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FIRST ORDER OPERATORS ON MANIFOLDS WITH A GROUP ACTION

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ABSTRACT. We investigate questions of spectral symmetry for certain first order differential operators acting on sections of bundles over manifolds which have a group action. We show that if the manifold is in fact a group we have simple spectral symmetry for all homogeneous operators. Furthermore if the manifold is not necessarily a group but has a compact Lie group of rank 2 or greater acting on it by isometries with discrete isotropy groups, and let D be a split invariant elliptic first order differential operator, then D has equivariant spectral symmetry.

1. **Introduction.** Let D be a first order differential operator on a complex vector bundle **E** over a compact Riemannian manifold M. The purpose of this work is to show that D has spectral symmetry when certain conditions are satisfied. These conditions are either M is a Lie group and D is a homogeneous operator; or a Lie group of rank 2 (or greater) acts suitably on M and D is an invariant elliptic operator. A more detailed description is given later.

For a self adjoint elliptic differential operator D on a compact Riemannian manifold M the eta function is defined as

$$\eta(s) = \sum_{\lambda \in \operatorname{spec}'(D)} \operatorname{sign}(\lambda)|\lambda|^{-s} \tag{1.1}$$

where spec(D) is the spectrum of D and $\operatorname{spec}'(D) = \operatorname{spec}(D) - \{0\}$. This converges for $Re(s)$ sufficiently large and the resulting function has a memomorphic extension, with simple poles, to the whole plane. The η-invariant is essentially the value of this function at $s = 0$, $\eta(0)$. The η-function and η-invariant were introduced in [1], where the η-invariant is the most important boundary contribution in the extension of the Atiyah-Singer index theorem to manifolds with boundary. Clearly if D has spectral symmetry then $\eta(s) = 0$ for all s and the η-invariant vanishes. Since its introduction much work has been done on the η-invariant, see [5] for further comments on this.

Our first result on spectral symmetry occurs in Section 5. Here we use the inversion map. The idea is similar to that in [2] but the details are somewhat different. The result is proved as Theorem 5.3.

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THEOREM 1.1. *Let D be a first order homogeneous invariant differential operator on a homogeneous bundle over a compact Lie group. Then D has spectral symmetry.*

In order to state and prove this theorem we need some background material. In Section 2 we give the construction of homogeneous operators and show that a first order operator splits into a first order part and a constant part. This is followed by Section 3 where we give a classification of the invariant operators both of first order and higher order. While the spirit of this is well known, many of the details are particular to this work; especially the characterization in the first order case in terms of the Hopf algebra structure on the symmetric algebra $\mathcal{S}(\mathfrak{g})$.

The main interest in η-invariants is for elliptic operators, see [10] for more information in this case. The series (1.1) converges for $Re(s)$ sufficiently large and has a meromorphic continuation for such operators. In general this is not the case for non elliptic operators, see [4] for an example of such. In Section 4 we give a characterization of first order elliptic operators over a compact Lie group, see Theorem 4.2, which is interesting in its own right.

THEOREM 1.2. *Let D be an invariant elliptic first order homogeneous differential operator over a simply connected Lie group. Then D is a twisted Dirac operator plus a bundle map.*

If D is an operator over a compact Lie group, G, there is an associated operator D' over the homogeneous space G/Γ. A natural question to ask is under what condition does spectral symmetry of D transfer to spectral symmetry of D'. In Section 6 we give an example to show that the conditions of Section 5 are too weak for this transfer. We then show that a stronger condition, equivariant symmetry where the eigenspaces are isomorphic as G-spaces, is sufficient. However, an other example of an operator (R) from [4] shows that there are interesting operators which have symmetry but not equivariant symmetry. Thus the work of later sections does not make that of Section 5 redundant.

This raises the question which operators have equivariant symmetry. The answer is given in Sections 7, 8, and 9. In Section 7 we prove:

THEOREM 1.3. *Let D be a first order, elliptic, invariant differential operator on a homogeneous bundle over a compact simply connected Lie group of rank ≥ 2. Then D has equivariant spectral symmetry.*

This is actually proved in the case where D is a twisted Dirac operator. Then the result of Section 4 is used to show that the class of twisted Dirac operators is essentially the same as that of elliptic operators over a group.

In Section 8 we treat the more general case of a manifold which has a group action. Specifically, we consider the case of a compact, simply connected nonabelian Lie group of rank 2 or greater acting on a manifold by isometries with discrete isotropy groups. To illustrate the type of group action which is considered here let $n \geq 4$. Then regard $\mathrm{SO}(2n-4)$ as a subgroup of $\mathrm{SO}(2n)$. If Γ is a discrete, and hence finite, subgroup of $\mathrm{SO}(2n)$ then we take $\Gamma \setminus \mathrm{SO}(2n)/\mathrm{SO}(2n-4)$ as the manifold. The group which acts is

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SO(4). This is regarded as a subgroup of SO($2n$) complimentary to SO($2n-4$), that is so that SO($2n-4$) $\times$ SO(4) embeds in SO($2n$). The main result, Theorem 8.12, can be summarized as

THEOREM 1.4. *Let D be a split, invariant elliptic first order differential operator. Then D has equivariant spectral symmetry.*

The notion of a split operator is given in Definition 8.3. A special case of split operators is that of twisted Dirac operators defined using a split connection.

In Section 9 we see that the same result holds when the group acting is the ℓ-dimensional torus with $\ell \geq 2$. The statement of the result is then the same as Theorem 1.4. However, the proof has some notable differences from that in Section 8.

2. **Invariant first order homogeneous operators.** We start by describing the general construction, following [12]. Let K be a compact Lie group and H a closed subgroup with Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$ respectively. Let $\pi: H \longrightarrow \mathrm{Aut}(E)$ be a finite dimensional representation of H. Then on $K \times E$ we have the equivalence relation $(k, v) \sim \big(kh^{-1}, \pi(h)v\big)$. The homogeneous bundle $\mathbf{E}$ is constructed by making the following diagram commute:

$$\begin{array}{ccc} K \times E & \longrightarrow & K \times_\pi E = K \times E/\sim = \mathbf{E} \\ \downarrow & & \downarrow \\ K & \longrightarrow & K/H. \end{array} \tag{2.1}$$

A section of $\mathbf{E}$ is represented by a map $\tilde{s}: K \longrightarrow E$ which satisfies $\tilde{s}(kh^{-1}) = \pi(h)s(k)$ for all $h \in H$ and $k \in K$. Then the section $s: K/H \longrightarrow \mathbf{E}$ is such that $\tilde{s}$ is its pullback. The map $\tilde{s}$ is equivariant. Let $\nu: H \longrightarrow \mathrm{Aut}\, F$ be another representation which defines the bundle $\mathbf{F}$. Then a homogeneous operator D: $C^\infty(\mathbf{E}) \longrightarrow C^\infty(\mathbf{F})$ is defined as follows. Let $\tilde{D}$ be an equivariant operator on K: that is $\tilde{D}\tilde{s}$ is an equivariant map into F whenever $\tilde{s}$ is an equivariant map into E. The operator D is homogeneous if $\tilde{D}\tilde{s}$ is the pullback of Ds. Symbolically we write $Ds = \tilde{D}\tilde{s}/H$.

In this paper we are interested in the case when $K/H = G$ is a compact Lie group. In particular we require that $K = G \times G$ and $H = G$ where H is represented as the diagonal subgroup of K: $H = \{(g, g) : g \in G\}$. The group K acts on K/H. In this particular case when $K = G \times G$ the left factor acts as left multiplication G and the right factor as right multiplication. If we embed $G \longrightarrow K$ by $g \longrightarrow (g, 1)$ we obtain a trivialization. In this context the homogeneous operators are the bi-invariant operators on $\mathbf{E}$ over G.

Now restrict attention to the case when D is a first order bi-invariant operator. Then D is a linear map on the first jet bundle D:$j^1(\mathbf{E}) \longrightarrow \mathbf{F}$. Since D is bi-invariant this is equivalent to giving an invariant map on the fibre over the identity: D:$j^1(E) \longrightarrow F$. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Then, as in [3], D is a splitting of the projected jet bundle sequence (2.2):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E \otimes \mathfrak{g}^* & \xrightarrow{\alpha} & j^1(E) & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & \overset{D}{\swarrow} & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & E & \longrightarrow & 0. \end{array} \tag{2.2}$$

The symbol is $D \circ \sigma: E \otimes \mathfrak{g}^* \longrightarrow F$. The lie algebra $\mathfrak{g}$ acts on $E \otimes \mathfrak{g}^*$ and E while the action on $j^1(E)$ is by $\mathfrak{g} \oplus \mathfrak{g}^{(1)}$ where $\mathfrak{g}^{(1)}$ is the first prolongation of $\mathfrak{g}$, see [11]. Thus the action of $\mathfrak{g}^{(1)}$ gives rise to an obstruction to find a first order invariant differential operator. However, since G is a compact we have $\mathfrak{g}^{(1)} = 0$. Thus we have proved:

THEOREM 2.1. *A first order homogeneous differential operator D can be constructed from any pair of invariant maps*

$$S: E \otimes \mathfrak{g}^* \longrightarrow F \text{ and } T: E \longrightarrow F.$$

REMARK 2.2. *These maps are projections which can be found by using Schur's lemma. The main application of this result is that the top order part of D is a bi-invariant first order operator in its own right.*

3. **Classification of first order bi-invariant operators.** Since we are interested in questions relating to the eigenvalues of the operators involved we only consider $D: C^\infty(\mathbf{E}) \longrightarrow C^\infty(\mathbf{E})$. That is we take $\mathbf{E} = \mathbf{F}$.

In the first instance let us only consider the case when $\pi: G \longrightarrow \operatorname{Aut} E$ is an irreducible representation. Then the map $T: E \longrightarrow E$ is a multiple of the identity. The interest is in the map $S: E \otimes \mathfrak{g}^* \longrightarrow E$.

Let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathfrak{g}$ relative to the Killing form innerproduct. Then using this basis we see that if $S: E \otimes \mathfrak{g}^* \longrightarrow E$ then

$$S = \sum_i \varphi_i \otimes X_i \tag{3.1}$$

here $\varphi: E \longrightarrow E$ and $X_i: \mathfrak{g}^* \longrightarrow \mathbb{R}$. Note that the linear maps φ_i are not invariant: invariance only holds for the whole sum not the individual terms. Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Then by the Jacobson density lemma, which is a form of a theorem of Burnside see [8], we have that $\varphi_i = \pi(p_i)$ for suitable $p_i \in \mathcal{U}$ and π has to be lifted to $\mathfrak{g}$ and extended to $\mathcal{U}$.

Let ℓ denote the rank of G and Z the centre of $\mathcal{U}$. Then $Z = \mathbb{C}[\Omega_1, \ldots, \Omega_\ell]$ is a polynomial algebra on ℓ generators, the Casimir operators. Define p_{ij} so $\sum_i p_{ij} X_i = \Omega_j$ and let $D_j = \sum_i \pi(p_{ij}) \otimes X_i$.

THEOREM 3.1. *The operators D_j span the vector space of invariant first order differential operators.*

PROOF. By the previous discussion we first need to describe all p_i such that $\sum p_i \otimes X_i$ is invariant. Now as vector spaces we can identify $\mathcal{U} \cong \mathcal{S}(\mathfrak{g}^*)$, the polynomial algebra on $\mathfrak{g}$. Thus the problem is to identify $\big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathfrak{g}\big)^G$. Now G acts on $\mathfrak{g}$ by the adjoint representation so by Schur's lemma we need only identify subrepresentations of $\mathcal{S}(\mathfrak{g}^*)$ which are equivalent to $\mathfrak{g}$. By the results of [7] these representations occur in degrees m_j which are the eigenvalues of a particular operator on $\mathfrak{g}^T$. Since $\mathfrak{g}^T$ has dimension ℓ it follows that $\dim\big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathfrak{g}\big)^G = \ell$.

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Under the identification $\mathcal{U} \cong \mathcal{S}(\mathfrak{g}^*)$ we have $Z \cong \mathcal{S}(\mathfrak{g}^*)^G$. Thus we can identify the Casimirs as elements $\Omega_j \in \mathcal{S}(\mathfrak{g}^*)^G$. There are ℓ of these and since they are algebraically independent they are also linearly independent. Using the Killing form inner product we can identify $X_i \in \mathfrak{g}$ as a degree one polynomial on $\mathfrak{g}$, *i.e.*, we can identify $X_i \in \mathcal{S}(\mathfrak{g}^*)$. This gives a multiplication map $m: \mathcal{S}(\mathfrak{g}^*) \otimes \mathfrak{g} \longrightarrow \mathcal{S}(\mathfrak{g}^*)$ with $m(\sum_i p_{ij} \otimes X_i) = \sum_i p_{ij} X_i$. Thus $\sum_i p_{ij} \otimes X_i$ are ℓ linearly independent elements of $\big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathfrak{g}\big)^G$ and by counting dimension they form a basis.

While the elements $\sum p_{ij} \otimes X_i$ are a basis for $\big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathfrak{g}\big)^G$ the $D_j = \sum_i \pi(p_{ij}) \otimes X_i$ need not be linearly independent but they still form a spanning seat for the invariant first order operators.

We can now write down the symbol $\sigma D_j : E \otimes \mathfrak{g}^* \longrightarrow E$. This is

$$\sigma\Big(\sum_i (p_{ij}) \otimes X_i\Big)(v \otimes x) = \sum \pi(p_{ij}) v x(X_i). \tag{3.2}$$

In more traditional notation

$$\sigma\Big(\sum \pi(p_{ij}) \otimes X_i\Big)(\xi) = \sum_i \xi_i \pi(p_{ij}), \tag{3.3}$$

where $\xi = \sum \xi_i X_i \in \mathfrak{g}$ and we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form.

The above results have been given for first order operators. It is routine to generalize them to operators of any order. We shall describe the results here but leave the proofs largely to the reader.

THEOREM 3.2. *The invariant differential operators on* $\mathbf{E}$ *all have the form* $D = \sum_i \pi(p_I) \otimes X_I$ *where* $p_I \in \mathcal{U} \approxeq \mathcal{S}(\mathfrak{g}^*)$ *and* $X_I = X_{i_1} \cdots X_{i_k} \in \mathcal{S}(\mathfrak{g})$, *which satisfy the condition* $\sum_i p_I \otimes X_I \in \big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g})\big)^G$.

REMARK 3.3. The operator described here with $X_I = X_{i_1} \cdots X_{i_k}$ is a k-th order operator. Here the upper case subscript is a multi-index while a lower subscript is a single index: $I = (i_1, \ldots, i_k)$.

From [7], we see that $\mathcal{S}(\mathfrak{g}) = Z \otimes H$ where $Z = \mathcal{S}(\mathfrak{g})^G$ and H is the space of harmonic polynomials. Then we can decompose $H = \sum H^k$, where H^k is the component which is homogeneous of degree k. If V_λ is the irreducible representation with highest weight λ, let V_λ^o denote the zero weight space of V_λ. There is an operator on V_λ^o with eigenvalues $m_j(\lambda)$ which are the generalized exponents of V_λ. These give the decomposition

$$H^k = \sum_\lambda V_\lambda m_j(\lambda) = k \text{ for some } j. \tag{3.4}$$

It is convenient to use the notation V_λ^j to denote the copy of V_λ occurring in H^k with $k = m_j(\lambda)$.

THEOREM 3.4. $\big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g})\big)^G = Z \otimes Z(\sum_\lambda \sum_{j_1} \sum_{j_2} \mathbb{C})$, *where the sum is over all eigenvalues* $m_{j_1}(\lambda)$, $m_{j_2}(\lambda)$ *for each highest weight* λ.

PROOF. Since $\mathcal{S}(\mathfrak{g}) = Z \otimes H$ we have

$$\big(\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g})\big)^G = Z \otimes Z \otimes (H \otimes H)^G. \tag{3.5}$$

By (3.4) we have $H = \sum_\lambda \sum_j V_\lambda^j$ and by Schur's lemma

$$(V_\lambda \otimes V_\mu)^G = \begin{cases} \mathbb{C} & \lambda^* = \mu \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Thus we calculate

$$\begin{aligned}(H \otimes H)^G &= \Big(\sum_\lambda \sum_{j_1} V_\lambda^{j_1} \otimes \sum_\mu \sum_{j_2} V_\mu^{j_2}\Big)^G \\ &= \sum_\lambda \sum_\mu \sum_{j_1} \sum_{j_2} (V_\lambda^{j_1} \otimes V_\mu^{j_2})^G \\ &= \sum_\lambda \sum_{j_1} \sum_{j_2} \mathbb{C}.\end{aligned} \tag{3.7}$$

This completes the proof.

From this we can write down the symbol of these operators.

THEOREM 3.5. *$\sigma\big(\sum_I \pi(p_I) \otimes X_I\big)(\xi) = \sum_I \xi_1 \pi(p_I)$ where $\xi = \sum \xi_i X_i$ and $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$.*

The algebra $\mathcal{S}(\mathfrak{g})$ is a Hopf algebra with comultiplication given by

$$df(x) = f(X \otimes 1 + 1 \otimes X), \tag{3.8}$$

where $d: \mathcal{S}(\mathfrak{g}) \longrightarrow \mathcal{S}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g})$. The algebra $\mathcal{S}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g})$ has a bigrading and let

$$P_{ij}: \mathcal{S}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g}) \longrightarrow \mathcal{S}^i(\mathfrak{g}) \otimes \mathcal{S}^j(\mathfrak{g}) \tag{3.9}$$

be the projection. Then the invariant elements $\sum_J p_{IJ} \otimes X_J$ are given by

$$\sum_J p_{IJ} \otimes X_J = P_{ij} d\Omega_k = \Omega_{ij} \tag{3.10}$$

for $i + j =$ degree Ω_k and $|I| = i$, $|J| = j$. Let PdZ be the algebra spanned by $\{\Omega_{ij}\}$. In the case of $G = \mathrm{SU}(2)$ it happens that $PdZ \cong \big(\mathcal{S}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g})\big)^G$. However, the situation is more complicated for SU(3).

The group SU(3) has two Casimirs: Ω_1 of degree 2 and Ω_2 of degree 3. Thus PdZ has 7 generators $\Omega_1 \otimes 1$, Ω_{11}, $1 \otimes \Omega_1$, $\Omega_2 \otimes 1$, Ω_{21}, Ω_{12} and $1 \otimes \Omega_2$ in the above notation. A calculation shows that, for $G = \mathrm{SU}(3)$, in addition to these there are at least 3 more generators, one in $\mathcal{S}^2(\mathfrak{g}) \otimes \mathcal{S}^2(\mathfrak{g})$ and two in $\mathcal{S}^3(\mathfrak{g}) \otimes \mathcal{S}^3(\mathfrak{g})$. Furthermore since $V_{3\rho}$, the irreducible representation with highest weight 3ρ, occurs uniquely in H^6 it follows that $\Omega_{21}^3(1 \otimes \Omega_2) = \Omega_{12}^3(\Omega_2 \otimes 1) \in \mathcal{S}^6(\mathfrak{g}) \otimes \mathcal{S}^6(\mathfrak{g})$ and so PdZ is not a polynomial algebra.

4. Elliptic first order operators.

We have that the symbol of a first order operator is a bilinear map

$$\sigma: \mathfrak{g} \otimes E \longrightarrow E. \tag{4.1}$$

In the case of a homogeneous operator, E is a complex representation space of G while the Lie algebra is, of course, a real vector space. The operator is elliptic if and only if σ

is a nondegenerate bilinear map, that is for each nonzero $X \in \mathfrak{g}$ the map $\nu \longrightarrow \sigma(X \otimes \nu)$ is invertible. Equivalently if $\sigma(X \otimes \nu) = 0$ then either $X = 0$ or $\nu = 0$. Thus we need to study nonsingular bilinear maps $\sigma: \mathbb{R}^k \otimes \mathbb{C}^n \longrightarrow \mathbb{C}^n$.

First we need to define the Clifford algebra. Let $T(\mathbb{R}^k)$ be the tensor algebra:

$$T(\mathbb{R}^k) = \mathbb{R} \oplus \mathbb{R}^k \oplus \mathbb{R}^k \otimes \mathbb{R}^k \oplus \cdots \tag{4.2}$$

Let I be the ideal in $T(\mathbb{R}^k)$ generated by elements of the form $X \otimes X + \langle X, X \rangle 1$ where $\langle , \rangle$ is an innerproduct on $\mathbb{R}^k$. Then the Clifford algebra is defined as

$$\mathrm{Cliff}(\mathbb{R}^k) = T(\mathbb{R}^k)/I, \tag{4.3}$$

see [6].

The spin group $\mathrm{Spin}(k) = \mathrm{Spin}(\mathbb{R}^k)$ which is a subgroup of the group of units of $\mathrm{Cliff}(\mathbb{R}^k)$, is a double cover of $\mathrm{SO}(k)$. This group, $\mathrm{Spin}(k)$, has a complex representation space S. Further S is a module for $\mathrm{Cliff}(\mathbb{R}^k)$ and the action of $\mathrm{spin}(n)$ is given by Clifford multiplication. If n is even then $S = S^+ \oplus S^-$ and $\dim S^+ = \dim S^-$. Clifford multiplication by a single element $X \in \mathbb{R}^k$ acts by interchanging S^+ and S^-. The elements of $\mathrm{Cliff}(\mathbb{R}^k)$ which generate $\mathrm{Spin}(k)$ have the form $X \otimes Y$ for X and $Y \in \mathbb{R}^k$.

We rewrite the bilinear map $\sigma: \mathbb{R}^k \otimes \mathbb{C}^n \longrightarrow \mathbb{C}^n$ as $\sigma: \mathbb{R}^k \longrightarrow \mathrm{End}(\mathbb{C}^n)$, where $\mathrm{End}(\mathbb{C}^n)$ is the space of $n \times n$ complex matrices. The nondegeneracy condition is $\sigma(\mathbb{R}^k - 0) \subset \mathrm{GL}(n, \mathbb{C})$, the invertible matrices.

LEMMA 4.1. *If the nonsingular bilinear map $\sigma: \mathbb{R}^k \longrightarrow \mathrm{End}(\mathbb{C}^n)$ satisfies the condition $\sigma(S^{k-1}) \subset U(n)$ then σ induces an action φ of* $\mathrm{spin}(k)$ *on $\mathbb{C}^n$. Further φ is the standard action so that $\mathbb{C}^n = mS$ (if k is odd) or $\mathbb{C}^n = m_+S^+ \oplus m_-S^-$ (if k is even).*

PROOF. As we noted earlier the group $\mathrm{Spin}(k)$ is generated by elements of the form $X \otimes Y$ in the Clifford algebra. Thus we need to define $\varphi: \mathbb{R}^k \otimes \mathbb{R}^k \longrightarrow \mathrm{End}(\mathbb{C}^n)$. We do this by

$$\varphi(X \otimes Y) = -\sigma(X)\overline{\sigma(Y)}^t. \tag{4.4}$$

Then

$$\varphi(X \otimes Y) = -\|X\|\,\|Y\|\sigma(e_X)\overline{\sigma(e_Y)}^t, \tag{4.5}$$

where $e_X = X/\|X\| \in S^{k-1}$ for $x \neq 0$ and is not defined for $X = 0$. Thus, since $\sigma(e_X) \in U(n)$ we have

$$\varphi(X \otimes Y) = -\|X\|\,\|Y\|\sigma(e_X)\sigma(e_Y)^{-1}. \tag{4.6}$$

Now this gives $\varphi(X \otimes X) + \langle X, X \rangle 1 = 0$ and so φ is an action of $\mathrm{spin}(k)$ on $\mathbb{C}^n$.

Furthermore since σ is nonsingular and linear this construction gives that $\mathbb{C}^n$ is just a number of copies of the basic spin representation. In the case when k is odd there is only one such representation and $\mathbb{C}^n = mS$. If k is even $\mathbb{C}^n$ splits as copies of S^+ and S^-: $\mathbb{C}^n = m_+S^+ \oplus m_-S^-$.

THEOREM 4.2. *Let $\mathcal{D}$ be an invariant, elliptic first order homogeneous differential operator on* $\mathbf{E}$ *over a simply connected group G. Then D is a twisted Dirac operator plus a bundle map.*

PROOF. Let $\pi\colon G \longrightarrow \mathrm{Aut}\,E$ be the representation which defines $\mathbf{E}$ and let $\sigma\colon \mathfrak{g}\otimes E \longrightarrow E$ be the symbol D.

First we consider the case when $\dim G$ is odd. Then the spin representation Δ: $\mathrm{Spin}(\mathfrak{g}) \longrightarrow \mathrm{Aut}(S)$ is irreducible. Since G is compact we define an invariant innerproduct on E and scale it so that $\sigma(S^{k-1}) \subset U(E)$. This is possible since σ and π commute by the invariance of D. By Lemma 4.1 σ induces an action $\mathrm{Spin}(\mathfrak{g})$ on E and $E = mS$, that is E is m copies of S with the action of $\mathrm{Spin}(\mathfrak{g})$ induced by σ being Δ on each summand.

Since G is simply connected the adjoint representation $\mathrm{Ad}\colon G \longrightarrow \mathrm{SO}(\mathfrak{g})$ lifts to spin

$$\tilde{s}\,\mathrm{Ad}\colon G \longrightarrow \mathrm{Spin}(\mathfrak{g}). \tag{4.7}$$

Differentiating this gives a map $\tilde{s}\,\mathrm{Ad}\colon \mathfrak{g} \longrightarrow \mathrm{spin}(\mathfrak{g})$ and then the spin representation $\chi\colon \mathfrak{g} \longrightarrow \mathrm{End}(S)$ by

$$\chi(X) = \Delta\big(\tilde{s}\,\mathrm{Ad}(X)\big). \tag{4.8}$$

Since σ and π commute and Δ is induced from σ it follows that χ and π commute. Thus on each summand π is a multiple of χ. This can be restated as $E = S\otimes V$, with $\dim V = m$, and $\pi = \chi \otimes \tilde{s}\pi$ for a representation $\tilde{s}\pi\colon G \longrightarrow \mathrm{Aut}\,V$. Thus $\mathbf{E}$ is a twisted spin bundle.

The symbol D is the same as that of a twisted Dirac operator, see Section 7, and the result follows from Lemma 2.1.

The case when $\dim G$ is even is similar. Here the spin bundle splits: $S = S^+ \oplus S^-$ and we have two Dirac operators P^+ and P^-. The result is now $D = P^+ \otimes \tilde{s}\pi_+ + P^- \otimes \tilde{s}\pi_- + \varphi$ for twistings $\tilde{s}\pi_+$ and $\tilde{s}\pi_-$ and bundle map φ.

5. **The inversion map.** On the Lie group G define the inversion map $i\colon G \longrightarrow G$ by $i(x) = x^{-1}$. For $x \in \mathfrak{g}$ let $\mathcal{L}_x$ be the Lie derivative in the direction X. Then we have

$$\begin{aligned} \mathcal{L}_x f(g) &= \frac{d}{dt} f\big(\exp(tX\cdot g)\big)\big|_{t=0} \\ &= \lim_{t\to 0} f\big(\exp(tX\cdot g) - f(g)\big)/t. \end{aligned} \tag{5.1}$$

If the homogeneous bundle $\mathbf{E}$ is trivialized by using left translation we can decompose the space of sections as

$$C^\infty(\mathbf{E}) = C^\infty(G)\otimes E. \tag{5.2}$$

Relative to this decomposition the operator $D = \sum_i \varphi_i \otimes X_i$, as given in (3.1), becomes

$$D = \sum_i \mathcal{L}_{X_i} \otimes \varphi_i. \tag{5.3}$$

The Lie derivative and inversion map interact as follows.

THEOREM 5.1. $\mathcal{L}_X(f \circ i)(g) = -(\mathcal{L}_{\mathrm{Ad}\, gX} f) \circ i(g)$.

PROOF. By the Baker-Campbell-Hausdorff formula

$$\exp(X \cdot g)^{-1} = \exp\big(-t\,\mathrm{Ad}(g)X\big) \cdot g^{-1} + 0(t^2). \tag{5.4}$$

The lemma now follows by using the definition (5.1).

The representation $\pi\colon G \longrightarrow \mathrm{Aut}\, E$ which gives rise to the homogeneous bundle $\mathbf{E}$, can be lifted to the Lie algebra $\mathfrak{g}$ and extended to the universal enveloping algebra $\mathcal{U}$. This gives an action of $\mathfrak{g}$, and hence $\mathcal{U}$, on $\mathbf{E}$. For $X \in \mathfrak{g}$ we have the map $\pi(X)\colon E \longrightarrow E$ which, by the trivialization of $\mathbf{E}$ using left translation, induces a map $\pi(X)\colon \mathbf{E} \longrightarrow \mathbf{E}$. If $s\colon G \longrightarrow \mathbf{E}$ is a section then $s \circ i$ is another section. The relationship between $\pi(x)$ and i is given by the next result.

THEOREM 5.2. $\pi(X)(s \circ i)(g) = \big(\pi(\mathrm{Ad}\, gX)s\big) \circ i(g)$.

PROOF. Let ℓ_y and r_y be left and right translation by $y \in G$, that is

$$\ell_y(g) = yg, \quad r_y(g) = gy^{-1}. \tag{5.5}$$

Then we have $i \circ \ell_y = r_y \circ i$. Now $\pi(X)$ on $\mathbf{E}$ is defined by left translation so we have

$$\pi(X) r_{y^{-1}} = \ell_{y^{-1}}(\mathrm{Ad}\, y \circ X). \tag{5.6}$$

The result of the lemma follows from the computation:

$$\begin{aligned} \pi(X)(s \circ i)(g) &= \ell_g \pi(X) s i \ell_{g^{-1}}(g) \\ &= \big(\pi(\mathrm{Ad}\, gX)s\big) i(g). \end{aligned} \tag{5.7}$$

We can now prove the main theorem of this section.

THEOREM 5.3. *Let D be a first order invariant differential operator on $\mathbf{E}$ of the form $D = \sum_i \mathcal{L}_{X_i} \otimes \pi(p_i)$. Then D has spectral symmetry.*

PROOF. By applying Lemma 5.1 and Lemma 5.2 we have

$$D(f \circ i)(g) = -\big((\mathrm{Ad}\, g \cdot D)f\big) i(g). \tag{5.8}$$

Since D is invariant $\mathrm{Ad}\, gD = D$ ad so we have

$$D(f \circ i) = -(Df) \circ i \tag{5.9}$$

Thus i intertwines the positive and negative eigenspaces.

REMARK 5.4. *The same proof gives that any odd order operator D of the form $D = \sum \mathcal{L}_{X_{i_1}} \cdots \mathcal{L}_{X_{i_k}} \otimes \pi(p_I)$ has spectral symmetry. The results of Section 3 using the Jacobson density lemma show that in the case when $\pi\colon G \longrightarrow \mathrm{Aut}\, E$ is irreducible all operators have the appropriate form. Here the upper case subscript is a multiindex: $I = (i_1, \ldots, i_k)$.*

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6. **Spectral symmetry and equivariant symmetry.**

Let Γ be a closed subgroup of G. Then, by integration, if D is an operator on a bundle $\mathbf{E}$ over G we obtain an operator D' on the induced bundle $\mathbf{E}'$ over G/Γ. However, it is possible for D to have spectral symmetry while D' does not. It follows from [6] that the Dirac operator on SU(2) has spectral symmetry while it is well known that the Dirac operator on SO(3) does not have spectral symmetry. Since the adjoint of SO(3) does not lift to spin, the spin bundle on SO(3) is not homogeneous for $\mathrm{SO}(3) = \mathrm{SO}(3) \times \mathrm{SO}(3)/\,\mathrm{SO}(3)$. The two spin structures on SO(3) are homogeneous for $\mathrm{SO}(3) = \mathrm{SO}(3)/\{1\}$, the trivial structure, and $\mathrm{SO}(3) = \mathrm{SU}(2)/\{1,-1\}$, the nontrivial structure. To answer the question of the descent of spectral symmetry we need the notion of equivariant symmetry.

Let the spectrum of D have eigenvalues $\{k\}$ with associated eigenspaces $\{U_k\}$. Then since D is invariant we have an action of G on each U_k. The spectrum of D has equivariant symmetry if $U_k \cong U_{-k}$ as representations of G for all k.

THEOREM 6.1. *If D has equivariant spectral symmetry on a homogeneous bundle* $\mathbf{E}$ *over G then D′ has spectral symmetry on* $\mathbf{E}'$ *over* G/Γ.

PROOF. By left translation we decompose the space of sections

$$L^2(\mathbf{E}) = \widehat{\textstyle\sum_\lambda} n_\lambda V_\lambda \otimes E, \text{ all } \lambda, \tag{6.1}$$

here the sum is over all highest weights λ with n_λ its multiplicity and the $\hat{}$ over the sum denotes completion. From this we get the decomposition

$$L^2(\mathbf{E}') = \widehat{\textstyle\sum_\lambda} n_\lambda V_\lambda \otimes E, \quad \lambda \text{ such that } \Gamma \subset \operatorname{Ker} \pi_\lambda, \tag{6.2}$$

where the sum is over those highest weights λ such that $\pi_\lambda|\Gamma$ is trivial, or equivalently $\Gamma \subset \operatorname{Ker} \pi_\lambda$. We then have

$$D'|n_\lambda V_\lambda \otimes E = \begin{cases} D|n_\lambda V_\lambda \otimes E & \Gamma \subset \operatorname{Ker} \pi_\lambda \\ \text{not defined} & \text{otherwise.} \end{cases} \tag{6.3}$$

Since D has equivariant symmetry $D|n_\lambda V_\lambda \otimes E$ is symmetric and so D' is symmetric.

To complete this section we give the following example to show that there are operators on groups other than SU(2) which have spectral symmetry but not equivariant symmetry.

The simplest first order differential operator to write down is $R = \sum_i \mathcal{L}_{X_i} \otimes \pi(X_i)$ summed over a basis $\{X_i\}$ of $\mathfrak{g}$. Decompose $C^\infty(\mathbf{E})$ as follows:

$$L^2(\mathbf{E}) = L^2(G) \otimes E = \widehat{\textstyle\sum_\lambda} V_\lambda \otimes E = \widehat{\textstyle\sum_\theta} V_{\theta(\lambda)}. \tag{6.4}$$

Here V_λ is irreducible representation with highest weight λ (and $V_{\theta(\lambda)}$ has highest weight θ) with each term repeated as often as its multiplicity. Now by invariance V_θ is an invariant space for R and indeed for any invariant operator. By completing the square we see R is constant on V_θ:

$$R|V_{\theta(\lambda)} = \frac{1}{2}(\|\lambda + \rho\|^2 + \|\mu + \rho\|^2 - \|\theta + \rho\|^2 - \|\rho\|^2) \tag{6.5}$$

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where $\|\ \|^2$ is the square of the Killing form norm (with the positive sign), μ is highest weight of $\pi: G \longrightarrow \operatorname{Aut} E$ (now taken to be irreducible) and θ is a highest weight in $\pi_\lambda \otimes \pi_\mu$. In fact θ has the form $\theta = \lambda + \alpha$ where α is a weight of μ (not all such weights necessarily occur). Then the eigenvalue of R is

$$(6.6)\qquad \begin{aligned} -2\langle \alpha, \lambda\rangle + 2\langle \mu - \alpha, \rho\rangle + \|\mu\|^2 - \|\alpha\|^2 = -2\langle \alpha, \lambda\rangle \\ + \langle \mu - \alpha, \mu + \alpha + 2\rho\rangle. \end{aligned}$$

Note that $\langle \mu - \alpha, \mu + \alpha + 2\rho\rangle$ is bounded and nonnegative. It is clear from this that R does not have equivariant symmetry.

7. **Twisted Dirac operators over a lie group.** Let G be a compact simply connected nonabelian Lie group and let $\chi: G \longrightarrow \operatorname{Aut} S$ be the spin representation of G. The spin bundle over G is denoted by $\mathbf{S}$ and has fiber S. Let $\pi: G \longrightarrow \operatorname{Aut} V$ be another representation of G and let $\mathbf{V}$ be the homogeneous vector bundle of G associated with π with fiber V. By left translation we can trivialize both $\mathbf{S}$ and $\mathbf{V}$ as $\mathbf{S} \cong G \times S$ and $\mathbf{V} \cong G \times V$. Let $\mathfrak{g}$ be the Lie algebra of G and then, using left translation, we trivialize the tangent bundle $T(G) \cong G \times \mathfrak{g}$. Thus we can regard elements $X \in \mathfrak{g}$ as a left invariant vector field, $s \in S$ as a left invariant spinor and $v \in V$ as a left invariant section of $\mathbf{V}$.

The bundle $\mathbf{V}$ has a connection ∇ which is associated to the Levi-Civita connection of G. This map

$$(7.1)\qquad \nabla: \Gamma\big(T(G) \times \mathbf{V}\big) \longrightarrow \Gamma(\mathbf{V}),$$

which restricted to left invariant section to give a map:

$$(7.2)\qquad \nabla: \mathfrak{g} \otimes V \longrightarrow V.$$

The twisted spin bundle if $\mathbf{S} \otimes \mathbf{V}$ which has the connection given by

$$(7.3)\qquad \nabla_X(s \otimes v) = \nabla_X(s) \otimes v + s \otimes \nabla_X(v),$$

where $\nabla_X(s)$ is the spin connection and $\nabla_X(v)$ is the connection of (7.1). If X, s and v are left invariant we can identify them as elements of $\mathfrak{g}$, S and V. Thus we have the spin connection as a map.

$$(7.4)\qquad \nabla: \mathfrak{g} \otimes S \otimes V \longrightarrow S \otimes V.$$

An easy calculation simliar to one in [5], gives the following result.

LEMMA 7.1. *a. The associated connection ∇ when restricted to left invariant section is given by $\nabla_X(v) = \frac{1}{2}\pi(X)v$.*

b. The twisted spin connection on $\mathbf{S} \otimes \mathbf{V}$ when restricted to left invariant sections is given by $\nabla_X(s \otimes v) = \frac{1}{2}\big(\chi(X)s\big) \otimes v + \frac{1}{2}s \otimes \pi(X)v$.

Let $E_1, \ldots, E_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. Then the twisted Dirac operator is given by

$$(7.5)\qquad P = \sum_i E_i \nabla_{E_i},$$

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where E_i acts by Clifford multiplication on **S** and ∇_{E_i} is the twisted spin connection. For a vector field X let $\mathcal{L}_X$ be a Lie differentiation in the direction X. Using left invariance we can trivialize the setions of the twisted spin bundle:

$$\Gamma(\mathbf{S}\otimes\mathbf{V})\cong C^\infty(G)\otimes S\otimes V. \tag{7.6}$$

PROPOSITION 7.2. *Relative to the trivialization of (7.6) the twisted Dirac operator is given by* $P(f\otimes s\otimes v)=\sum_i(\mathcal{L}_{E_i}f)\otimes E_is\otimes v+\frac{1}{2}\sum_i f\otimes\big(E_i\chi(E_i)s\big)\otimes v+\frac{1}{2}\sum_i f\otimes E_is\otimes\pi(E_i)v.$

PROOF. This is a routine calculation using Lemma 7.1.

Define the operators Q, M and T by the following formula relative to the trivialization (2.6):

$$\begin{aligned} Q&=\sum_i \mathcal{L}_{E_i}\otimes E_i\otimes 1,\\ M&=\sum_i 1\otimes E_i\chi(E_i)\otimes 1,\\ T&=\sum_i 1\otimes E_i\otimes \pi(E_i). \end{aligned} \tag{7.7}$$

Then Proposition 7.2 can be restated as follows.

COROLLARY 7.3. $P=Q+\frac{1}{2}M+\frac{1}{2}T.$

A key step in establishing spectral symmetry of P is to calculate the anticommutators of Q, M and T. Recall that for two operators, A and B, the anticommutator is $\{A,B\}=AB+BA$. We need some notations: Δ is the Laplacian, Ω is the Casimir element of the universal enveloping algebra and three other operators are defined as follows:

$$\begin{aligned} R_X&=\sum_i \mathcal{L}_{E_i}\otimes \chi(E_i)\otimes 1,\\ R_\pi&=\sum_i L_{E_i}\otimes 1\otimes \pi(E_i),\\ R_T&=\sum_i 1\otimes \chi(E_i)\otimes \pi(E_i). \end{aligned} \tag{7.8}$$

The anticommutators are then given by:

LEMMA 7.4. *i.* $\{Q,M\}=-6R_X$
ii. $\{Q,T\}=-2R_\pi$
iii. $\{M,T\}=-6R_T$
iv. $Q^2=-\Delta\otimes 1\otimes 1+2R_X$
v. $M^2=9\|\rho\|^2$
vi. $T^2=2R_T+1\otimes 1\otimes\pi(\Omega)$,
where ρ is half the sum of the postive roots and $\|\ \|$ is the negative of the Killing form.

PROOF. Parts i., iv. and v. can be found in [5]. The other parts follow by direct calculation. We shall illustrate this in the case of ii. and omit the others. The calculation is:

$$\{Q,T\} = \sum_{ij}\big(\mathcal{L}_{E_i}\otimes E_iE_j\otimes\pi(E_j) + \mathcal{L}_{E_i}\otimes E_jE_i\otimes\pi(E_j)\big)$$

(7.9)
$$= \sum_{ij}\mathcal{L}_{E_i}\otimes(E_iE_j+E_jE_i)\otimes\pi(E_j)$$
$$= -2\sum\mathcal{L}_{E_i}\otimes 1\otimes\pi(E_i).$$
$$= -2R_\pi$$

Now decompose the sections of $\mathbf{S}\otimes\mathbf{V}$ under the action of G. First note that we have identified

(7.10)
$$\Gamma(\mathbf{S}\otimes\mathbf{V}) = C^\infty(G)\otimes S\otimes V.$$

Now by the Peter-Weyl theorem $C^\infty(G) = \hat{\bigoplus} V_\lambda$, where V_λ is the isotopic component of the type λ. Next decompose each $V_\lambda\otimes S\otimes V = \oplus S_\theta$ into isotopic components where S_θ has type θ. By the results of [5] we see that each S_θ is invariant under M and Q. Hence these spaces are also invariant under T. Furthermore from [5] we have:

PROPOSITION 7.5. *When rank $G > 1$,* $\operatorname{tr} M|S_\theta = 0$.

On each space S_θ the operators the operators R_χ, R_π and R_T are constant.

PROPOSITION 7.6. *The restrictions of R_χ, R_π and R_T to S_θ are constant.*

PROOF. This is an immediate consequence of the following polarization identities:

$$R_\chi = \frac{1}{2}\big(-(L\otimes\chi\otimes 1)(\Omega) + L(\Omega)\otimes 1\otimes 1 + 1\otimes\chi(\Omega)\otimes 1\big),$$

(7.11)
$$R_\pi = \frac{1}{2}\big(-(L\otimes 1\otimes\pi)(\Omega) + L(\Omega)\otimes 1\otimes 1 + 1\otimes\chi(\Omega)\otimes\pi(\Omega)\big),$$
$$R_T = \frac{1}{2}\big(-(1\otimes\chi\otimes\pi)(\Omega) + 1\otimes\chi(\Omega)\otimes 1 + 1\otimes 1\otimes\pi(\Omega)\big),$$

since the Casimir element Ω is constant on isotopic components.

COROLLARY 7.7. *The restrictions of $\{P,M\}$ and P^2 to S_θ are constant.*

PROOF. By Lemma 6.4 both $\{P,M\}$ and P^2 can be expressed in terms of Δ, $1\otimes 1\otimes\pi(\Omega)$ and the operators R_χ, R_π and R_T. Each of these is constant on S_θ.

THEOREM 7.8. *The operator P has spectral G-symmetry.*

PROOF. We establish that P has spectral symmetry on S_θ, proceeding as in [5]. Let $\alpha = 3\|\rho\|$. Then since $\operatorname{tr} M|S_\theta = 0$ and $M|S_\theta = \alpha^2$ we can decompose $S_\theta = S_\theta^+\otimes S_\theta^-$ into the α and $-\alpha$ eigenspaces of M, with $\dim S_\theta^+ = \dim S_\theta^-$. Let $P|S_\theta$ have block matrix with $\begin{pmatrix} A & B\\ C & D\end{pmatrix}$ relative to this decomposition. Then we calculate:

(7.12)
$$\{P,M\} = \begin{pmatrix} 2\alpha A & 0\\ 0 & -2\alpha D\end{pmatrix} = kI,$$

for some constant k. Thus $D = -A$ and so $\operatorname{tr} P|S_\theta = 0$. Since $P^2|S_\theta = \beta$, a constant, the eigenvalues of $P|S_\theta$ are $\sqrt{\beta}$ and $-\sqrt{\beta}$. Since $\operatorname{tr} P|S_\theta = 0$ these have equal multiplicity and the theorem is proved.

8. **Elliptic operators on a manifold with a group action.** Let N be a manifold and G a compact simply connected nonabelian Lie group of rank 2 or greater which acts on N by isometries such that all the isotropy groups are discrete. We use the local basis, $E_1, \ldots, E_r, E_{r+1}, \ldots, E_n$ for $T(N)$ which is given in [5]. For convenience we summarize its construction.

Let $E_1, \ldots, E_r$ be an orthonormal basis for $\mathfrak{g}$. Then the formula

$$Ef(\rho) = \lim_{t \to 0} \frac{1}{t}\big(f \exp(tE_\rho) - f(\rho)\big) \tag{8.1}$$

defines a global vector field on N associated to each element of $\mathfrak{g}$. We use the same letters $E_1, \ldots, E_r$ to denote the vector fields on N associated to the elements $E_1, \ldots, E_r$ of $\mathfrak{g}$. For $p \in N$ we have the isotropy group G_p and hence a G_p-slice D in N, see [9]. We may take D to be the disc with center p and $E_{r+1}, \ldots, E_n$ to be an orthonormal basis of $T(D)$. Let $U = GD$, an open set in N, and use the group action to extend $E_{r+1}, \ldots, E_n$ to be a vector field on U. Thus we have an orthonormal basis $E_1, \ldots, E_n$ for $T(U)$. We shall use the convention that Latin subscripts run from 1 to r and Greek subscripts from $r+1$ to n. The subbundle $V \subset T(N)$ spanned by $E_1, \ldots, E_r$ is the vertical bundle and is trivial: $V \cong N \times \mathfrak{g}$. The orthogonal complement H to V is the horizontal bundle and locally has the basis $E_{r+1}, \ldots, E_n$. There is the decomposition

$$T(N) = V \oplus H. \tag{8.2}$$

Let W be a complex vector bundle over N such that the action of G on N lifts to an action on W. Using the action of G we can decompose

$$W = \sum W_H \otimes W_G. \tag{8.3}$$

Without loss of generality we may suppose that there is only one term in the sum (8.3). Further we obtain this decomposition by having G act trivially on W_H. We also require that W_G is a homogeneous bundle on G which has been transferred to N by the G action. On this bundle W we have the split connection:

$$\nabla = \nabla^H \otimes 1 + 1 \otimes \nabla^G. \tag{8.4}$$

For this connction we note $\nabla^H E_i = 0$ and $\nabla^G E_\alpha = 0$, using the convention $1 \leq i \leq r$ and $r+1 \leq \alpha \leq n$.

Let $D\colon C^\infty(W) \longrightarrow C^\infty(W)$ be an elliptic invariant first order differential operator. Then the symbol of D is $\sigma(D)\colon T(N) \otimes W \longrightarrow W$. Using the splitting (8.2) we have $\sigma(D) = \sigma(D)_H + \sigma(D)_V$ where

$$\begin{aligned} &\sigma(D)_H|H \otimes W = \sigma(D)|H \otimes W \\ &\sigma(D)_H|V \otimes W = 0 \\ &\sigma(D)_V|H \otimes W = 0 \\ &\sigma(D)_V|V \otimes W = \sigma(D)|H \otimes W. \end{aligned} \tag{8.5}$$

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LEMMA 8.1. *The operator D splits $D = D_H + D_G + \varphi$ where $D_H = \sigma(D)_H \nabla^H \otimes 1$, $D_V = \sigma(D)_V| \otimes \nabla^G$ and φ is a bundle map.*

PROOF. From the basic definitions of a connection and the symbol, since D is first order, we have

$$D = \sigma(D)\nabla + \varphi. \tag{8.6}$$

Now $\sigma(D) = \sigma(D)_H + \sigma(D)_V$ and since ∇ is the split connection the result follows.

COROLLARY 8.2. *The operator D_V is obtained from a twisted Dirac operator on G.*

PROOF. By construction the operator D_V is obtained from an operator on G. Since D is elliptic so is D_V on G. The corollary follows for the results of Section 4.

DEFINITION 8.3. The operator D is a split operator if $\varphi = 0$.

COROLLARY 8.4. *A twisted Dirac operator on N defined by using the split connection is split.*

As in Section 7 we can write $D_V = Q_G + \frac{1}{2}M + \frac{1}{2}T$ where

$$\begin{aligned} Q_G &= \sum_i \mathcal{L}_{E_i} \otimes E_i \otimes 1, \\ M &= \sum_i 1 \otimes E_{i\chi}(E_i) \otimes 1, \\ T &= \sum_i 1 \otimes E_i \otimes \pi(E_i), \end{aligned} \tag{8.7}$$

act on $C^\infty(W_H) \otimes S_G \otimes E_G$. Here $W_G = S_G \otimes E_G$ is the prepresentation of W_G as a twisted spin bundle obtained from Corollary 8.2.

An easy calculation then gives the following anticommutators.

LEMMA 8.5. *Let ω be the volume form on N and X a vector field on N associated to an element $X \in \mathfrak{g}$ then*

i. $\{D_H, \omega X\} = 0$
ii. $\{Q_G, \omega X\} = -2(\mathcal{L}_X \otimes 1 \otimes 1)$
iii. $\{M, \omega X\} = -6\big(1 \otimes \chi(X) \otimes 1\big)$
iv. $\{T, \omega X\} = -2\big(1 \otimes 1 \otimes \pi(X)\big)$.

PROOF. The first part follows from the use of the split connection. The other parts are straightforward computations.

COROLLARY 8.6. $\{D, \omega X\} = -2(\mathcal{L}_X \otimes 1 \otimes 1) - 3\big(1 \otimes \chi(X) \otimes 1\big) - (1 \otimes 1 \otimes \pi(X)\big)$ *for a split operator D.*

LEMMA 8.7. *The commutator $[D^2, \omega X] = -\sum_i \mathcal{L}_{E_i} \otimes [E_i, X] \otimes 1 - \sum_i 1 \otimes [E_i, X] \otimes \pi(E_i)$ for a split operator D.*

PROOF. We start with

$$
\begin{aligned}
[D^2,\omega X] &= [D,\{D,\omega X\}] = [D_V^2,\omega X] \\
&= -2\sum_i \mathcal{L}_{[E_i,x]}\otimes E_i\otimes 1 - 3\sum_i \mathcal{L}_{E_i}\otimes [E_i,X]\otimes 1 \\
&\quad -\frac{3}{2}\sum_i 1\otimes [E_i\chi(E_i),\chi(X)]\otimes 1 \\
&\quad -\frac{3}{2}\sum_i 1\otimes [E_i,X]\otimes \pi(E_i) \\
&\quad -\frac{1}{2}\sum_i 1\otimes 1\otimes \pi([E_i,X]).
\end{aligned}
\tag{8.8}
$$

Now an easy calculation gives

$$\sum_i [E_i\chi(E_i),\chi(X)] = 0 \tag{8.9}$$

and

$$\sum_i 1\otimes [E_i,X]\otimes \pi(E_i) = -\sum_i 1\otimes E_i\otimes \pi([E_i,X]). \tag{8.10}$$

Using $[E_i,X]=\sum_k\langle E_k,[E_i,X]\rangle E_k$, and the invariance of the innerproduct:

$$\langle E_i,[E_k,X]\rangle = -\langle [E_i,X],E_k\rangle \tag{8.11}$$

reduces (8.8) to the result of the lemma.

The immediate use of this lemma is to investigate the commutator of D^2 with the spin representation. As in the previous lemma and unless otherwise stated we shall restrict our attention to the case when D is a split operator.

THEOREM 8.8. *$[D^2,XY] = -[R_\chi + R_T, XY]$, where R_χ and R_T are given by the formulae (7.8).*

PROOF. We calculate:

$$
\begin{aligned}
[D^2,XY] &= [D^2,\omega X]\omega Y + \omega X[D^2,\omega Y] \\
&= -\sum_i \mathcal{L}_{E_i}\otimes ([E_i,X]Y + X[E_i,Y])\otimes 1 \\
&\quad -\sum_i 1\otimes ([E_i,X]Y + X[E_i,Y])\otimes \pi(E_i) \\
&= -\sum_i \mathcal{L}_{E_i}\otimes [\chi(E_i),XY]\otimes 1 - \sum_i 1\otimes [\chi(E_i),XY]\otimes \pi(E_i) \\
&= -[R_\chi,XY] - [R_T,XY].
\end{aligned}
\tag{8.12}
$$

COROLLARY 8.9. *$D^2 + R_\chi + R_T$ commutes with the action of* spin($\mathfrak{g}$) *via* $1\otimes\Delta\otimes 1$ *where Δ is the spin representation* $\Delta\colon \mathrm{spin}(\mathfrak{g}) \longrightarrow \mathrm{End}\, S_G$.

Now decompose $L^2(W) = \sum V_\lambda$ into eigenspaces of $D^2 + R_\chi + R_T$. Since the operator commutes with the representation $1\otimes\Delta\otimes 1$ we have

$$V_\lambda = V_\lambda^H \otimes S_G, \tag{8.13}$$

where $V_\lambda^H \subset L^2(W_H) \otimes E_G$. Next we can decompose

$$V_\lambda = \sum_\theta S_{\lambda\theta} \tag{8.14}$$

into isotopic components under the $\chi \otimes \pi$ action of $\mathfrak{g}$.

LEMMA 8.10. $\operatorname{tr} M|S_{\lambda\theta} = 0$.

PROOF. As in the case of M defined on a group we find

$$\operatorname{tr} M = 0, \quad M^2 = 9\|\rho\|^2 \tag{8.15}$$

Thus $S_G = S_G^+ \oplus S_G^-$ decomposes into eigenspaces of M with $S_G^+ \cong S_G^- \cong 2^{k-1} V_\rho\big(k = \frac{1}{2}\ell$ for ℓ even or $k = \frac{1}{2}(\ell - 1)$ for ℓ odd$\big)$. Then decomposing V_λ gives

$$\begin{aligned} V_\lambda = V_\lambda^H \otimes S_G &= (V_\lambda^H \otimes S_G^+) \oplus (V_\lambda^H \otimes S_G^-) \\ &= \sum_\theta (S_{\lambda\theta}^+ \oplus S_{\lambda\theta}^-). \end{aligned} \tag{8.16}$$

Since $S_G^+ \cong S_G^-$ we have $S_{\lambda\theta}^+ \cong S_{\lambda\theta}^-$ and so $\operatorname{tr} M|S_{\lambda\theta} = 0$.

LEMMA 8.11. *i.* $\{M, D_H\} = 0$
ii. $\{M, DV\} = -6R_\chi - 3R_T + 9\|\rho\|^2$.

PROOF. i. This follows from the splitting into vertical and horizontal components.
ii. This follows from $D_V = Q_G + \frac{1}{2}M + \frac{1}{2}T$ and Lemma 7.4.
By the polarization identities:

$$\begin{aligned} R_\chi &= -\frac{1}{2}\big((L \otimes 1 \otimes 1 + 1 \otimes \chi \otimes 1)(\Omega) - L(\Omega) \otimes 1 \otimes 1 - 1 \otimes \chi(\Omega) \otimes 1\big), \\ R_T &= -\frac{1}{2}\big((1 \otimes \chi \otimes 1 + 1 \otimes 1 \otimes \pi)(\Omega) - 1 \otimes \chi(\Omega) \otimes 1 - 1 \otimes 1 \otimes \pi(\Omega)\big), \end{aligned} \tag{8.17}$$

where Ω is the Casimir element, we see that both R_χ and R_T are constants on $S_{\lambda\theta}$. Thus we have $\{M, D\}|S_{\lambda\theta}$ is constant.

THEOREM 8.12. *The operator D has G spectral symmetry.*

PROOF. This proceeds as in the proof of Theorem 7.8. We note that, since $S_{\lambda\theta} \subset V_\lambda$, $S_{\lambda\theta}$ is a subspace of an eigenspace of $D^2 + R_\chi + R_T$. Thus $D^2|S_{\lambda\theta}$ is constant. By the linear algebra used in the proof of Theorem 7.8 $\operatorname{tr} D|S_{\lambda\theta} = 0$. Thus P has two eigenvalues μ and $-\mu$ on $S_{\lambda\theta}$ and these have equal multiplicity.

9. **The case of an abelian group action.** Let T^ℓ (with $\ell \geq 2$) act on N with discrete isotropy subgroups. Let $\{E_1, \ldots, E_\ell\}$ be an orthonormal basis for the Lie algebra $\mathfrak{t}$ of T^ℓ; otherwise we keep the notation of Section 8. With the exception of the proofs of Lemma 8.10 and Theorem 8.12 the calculations of Section 8 still hold. However, some of these hold in a trivial fashion since we have

$$M = R_\chi = R_T = 0. \tag{9.1}$$

Let $E = E_1, \ldots, E_{\ell-1}$ and $F = E_\ell$. Then we have the following calculation.

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LEMMA 9.1. *i.* $\{\omega_H E, D_H\} = 0$
ii. $\{\omega_H E, Q_G\} = 2\mathcal{L}_F \otimes 1 \otimes 1$
iii. $\{\omega_H E, T\} = 2\big(1 \otimes 1 \otimes \pi(F)\big)$.

PROOF. This follows as in [5].

COROLLARY 9.2. $\{\omega_H E, D\} = 2\mathcal{L}_F \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \pi(F)$.

PROOF. We calculate:

$$\begin{aligned} [D^2, \omega_H E] &= [D, \{D, \omega_H E\}] \\ &= 2[D, L_F \otimes 1 \otimes 1] + [D, 1 \otimes 1 \otimes \pi(F)] \\ &= 0, \end{aligned} \tag{9.2}$$

since D is invariant and T^ℓ is abelian.

Now we decompose $L^2(S \otimes E) = \sum V_\lambda$ into eigenspaces of D^2 and further decompose V_λ under the action of T^ℓ:

$$V_\lambda = \sum S_{\theta\lambda}. \tag{9.3}$$

Now both $\mathcal{L}_F$ and $\pi(F)$ act as constants on any irreducible representation space of T^ℓ, remembering that since T^ℓ is abelian such spaces are one dimensional over $\mathbb{C}$. Thus we can proceed as in [5] and conclude the following result

LEMMA 9.4. *i.* $\{\omega_H E, D\}|S_{\lambda\theta}$ *is constant.*
ii. $\operatorname{tr} \omega_H E|S_{\lambda\theta} = 0$.

As a consequence of this we have the main result of this section.

THEOREM 9.5. *On $S_{\lambda\theta}, D$ has two eigenvalues μ and $-\mu$ which have the same multiplicity.*

PROOF. This is similar to the result in [5]. We note that the proof proceeds in the same way as that of Theorem 7.8 and $\omega_H E$ in place of M.

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