V^\alpha \cap V$ is empty,*

(D) *for any nonzero closed ideal I of A, there is a nonzero ideal $J \subset I$ such that $J^\alpha \cap J = 0$,*

(E) *for any nonzero closed ideal I of A, there is a nonzero self-adjoint element $x \in I$ such that $x^\alpha x = 0$.*

Proof. (A) implies (B): If the set F of all invariant characters is not nondense, then there exists a nonempty open subset $W \subset F$. Take a nonzero element a having the support in W. Then a is a dependent element of α, since $\chi(ax^\alpha) = \chi^\alpha(ax) = \chi(ax)$ for $\chi \in W$ and $\chi(ax^\alpha) = 0 = \chi(ax)$ for $\chi \notin W$. Hence we have a contradiction.

(B) implies (C): For a nonempty open set $U \subset X$, there is $\chi \in U \cap F^c$ by the assumption. Since $\chi \neq \chi^\alpha$, there are a neighborhood W of χ^α and a neighborhood W' of χ such that $W \cap W'$ is empty. Since α is a homeomorphism, there is a neighborhood V' of χ such that V'^α is contained in W. If we put $V = V' \cap W' \cap U$, then V is the desired one.

(C) implies (D): For a closed ideal I, let U be the complement of the set of all characters annihilating I. Then U is open in X. Hence there is an open set V which satisfies the conditions of (C). If J is the set of all element of A which have their supports in V, then J is an ideal and satisfies $J \cap J^\alpha = 0$.

(D) implies (E): If $J \subset I$ is a nontrivial ideal with $J \cap J^\alpha = 0$, then there is a nonzero self-adjoint element $x \in J$ and we have $xx^\alpha = 0$.

(E) implies (A): Suppose that a is a dependent element of α. If I is the nonzero (closed) ideal of A generated by a, then there is a nonzero self-adjoint element $x \in I$ such that $xx^\alpha = 0$. Hence we have $ax^2 = ax^\alpha x = 0$. Therefore, in the support of x, a is 0, which is a contradiction. Hence $a = 0$ and α is freely acting.

Remark. Prof. M. Choda kindly pointed out that the above conditions are also equivalent to the following one:

(F) In the subalgebra of all invariant elements of α, there is no proper ideal of A included in it.

3. Transversal group.

Let u be a unitary operator and $\{v_s\}$ be a one-parameter group of unitary operators on a separable Hilbert space H. $\{v_s\}$ is said to be a transversal group for u, if

$$uv_s = v_{\alpha s}u \tag{3}$$

is satisfied for every s by a real number α with $|\alpha| \neq 1$. The notion of transversal groups for unitary operators is due to Kowada [5]. The origin of the notion goes back to Sinai who introduced for measure preserving transformations. By an inductive argument, we can easily prove that u^n has a transversal group $\{v_s\}$ for every n. In this section, we shall discuss a unitary operator u with a transversal group $\{v_s\}$ in a connection with Theorem 1.

Let $\boldsymbol{R}$ be the additive group of real numbers. Then we can construct a representation of the group algebra $L^1(\boldsymbol{R})$ using the given one-parameter unitary group $\{v_s\}$ by

$$t(x) = \int_{-\infty}^{+\infty} x(s)v_s ds \ , \tag{4}$$

where $x \in L^1(\boldsymbol{R})$ and the integration ranges over $(-\infty, +\infty)$.

In the next place, we assume that there exists a real number t_0 such that 1 is not contained in the proper value of v_{t_0}.

THEOREM 2. *Let A be the C^*-algebra generated by the identity and $\{t(x)\,; x \in L^1(\boldsymbol{R})\}$. If*

$$t^\alpha(x) = \int_{-\infty}^{+\infty} x(s)v_{\alpha s} ds \tag{5}$$

for $x \in L^1(\boldsymbol{R})$, then α becomes a freely acting automorphism of A.

Proof. Let X be the character space of A. Then X is homeomorphic to a compact subset of the one point compactification of the real line. By the Stone theorem, $\{v_s\}$ is represented as follows;

$$v_s = \int_{-\infty}^{+\infty} e^{-ist} dE(t) \ . \tag{6}$$

By (4) and (6), we have

$$t(x) = \int_{-\infty}^{+\infty} x(s)v_s ds = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(s)e^{-ist}dE(t)ds = \int_{-\infty}^{+\infty} \hat{x}(t)dE(t) \ ,$$

where $\hat{x}$ is the Fourier transform of x. Considering the correspondence $t \to t/\alpha$ on the real line, we have the correspondence $E(t) \to E(t/\alpha)$. On the other hand, we have

$$(7) \qquad t^{\alpha}(x) = \int_{-\infty}^{+\infty} \hat{x}(t) dE\left(\frac{t}{\alpha}\right),$$

which is a consequent of direct computation. It is clear that α is an automorphism of A and induces the homeomorphisms $t \to t/\alpha$ on X.

By the assumption of v_{t_0}, $\{t(x); x \in L^1(\boldsymbol{R})\}$ is not isomorphic to the complex number field. Therefore there are at least two characters which do not vanish on $\{t(x); x \in L^1(\boldsymbol{R})\}$. Thus we can conclude that there exists an element s in X which is neither 0 nor ∞. Since $|\alpha| \neq 1$, there is no fixed points up to 0 and ∞. Moreover, 0 and ∞ are not isolated points of X, since $\alpha^n s$ or $\alpha^{-n} s$ converge to 0 and ∞ as $n \to \infty$. Hence, by Theorem 1(B), we can conclude that α is freely acting on A.

By (3), we have

$$(3') \qquad u v_s u^* = v_{\alpha s}$$

so that we have by (4) and (5)

$$t^{\alpha}(x) = \int_{-\infty}^{+\infty} x(s) v_{\alpha s} ds = \int_{-\infty}^{+\infty} x(s) u v_s u^* ds$$

and consequently we have

$$(5') \qquad t^{\alpha}(x) = u t(x) u^*$$

Therefore, we have the following

COROLLARY 3. *The automorphism α of A induced by the unitary operator u by (5′) is freely acting.*

Remark. Similary, we can show that α^n is freely acting ($n = \pm 1, \pm 2, \cdots$).

COROLLARY 4 [3: Theorem 10]. *The spectrum of the unitary operator u is the entire unit circle.*

Proof. By the fact that α^n is freely acting ($n = \pm 1, \pm 2, \cdots$), there exists nonzero self-adjoint element x such that $x^{\alpha^{-n}} x = 0$ for $n = 1, 2, \cdots$, k. Take an element $\xi \neq 0$ in H such that $x\xi \neq 0$. Then, we have $(u^n x\xi \mid x\xi) = 0$ for $n = 1, 2, \cdots, k$. Therefore, u is nondegenerate. By the Arveson's

theorem [1: Theorem 1], the spectrum of u is the entire unit circle.
At this end, we wish express our hearty thanks to Mr. Takai to whom we are indebted the proof (C) – (D) of Theorem 1.

References

[1] W. B. Arveson, A theorem on the action of abelian unitary groups, Pacific J. Math., **16** (1966), 205–212.
[2] H. Choda, On freely acting automorphisms of operator algebras, to appear.
[3] M. Choda, I. Kasahara and R. Nakamoto, Dependent element of an automorphism of a C*-algebra, Proc. Japan Acad., **48** (1972), 561–565.
[4] R. Kallmann, A generalization of free action, Duke Math. J., **36** (1969), 781–789.
[5] M. Kowada, Spectral type of one-parameter group of unitary operators with transversal group, Nagoya Math. J., **32** (1968), 141–153.

Department of Mathematics, Osaka Kyoiku University.