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Reducibility of Generalized Principal Series

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Abstract. In this paper we describe reducibility of non-unitary generalized principal series for classical p-adic groups in terms of the classification of discrete series due to Mœglin and Tadić.

Introduction

In this paper we describe reducibility of non-unitary generalized principal series for classical p-adic groups in terms of the classification of discrete series due to Mœglin and Tadić [Mœ, MT]. Their results, and consequently ours, are complete only assuming that discrete series have generic supercuspidal support, thanks to the work of Shahidi [Sh1] on rank-one supercuspidal reducibilities.

Even then, the structure of discrete series with generic supercuspidal support is complicated and many of them are not generic themselves. Thus, our paper generalizes known results (see [J, M1, M2, T1], for example) and the main application of our results is to the determination of the unitary duals of classical p-adic groups.

To describe our results, we introduce some notation. Let G_n be a symplectic or (full) orthogonal group having split rank n. Let $\sigma \in \operatorname{Irr} G_n$ be a discrete series. We write $(\mathrm{Jord}, \sigma', \epsilon)$ for the admissible triple associated to σ by Mœglin (see [Mœ] or Section 1 here). Let $\delta \in \operatorname{Irr} GL(M_\delta, F)$ (this defines M_δ) be an essentially square integrable representation. According to [Ze], δ is attached to a segment. We may (and will) write this segment as follows, $[\nu^{-l_1}\rho, \nu^{l_2}\rho]$, $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$, with $\rho \in \operatorname{Irr} GL(m_\rho, F)$ (this defines m_ρ) unitary.

In this paper we describe the reducibility of the following induced representation (see [T1] for notation):

$$\delta \rtimes \sigma.$$

Since the reducibility of unitary generalized principal series is an integral part of the classification of discrete series [Mœ, MT] we consider only non-unitary generalized principal series. Thus, we may assume

$$(*) \qquad l_2 - l_1 > 0,$$

since $\delta \rtimes \sigma$ and $\widetilde{\delta} \rtimes \sigma$ have the same composition factors.

To describe the reducibility, we write Jord_ρ for the set of all $2a + 1 \in (1/2)\mathbb{Z}$ such that $(2a + 1, \rho) \in \mathrm{Jord}$.

The reducibility can be described in two steps. First, we have some simple reducibility criteria that were established in the preliminary section of [M4] (see Section 2 in that paper or Theorem 2.1 in this paper). We should point out that many

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of them were known earlier. (See for example [T1] and references therein.) We now recall these simple criteria.

- Assume that $\rho \not\cong \widetilde{\rho}$ or $2l_1 + 1 \notin \mathbb{Z}$, then $\delta \rtimes \sigma$ is irreducible.
- If $\mathrm{Jord}_\rho \neq \varnothing$ but $2l_1 + 1 - a \notin 2\mathbb{Z}$, $a \in \mathrm{Jord}_\rho$, then $\delta \rtimes \sigma$ is irreducible.
- Assume $\mathrm{Jord}_\rho = \varnothing$. Then $\delta \rtimes \sigma$ is reducible if and only if $l_1 \geq -1/2$, $2l_1 + 1 \in \mathbb{Z}$, and $2l_1 + 1$ is even if and only if $\nu^{1/2}\rho \rtimes \sigma'$ is reducible. (Note that the other possibility is $\rho \rtimes \sigma'$ is reducible.)

This enables us to make the following assumption:

$$(**) \qquad \begin{cases} \mathrm{Jord}_\rho \neq \varnothing \\ 2l_1 + 1 - a \in 2\mathbb{Z}, \quad \forall a \in \mathrm{Jord}_\rho \,. \end{cases}$$

We are now ready to describe all other reducibility points for the remaining cases. We assume that $(*)$ and $(**)$ hold and we consider four cases (see Theorems 4.1, 5.1, and 6.1). (All unexplained notation can be found in Section 1.)

- Assume $l_1 \geq 0$ and $]2l_1 + 1,\ 2l_2 + 1[\, \cap \mathrm{Jord}_\rho \neq \varnothing$. Then $\delta \rtimes \sigma$ is irreducible if and only if $2l_1 + 1, 2l_2 + 1 \in \mathrm{Jord}_\rho$ and there exists an alternated triple dominated by $(\mathrm{Jord}, \sigma', \epsilon)$ and contains all $(2a + 1, \rho)$, $2a + 1 \in \mathrm{Jord}_\rho \cap [2l_1 + 1, 2l_2 + 1]$.
- Assume $l_1 \geq 0$ and $]2l_1 + 1, 2l_2 + 1[\, \cap \mathrm{Jord}_\rho = \varnothing$. Then $\delta \rtimes \sigma$ is irreducible if and only if $2l_1 + 1, 2l_2 + 1 \in \mathrm{Jord}_\rho$ and $\epsilon(2l_2 + 1, \rho)\epsilon(2l_1 + 1, \rho)^{-1} = -1$.
- Assume $l_1 \leq -1/2$ and $[-2l_1 - 1, 2l_2 + 1[\, \cap \mathrm{Jord}_\rho = \varnothing$. Then $\delta \rtimes \sigma$ is irreducible unless $l_1 = -1/2$, and if $2l_2 + 1 \in \mathrm{Jord}_\rho$, then $\epsilon(2l_2 + 1, \rho) = 1$.
- Assume $l_1 \leq -1/2$ and $[-2l_1 - 1,\ 2l_2 + 1[\, \cap \mathrm{Jord}_\rho \neq \varnothing$. Then $\delta \rtimes \sigma$ is reducible unless there exists an alternated triple $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$ dominated by $(\mathrm{Jord}, \sigma', \epsilon)$ and contains all $(2a + 1, \rho)$, $2a + 1 \in [-2l_1 - 1,\ 2l_2 + 1[\, \cap \mathrm{Jord}_\rho$ and $(2l_2 + 1, \rho)$. Moreover, if such an alternated triple exists, the induced representation $\delta \rtimes \sigma$ is reducible if and only if $l_1 = -1/2$ and $\epsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = 1$.

This completes our description of the reducibility of generalized principal series. The actual determination of the composition series is a more complicated problem and it will be considered elsewhere. Our approach to the determination of the reducibility of $\delta \rtimes \sigma$ is straightforward. For each reducibility point we construct an irreducible subquotient non-isomorphic to the Langlands quotient of $\delta \rtimes \sigma$. This approach is used in many cases. (See the proofs of Theorems 3.1, 4.1, and 5.1). Theorem 3.1 shows that a composition series of $\delta \rtimes \sigma$ can be arbitrarily large, as opposed to the case of general linear groups [Ze]. Reducibilities of the type described in Theorem 3.1 were also known to Tadić, and following him we call them the independent reducibilities. The general case (see the proof of Theorem 4.1 and the proof of Theorem 6.1) constructs many more reducibilities and irreducible subquotients using the inductive construction of discrete series given in [MT].

The results of this paper rely on our previous work [M4], but not on the complete determination of composition series of $\delta \rtimes \sigma$.

Many particular results were established earlier. We refer to [T1] and references therein for the results before the classification of discrete series [Mœ, MT]. We also should point out that [M1, M2] contain, among other things, the reducibility of $\delta \rtimes \sigma$ for generic discrete series representations σ. They were done without reference to any classification of generic discrete series.

We should point out that Tadić, in an unpublished work [T2], worked out completely the case $l_2 = -l_1 > 0$, that is, the reducibility of $\nu^{l_2}\rho \rtimes \sigma$, also using Jacquet modules but in a different way. (See also [MT, Lemma 5.3].)

1 Preliminaries

Let F be a nonarchimedean field of characteristic different from 2. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively. If $x_1, x_2 \in \mathbb{R}$, we denote by $[x_1, x_2]$ (resp., $]x_1, x_2[$) the set of all $x \in \mathbb{R}$ such that $x_1 \leq x \leq x_2$ (resp., $x_1 < x < x_2$). Similarly, we define $]x_1, x_2]$ and $[x_1, x_2[$.

Now, we describe the groups that we consider. (See [MVW] for more details.) We look at the usual towers of orthogonal or symplectic groups $G_n = G(V_n)$ that are groups of isometries of F-spaces $(V_n, (\,\cdot\,,\,\cdot\,))$, $n \geq 0$, where the form $(\,\cdot\,,\,\cdot\,)$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise. We fix a set of standard parabolic subgroups in the usual way.

We now fix basic notation from the representation theory of general linear groups and use freely the well-known results of [Ze] throughout the paper. In particular, we write ν for the character obtained by the composition of the determinant character and (normalized as usual) absolute value of F.

If $\rho \in \operatorname{Irr} GL(M_\rho, F)$ is a supercuspidal representation and $k \in \mathbb{Z}_{\geq 0}$, then we define a segment $[\rho, \nu^k\rho]$ as the set $\{\rho, \nu\rho, \dots, \nu^k\rho\}$. This segment has associated to it a unique essentially square integrable representation $\delta([\rho, \nu^k\rho])$ given as the unique irreducible subrepresentation of $\nu^k\rho \times \cdots \times \nu\rho \times \rho$.

Now, we briefly describe the classification of discrete series in $\operatorname{Irr}' = \bigcup_{n\geq 1} \operatorname{Irr} G_n$. This has been done in [Mœ, MT] under some assumptions on rank-one reducibilities of the representation induced from supercuspidal. (See [MT] for the precise statement.)

We start the discussion of discrete series by recalling [Mœ] the definition of two invariants attached to a discrete series $\sigma \in \operatorname{Irr}'$. First, a partial supercuspidal support of $\sigma_{\text{cusp}} \in \operatorname{Irr}'$ is a supercuspidal representation such that there exists an irreducible representation $\pi \in GL(M_\pi, F)$ (this defines M_π) such that $\sigma \hookrightarrow \pi \rtimes \sigma_{\text{cusp}}$. This property determines $\sigma_{\text{cusp}} \in \operatorname{Irr}'$ uniquely.

Next, $\operatorname{Jord}(\sigma)$ is defined as a set of all pairs (a, ρ) ($\rho \cong \widetilde{\rho}$ is a supercuspidal representation of some $GL(m_\rho, F)$, $a > 0$ is integer) such that (a) and (b) hold:

(a) a is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi [Sh1, Sh2], where $r = \wedge^2\mathbb{C}^{m_\rho}$ is the exterior square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $GL(m_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group and $r = \operatorname{Sym}^2 \mathbb{C}^{m_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $GL(m_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group.

(b) The induced representation

$$\delta([\nu^{-(a-1)/2}\rho,\ \nu^{(a-1)/2}\rho]) \rtimes \sigma$$

is irreducible.

The main point of the classification is that discrete series are in one-to-one correspondence with admissible triples (*cf.* [Mœ]), and our results are also formulated in terms of admissible triples. So, we start recalling the definition of an admissible triple. This will be given in several steps.

First, we look at the collection Trip of all triples $(\mathrm{Jord}, \sigma', \epsilon)$, where

- $\sigma' \in \mathrm{Irr}'$ is a supercuspidal representation.
- Jord is a finite set (perhaps empty) of pairs (a, ρ) ($\rho \cong \widetilde{\rho}$ is supercuspidal of some $GL(M_\rho, F)$, $a > 0$ is integer) such that a is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$ (see (a) the above). We will also recall some notation from [MT]. We write $\mathrm{Jord}_\rho = \{a\,;\,(a, \rho) \in \mathrm{Jord}\}$, and for $a \in \mathrm{Jord}_\rho$ we write a_- for the largest element of Jord_ρ that is strictly less than a (if one exists).
- ϵ is a function defined on a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ into $\{\pm 1\}$ as follows. First, if $(a, \rho) \in \mathrm{Jord}$, then $\epsilon(a, \rho)$ is not defined if and only if a is odd and $(a', \rho) \in \mathrm{Jord}(\sigma')$ for some positive integer a'. Next, ϵ is defined on a pair $(a, \rho), (a', \rho') \in \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $a \neq a'$. This ends the definition of the domain of the definition of ϵ. The following compatibility conditions must hold for different $a, a', a'' \in \mathrm{Jord}_\rho$:
  - (i) If $\epsilon(a, \rho)$ is defined (hence $\epsilon(a', \rho)$ is also defined), then the value of ϵ on (a, ρ) and (a', ρ) is $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$. If $\epsilon(a, \rho)$ is not defined, then the value of ϵ on the pair (a, ρ) and (a', ρ) we shall, after [MT], denote also (formally) by $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$.
  - (ii) $\epsilon(a, \rho)\epsilon(a'', \rho)^{-1} = \big(\epsilon(a, \rho)\epsilon(a', \rho)^{-1}\big) \cdot \big(\epsilon(a', \rho)\epsilon(a'', \rho)^{-1}\big)$.
  - (iii) $\epsilon(a, \rho)\epsilon(a', \rho)^{-1} = \epsilon(a', \rho)\epsilon(a, \rho)^{-1}$.

Let $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ and $(a, \rho) \in \mathrm{Jord}$, such that a_- is defined, and

$$\epsilon(a, \rho) \cdot \epsilon(a_-, \rho)^{-1} = 1.$$

Now, it is easy to check the following: if we put $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(a, \rho), (a_-, \rho)\}$, and consider the restriction ϵ' of ϵ to $\mathrm{Jord}' \cup (\mathrm{Jord}' \times \mathrm{Jord}')$, then $(\mathrm{Jord}', \sigma', \epsilon') \in \mathrm{Trip}$. We say that the triple $(\mathrm{Jord}', \sigma', \epsilon')$ is subordinated to the triple $(\mathrm{Jord}, \sigma', \epsilon)$.

We say that $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ is an admissible triple of alternated type if $\epsilon(a, \rho) \cdot \epsilon(a_-, \rho)^{-1} = -1$ whenever a_- is defined and there is an increasing bijection $\phi_\rho\colon \mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where

$$\mathrm{Jord}'_\rho(\sigma') = \begin{cases} \mathrm{Jord}_\rho(\sigma') \cup \{0\} & \text{if } a \text{ is even and } \epsilon(\min \mathrm{Jord}_\rho, \rho) = 1; \\ \mathrm{Jord}_\rho(\sigma') & \text{otherwise.} \end{cases}$$

Here $\mathrm{Jord}_\rho(\sigma')$ is the set of all positive integers a such that $(\rho, a) \in \mathrm{Jord}(\sigma')$. We write $\mathrm{Trip}_{\mathrm{alt}}$ for the set of all triples in Trip that have alternated type.

We say that the triple $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ dominates the triple $(\mathrm{Jord}'', \sigma', \epsilon'') \in \mathrm{Trip}$ if we can find a sequence of triples $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, $1 \leq i \leq k$, such that

- $(\mathrm{Jord}, \sigma', \epsilon) = (\mathrm{Jord}_1, \sigma', \epsilon_1)$.
- $(\mathrm{Jord}_{i+1}, \sigma', \epsilon_{i+1})$, is subordinated to $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, for each i, $1 \le i \le k-1$.
- $(\mathrm{Jord}'', \sigma', \epsilon'') = (\mathrm{Jord}_k, \sigma', \epsilon_k)$.

Finally, we come to the definition of an admissible triple. We say that

$$(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$$

is an admissible triple if it dominates some triple of alternated type.

We write $\mathrm{Trip}_{\mathrm{adm}}$ for the set of all triples in Trip that are admissible. Obviously, we have

$$\mathrm{Trip}_{\mathrm{alt}} \subseteq \mathrm{Trip}_{\mathrm{adm}} \subseteq \mathrm{Trip}\,.$$

Now, the classification of discrete series [MT, Mœ] can be described as follows:

Theorem 1.1 *There exists a one-to-one correspondence between the set of all discrete series $\sigma \in \mathrm{Irr}'$ and the set of all triples $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{\mathrm{adm}}$ denoted by*

$$\sigma = \sigma_{(\mathrm{Jord}, \sigma', \epsilon)}$$

such that the following hold:

(i) $\mathrm{Jord}(\sigma) =$ *Jord and* $\sigma_{\mathrm{cusp}} = \sigma'$.

(ii) *Let* $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{\mathrm{alt}}$. *Then σ can be described explicitly as follows: For each ρ such that $\mathrm{Jord}_\rho \neq \emptyset$, we write the elements of Jord_ρ in increasing order $a_1^\rho < a_2^\rho < \cdots < a_{k_\rho}^\rho$. Now, σ is the unique irreducible subrepresentation of*

$$\times_\rho \times_{i=1}^{k_\rho} \delta([\nu^{(\phi_\rho(a_i^\rho)+1)/2}\rho, \nu^{(a_i^\rho-1)/2}\rho]) \rtimes \sigma'.$$

(iii) *Let* $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{\mathrm{adm}}$ *and* $(2b+1, \rho) \in \mathrm{Jord}$, *such that* $2b_- + 1 := (2b+1)_-$ *is defined, and* $\epsilon(2b+1, \rho) \cdot \epsilon(2b_- + 1, \rho)^{-1} = 1$. *We put* $\mathrm{Jord}'' = \mathrm{Jord} \setminus \{(2b+1, \rho), (2b_- + 1, \rho)\}$, *and consider the restriction ϵ'' of ϵ to* Jord''. *Then* $(\mathrm{Jord}'', \sigma', \epsilon'') \in \mathrm{Trip}_{\mathrm{adm}}$, *and*

$$\sigma \hookrightarrow \delta([\nu^{-b_-}\rho, \nu^b\rho]) \rtimes \sigma'', \tag{1.1}$$

where $\sigma'' = \sigma_{(\mathrm{Jord}'', \sigma', \epsilon'')}$. Moreover, the induced representation

$$\delta([\nu^{-(b_- -1)/2}\rho, \nu^{(b_- -1)/2}\rho]) \rtimes \sigma''$$

is a direct sum of two non-equivalent tempered representations $\tau_\pm$, and there exists a unique $\tau \in \{\tau_-, \tau_+\}$ such that

$$\sigma \hookrightarrow \delta([\nu^{(b_- +1)/2}\rho, \nu^{(b-1)/2}\rho]) \rtimes \tau. \tag{1.2}$$

Our main tool for computing composition series is Tadić's theory of Jacquet modules. We end this section by recalling his basic result.

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Let $R(G_n)$ be the Grothendieck group of admissible representations of finite length. Put

$$R(G) = \oplus_{n\geq 0} R(G_n).$$

We write $\geq$ or $\leq$ for the natural order on $R(G)$. In more detail, $\pi_1 \leq \pi_2$, $\pi_1, \pi_2 \in R(G)$, if and only if $\pi_2 - \pi_1$ is a linear combination of irreducible representations with non-negative coefficients.

We also define

$$R(GL) = \oplus_{n\geq 0} R(GL(n, F)).$$

Let $\sigma \in \operatorname{Irr} G_n$. Then for each standard proper maximal parabolic subgroup, *cf.* [MVW], P_j with Levi factor $GL(j, F) \times G_{n-j}$, $1 \leq j \leq n$, we can identify $R_{P_j}(\sigma)$ with its semisimplification in $R(GL(j, F)) \otimes R(G_{n-j})$. Thus, we can consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{j=1}^{n} R_{P_j}(\sigma) \in R(GL) \otimes R(G).$$

Now, the basic result of Tadić is the following theorem (see [MT] and references there):

Theorem 1.2 *Let $\sigma \in \operatorname{Irr} G_n$. We decompose in $R(G)$ into irreducible constituents (with repetitions possible):*

$$\mu^*(\sigma) = \sum_{\delta', \sigma_1} \delta' \otimes \sigma_1.$$

Assume that $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 + 1 \in \mathbb{Z}_{>0}$, and $\rho \in \operatorname{Irr} GL(m_\rho, F)$ (this defines m_ρ) is a supercuspidal representation. Then we have

$$\begin{aligned} (1.3) \qquad \mu^*\left(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\right) = \sum_{\delta', \sigma_1} \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} & \delta([\nu^{i-l_2}\widetilde{\rho}, \nu^{l_1}\widetilde{\rho}]) \\ & \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \\ & \times \delta' \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1, \end{aligned}$$

(We omit $\delta[\nu^\alpha\rho, \nu^\beta\rho])$ if $\alpha > \beta$.)

2 Some Simple Reductions

In this section we fix the notation that we use through the paper. We assume that σ is a discrete series representation attached to the triple $(\mathrm{Jord}, \sigma', \epsilon)$, and

$$\delta \in \operatorname{Irr} GL(M_\delta, F)$$

is an essentially square integrable representation. We study reducibility and composition series for $\delta \rtimes \sigma$.

By [Ze], δ is attached to a segment. We may (and will) write this segment as follows:

$$[\nu^{-l_1}\rho, \nu^{l_2}\rho], l_1, l_2 \in \mathbb{R}, l_1 + l_2 \in \mathbb{Z}_{\geq 0}, \rho \in \mathrm{Irr}\ GL(m_\rho, F) \text{ unitary.} \tag{2.1}$$

Next, since $\delta \rtimes \sigma = \widetilde{\delta} \rtimes \sigma$ in $R(G)$, we also assume

$$l_2 - l_1 > 0. \tag{2.2}$$

In this way $\delta \rtimes \sigma$ becomes a standard representation, and we denote by $\mathrm{Lang}(\delta \rtimes \sigma)$ its Langlands quotient.

In the remainder of the paper we assume that the following conditions hold:

$$\begin{cases} \mathrm{Jord}_\rho \neq \varnothing \\ 2l_1 + 1 - 2a \in \mathbb{Z}, \quad \forall a \in \mathrm{Jord}_\rho, \end{cases} \tag{2.3}$$

since a simple Jacquet module computation in [M4] established the following result:

Theorem 2.1

(i) *If $\rho \not\cong \widetilde{\rho}$ or $2l_1 + 1 \notin \mathbb{Z}$, then $\delta \rtimes \sigma$ is irreducible.*

(ii) *If $\mathrm{Jord}_\rho \neq \varnothing$ but $2l + 1 - 2a \notin 2\mathbb{Z}$, $a \in \mathrm{Jord}_\rho$, then $\delta \rtimes \sigma$ is irreducible.*

(iii) *Assume $\mathrm{Jord}_\rho = \varnothing$. Then $\delta \rtimes \sigma$ is reducible if and only if $l_1 \geq -1/2$, and $2l_1 + 1$ is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$.*

(iv) *Assume that (2.3) holds, $\mathrm{Jord}_\rho \cap [2l_1 + 1, 2l_2 + 1] = \varnothing$, and $l_1 \geq 0$. Then in the appropriate Grothendieck group we have an expansion of the form*

$$\delta \rtimes \sigma = \sigma_1 + \sigma_2 + \mathrm{Lang}(\delta \rtimes \sigma),$$

where σ_1 and σ_2 are discrete series obtained from σ by extending the triple of σ in the usual way, cf. [MT].

Also, a simple computation of Jacquet modules established in [M4, §2] shows the following:

Lemma 2.1 *Assume that π is a tempered (but not square-integrable) irreducible subquotient of $\delta \rtimes \sigma$. Then one of the following must hold:*

(a) *$\pi \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$, where σ_1 is tempered representation satisfying*

$$\mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1.$$

Moreover, unless $l_1 \geq 0$ and $2l_1 + 1 \in \mathrm{Jord}_\rho$, we have σ_1 in the discrete series and

$$\mathrm{Jord}(\sigma_1) = \begin{cases} \mathrm{Jord}(\sigma) \setminus \{(2l_2+1, \rho)\} \cup \{(2l_1+1, \rho)\}, & l_1 \geq 0. \\ \mathrm{Jord}(\sigma) \setminus \{(2l_2+1, \rho)\}, & l_1 = -1/2, \\ \mathrm{Jord}(\sigma) \setminus \{(2l_2+1, \rho), (-2l_1-1, \rho)\}, & l_1 \leq -1. \end{cases}$$

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If $l_1 \geq 0$ and $2l_1+1 \in \mathrm{Jord}_\rho$, then σ_1 is a tempered representation, given as an irreducible subrepresentation of

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_2,$$

where σ_2 is a discrete series representation satisfying

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1,\rho),(2l_2+1,\rho)\}.$$

(b) *$l_1 \geq 0$, and $\pi \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1$, where σ_1 is a tempered irreducible subquotient of $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. Moreover, if $2l_2+1 \notin \mathrm{Jord}_\rho$, then σ_1 is a discrete series representation satisfying*

$$\mathrm{Jord}(\sigma_1) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1,\rho)\} \cup \{(2l_2+1,\rho)\}.$$

If $2l_2+1 \in \mathrm{Jord}_\rho$, then σ_1 is tempered representation, given as an irreducible subrepresentation of

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_2,$$

where σ_2 is a discrete series representation satisfying

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1,\rho),(2l_2+1,\rho)\}.$$

Lemma 2.2 *In order for $\delta \rtimes \sigma$ to have an irreducible non-tempered subquotient non-isomorphic to $\mathrm{Lang}(\delta \rtimes \sigma)$ it is necessary that (2.3) holds and there exists $a \in \mathrm{Jord}_\rho$ such that $l_1, -l_1-1 < (a-1)/2 < l_2$.*

3 Independent Reducibilities

In this section we prove some straightforward reducibility results.

Assume that $2b+1 \in \mathrm{Jord}_\rho$ is such that $2b_-+1 := (2b+1)_- \in \mathrm{Jord}_\rho$ is defined and $\epsilon(2b+1,\rho)\cdot\epsilon(2b_-+1,\rho)^{-1} = 1$. We define the discrete series σ'' as in Theorem 1.1(iii). In particular, we have

$$\sigma \hookrightarrow \delta([\nu^{-b_-}\rho, \nu^{b}\rho]) \rtimes \sigma''. \tag{3.1}$$

As can be easily seen from the [MT] results recalled in Section 1, the induced representation in (3.1) has one more discrete series subrepresentation that we temporarily denote by $\hat{\sigma}$.

To motivate and explain the results of this section, let us look at some intertwining operators.

$$\begin{aligned}
\delta \rtimes \sigma &\hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{b}\rho]) \rtimes \sigma'' \\
&\to \delta([\nu^{-b_-}\rho, \nu^{b}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma'' \\
&\to \delta([\nu^{-b_-}\rho, \nu^{b}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma'' \\
&\to \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{b}\rho]) \rtimes \sigma''
\end{aligned} \tag{3.2}$$

Observe that the second intertwining operator is not an isomorphism if and only if the segments $[\nu^{-b_-}\rho, \nu^b\rho]$ and $[\nu^{-l_1}\rho, \nu^{l_2}\rho]$ are linked [Ze]. This happens exactly when one of the following holds:

$$\begin{cases} l_1 < b_- < b < l_2,\ l_1 \geq 0, \\ b_- < l_1 < l_2 < b,\ l_1 \geq 0, \\ -l_1 - 1 \leq b < l_2,\ l_1 \leq -1/2. \end{cases}$$

The fourth intertwining operator is not an isomorphism if and only if the segments $[\nu^{-b_-}\rho, \nu^b\rho]$ and $[\nu^{-l_2}\rho, \nu^{l_1}\rho]$ are linked [Ze]. This happens exactly when the following holds:

$$\begin{cases} b_- < l_2,\ l_1 < b,\ l_1 \geq 0, \\ -l_1 - 1 \leq b_- < l_2,\ l_1 \leq -1/2. \end{cases}$$

Note that Theorem 2.1 shows that $\delta \rtimes \sigma$ reduces if $l_1 \geq 0$ and $b_- < l_1 < l_2 < b$ ((2.3) holds). The first result of this section is the following theorem.

Theorem 3.1 *Assume that* (2.3) *holds and there exists* $2b+1 \in \mathrm{Jord}_\rho$ *such that* $2b_- + 1 := (2b+1)_- \in \mathrm{Jord}_\rho$ *is defined,* $\epsilon(2b+1, \rho) \cdot \epsilon(2b_- + 1, \rho)^{-1} = 1$, *and*

$$\begin{cases} l_1 < b_- < b < l_2,\ l_1 \geq 0, \\ -l_1 - 1 \leq b_- < b < l_2,\ l_1 \leq -1/2. \end{cases}$$

Then the unique irreducible quotient of

$$\delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{b_-}\rho]) \rtimes \sigma'' \cong \delta([\nu^{-l_1}\rho, \nu^{b_-}\rho]) \times \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \rtimes \sigma''$$

is an irreducible subquotient of $\delta \rtimes \sigma$ *and it appears with multiplicity one in its composition series. In particular,* $\delta \rtimes \sigma$ *reduces.*

Proof Let us denote by L the irreducible quotient mentioned. The proof of the theorem consists of two parts. First, assuming that L appears twice in the composition series of the standard representation

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^b\rho]) \rtimes \sigma'', \tag{3.3}$$

we complete the proof of the theorem. Then we prove that claim.

First, the induced representations $\delta \rtimes \sigma$ and $\delta \rtimes \hat{\sigma}$ are subrepresentations of the induced representation in (3.3) (apply (3.1)). Now, it is enough to show that the multiplicity of the irreducible representation $\delta([\nu^{-l_2}\rho, \nu^b\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{l_1}\rho]) \otimes \sigma''$ is one in both $\mu^*(\delta \rtimes \sigma)$ and $\mu^*(\delta \rtimes \hat{\sigma})$, and two in

$$\mu^* \left(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^b\rho]) \rtimes \sigma'' \right). \tag{3.4}$$

We show just the last claim, since using the same method we can check that it appears in $\mu^*(\delta \rtimes \sigma)$ and $\mu^*(\delta \rtimes \hat{\sigma})$ at least once, and this will complete the proof.

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Now, we expand (3.4) using Theorem 1.2. Thus, we take indices $0 \le j \le i \le l_1 + l_2 + 1$, $0 \le j' \le i' \le b_- + b + 1$, and irreducible constituents $\delta' \otimes \sigma_1$ of $\mu^*(\sigma'')$, and we obtain

$$\begin{aligned}(3.5)\quad \delta([\nu^{-l_2}\rho, \nu^{b}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{l_1}\rho]) \le \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) &\times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \\ &\times \delta([\nu^{i'-b}\rho, \nu^{b_-}\rho]) \\ &\times \delta([\nu^{b+1-j'}\rho, \nu^{b}\rho]) \times \delta' \\ \sigma'' \le \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{b+1-i'}\rho, \nu^{b-j'}\rho]) &\rtimes \sigma_1.\end{aligned}$$

The first formula in (3.5) shows that δ' is non-degenerate. In particular, it is fully induced from the tensor product of essentially square-integrable representations [Ze]. Now, if $i > 0$, then the first formula in (3.5) shows that one of those essentially square-integrable representations must be attached to a segment of the form $[\nu^{-l_2}\rho, \nu^{k}\rho]$, for some $k < l_2$. Since $\mu^*(\sigma'') \ge \delta' \otimes \sigma_1$, we obtain $\sigma'' \hookrightarrow \nu^{k}\rho \times \cdots \times \nu^{-l_2}\rho \rtimes \sigma_1'$, for some irreducible representation σ_1'. This contradicts the square-integrability criterion for σ''. Thus, $i = 0$, and since $0 \le j \le i$, we obtain $j = 0$. Now, the first formula in (3.5) is

$$\begin{aligned}\delta([\nu^{-l_2}\rho, \nu^{b}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{l_1}\rho]) \le \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) &\times \delta([\nu^{i'-b}\rho, \nu^{b_-}\rho]) \\ &\times \delta([\nu^{b+1-j'}\rho, \nu^{b}\rho]) \times \delta'.\end{aligned}$$

This implies that only possible terms in a supercuspidal support of δ' are

$$\nu^{-b_-}\rho, \dots, \nu^{b}\rho$$

(no repetition of the terms).

Next, $j' > 0$, or otherwise one of the segments that determine δ' would end with $\nu^{b}\rho$ and this would imply $2b+1 \in \mathrm{Jord}_\rho(\sigma'')$, a contradiction. Now, if $j' = b_- + b + 1$, then since $j \le i \le b_- + b + 1$, we obtain $i' = b_- + b + 1$. Since [Ze] implies that $\delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{b}\rho])$ contains $\delta([\nu^{-l_2}\rho, \nu^{b}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^{l_1}\rho])$ with multiplicity one, we see that δ' is trivial and $\sigma'' = \sigma_1$ (using the second formula of (3.5)). This produces the desired term once.

If $j' < b_- + b + 1$, then $b - j' = b_-$, or one of the segments that determine δ' would be $[\nu^{b_-+1}\rho, \nu^{b-j'}\rho]$, and this would imply that $\mathrm{Jord}_\rho(\sigma_b) \cap \,]2b_- + 1, 2b + 1[$ is not empty. This is contradicts Theorem 1.1(ii). Thus, $j' = b - b_-$. If $i' - b \ne -b_-$, then δ' must be attached to the segment of the form $[\nu^{-b_-}\rho, \nu^{k}\rho]$, for some $k < b_-$. As above, this contradicts the square-integrability criterion for σ''. Thus, $i' = j' = b - b_-$. Now, δ' is trivial and $\sigma_1 = \sigma''$. This produces the desired term once.

Now, we prove that L appears twice in the composition series of the induced representation in (3.3).

The above considerations of Jacquet modules show that it appears at most twice. We have

$$
\begin{aligned}
(3.6)\quad & \delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\rtimes\sigma''\to\\
&\qquad \delta([\nu^{-b_-}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\rtimes\sigma''\to\\
&\qquad \delta([\nu^{-b_-}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\rtimes\sigma''\to\\
&\qquad\qquad \delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\rtimes\sigma''
\end{aligned}
$$

Now, applying [Ze], the third intertwining operator above has kernel isomorphic to

$$\delta([\nu^{-l_2}\rho,\nu^{b}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{l_1}\rho])\rtimes\sigma'',$$

and this representation has L as the unique irreducible subrepresentation (Langlands subrepresentation).

More delicate is to show that the kernel of the first intertwining operator contains L as an irreducible subquotient. First, applying [Ze], the first intertwining operator in (3.6) has kernel isomorphic to

$$\delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{b}\rho])\rtimes\sigma''. \tag{3.7}$$

We use the following intertwining operators:

$$
\begin{aligned}
(3.8)\quad & \delta([\nu^{-l_1}\rho,\nu^{b_-}\rho])\times\delta([\nu^{-b}\rho,\nu^{l_2}\rho])\rtimes\sigma''\hookrightarrow\\
& \delta([\nu^{-l_1}\rho,\nu^{b_-}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-b}\rho,\nu^{-b_--1}\rho])\rtimes\sigma''\cong\\
& \delta([\nu^{-l_1}\rho,\nu^{b_-}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{b_-+1}\rho,\nu^{b}\rho])\rtimes\sigma''\cong\\
& \delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{b_-}\rho])\times\delta([\nu^{b_-+1}\rho,\nu^{b}\rho])\rtimes\sigma''\to\\
&\qquad\qquad \delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{b}\rho])\rtimes\sigma''.
\end{aligned}
$$

The claims implicit in the above sequence of intertwining operators are obvious, except the first isomorphism. It follows from the fact that the standard representation $\delta([\nu^{b_-+1}\rho,\nu^{b}\rho])\rtimes\sigma''$ is irreducible. This is easy since Lemma 2.1 and Lemma 2.2 immediately imply that it has no non-discrete series subquotients. Finally, since $2b_-+1\notin\mathrm{Jord}_\rho(\sigma'')$, a consideration of the sets of Jordan blocks as in [MT, §8] shows that it also has no discrete series subquotient.

Also, we note that the last intertwining operator in (3.8) is an epimorphism. To complete the proof, it is enough to show that L is not in the kernel of the last intertwining operator in (3.8). To accomplish that it is enough to show that

$$\delta([\nu^{-l_2}\rho,\nu^{b}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{l_1}\rho])\otimes\sigma''$$

appears exactly once in

$$\mu^*\left(\delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{b_-}\rho])\times\delta([\nu^{b_-+1}\rho,\nu^{b}\rho])\rtimes\sigma''\right)$$

and in

$$\mu^*\left(\delta([\nu^{-b_-}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-l_1}\rho,\nu^{b}\rho])\rtimes\sigma''\right).$$

Since this analysis is entirely analogous to the one given in the first part of the proof we omit the details.

We prove that constructed copies of L differ. We write $\delta=\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])$ as before (see the beginning of Section 2). Now, the long-intertwining operator

$$\delta\rtimes\sigma\oplus\delta\rtimes\hat{\sigma}\to\widetilde{\delta}\rtimes\sigma\oplus\widetilde{\delta}\rtimes\hat{\sigma} \tag{3.9}$$

has image isomorphic to

$$\mathrm{Lang}(\delta\rtimes\sigma)\oplus\mathrm{Lang}(\delta\rtimes\hat{\sigma}). \tag{3.10}$$

Next, we can fix embeddings

$$\begin{aligned}\delta\rtimes\sigma\oplus\delta\rtimes\hat{\sigma}&\hookrightarrow\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\rtimes\sigma''\\ \widetilde{\delta}\rtimes\sigma\oplus\widetilde{\delta}\rtimes\hat{\sigma}&\hookrightarrow\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\rtimes\sigma''\end{aligned} \tag{3.11}$$

such that the long-intertwining operator in (3.9) factors using (3.6). We write Π for the image of the first embedding in (3.11).

The computations of Jacquet modules given above show that Π must intersect the kernel of the first intertwining operator in (3.6) and that the intersection must contain L as an irreducible subquotient. Now, to complete the proof, it is enough to show that the image of Π under the composition of the first and the second intertwining operator in (3.6) intersects the kernel of the third (which has L as the unique irreducible subrepresentation). If not, then since the image of Π under the composition of all three intertwining operators in (3.6) is isomorphic to (3.10), we see that

$$\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\rtimes\sigma'' \tag{3.12}$$

has at least three different irreducible subrepresentations. Thus, by Frobenius reciprocity,

$$\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\otimes\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\otimes\sigma'' \tag{3.13}$$

appears at least three times in appropriate Jacquet module of (3.12). We show that this is not the case combining Theorem 1.2 and transitivity of Jacquet modules. First, we express

$$\mu^*\left(\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\rtimes\sigma''\right)$$

using Theorem 1.2, and arguing as before we can easily see that only the two terms

$$\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\otimes\sigma''$$

$$\delta([\nu^{-b_-}\rho,\nu^{b_-}\rho])\times\delta([\nu^{b_-+1}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])\otimes\sigma''$$

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can have (3.13) in appropriate Jacquet modules. Finally, the Jacquet modules of

$$\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])$$
$$\delta([\nu^{-b_-}\rho,\nu^{b_-}\rho])\times\delta([\nu^{b_-+1}\rho,\nu^{b}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])$$

can be computed easily and explicitly, *cf.* [Ze], showing that each of them contains $\delta([\nu^{-b_-}\rho,\nu^{b}\rho])\otimes\delta([\nu^{-l_2}\rho,\nu^{l_1}\rho])$ exactly once. ∎

In the remainder of this section we assume that $l_1 \geq 0$, and we analyze the other cases related to $b_- < l_2$ and $l_1 < b$.

Theorem 3.2 *Assume that* (2.3) *holds,* $l_1 \geq 0$*, and there exists* $2b+1 \in \mathrm{Jord}_\rho$ *such that* $2b_- + 1 := (2b+1)_- \in \mathrm{Jord}_\rho$ *is defined and* $\epsilon(2b+1,\rho)\cdot\epsilon(2b_-+1,\rho)^{-1} = 1$*. Let* σ'' *be defined by Theorem* 1.1(iii)*. Then we have the following:*

(i) *If* $b_- \leq l_1 < b < l_2$*, then the Langlands quotient* $\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^{l_2}\rho])\rtimes\tau)$ *is an irreducible subquotient of* $\delta\rtimes\sigma$ *for a unique irreducible subrepresentation* $\tau\hookrightarrow\delta([\nu^{-b_-}\rho,\nu^{l_1}\rho])\rtimes\sigma''$*, and it appears with multiplicity one in its composition series. Note that* τ *is in the discrete series if and only* $b_- < l_1$*, otherwise* τ *is tempered. Finally,* $\sigma\hookrightarrow\delta([\nu^{l_1+1}\rho,\nu^{b}\rho])\rtimes\tau$*.*

(ii) *If* $l_1 < b_- < l_2 \leq b$*, then the Langlands quotient* $\mathrm{Lang}(\delta([\nu^{-l_1}\rho,\nu^{b_-}\rho])\rtimes\tau)$ *is an irreducible subquotient of* $\delta\rtimes\sigma$ *for a unique irreducible subrepresentation* $\tau\hookrightarrow\delta([\nu^{-l_2}\rho,\nu^{b}\rho])\rtimes\sigma''$*, and it appears with multiplicity one in its composition series. Note that* τ *is in the discrete series if and only* $b > l_2$*, otherwise* τ *is tempered. Finally,* $\tau\hookrightarrow\delta([\nu^{b_-+1}\rho,\nu^{l_2}\rho])\rtimes\sigma$*.*

(iii) *If* $l_1 = b_- < l_2 \leq b$*, then there exists a unique irreducible subrepresentation* $\tau\hookrightarrow\delta([\nu^{-l_2}\rho,\nu^{b}\rho])\rtimes\sigma''$ *such that the induced representations* $\delta([\nu^{-b_-}\rho,\nu^{b_-}\rho])\rtimes\tau$ *and* $\delta\rtimes\sigma$ *have a common (tempered) irreducible subrepresentation. It appears with multiplicity one in the composition series of* $\delta\rtimes\sigma$*. Note that* τ *is in the discrete series if and only* $b > l_2$*, otherwise* τ *is tempered.*

(iv) *If* $b_- < l_1 < b = l_2$*, then there exists a unique irreducible subrepresentation* $\tau\hookrightarrow\delta([\nu^{-b_-}\rho,\nu^{l_1}\rho])\rtimes\sigma''$ *such that the induced representations* $\delta([\nu^{-b}\rho,\nu^{b}\rho])\rtimes\tau$ *and* $\delta\rtimes\sigma$ *have a common (tempered) irreducible subrepresentation. It appears with multiplicity one in the composition series of* $\delta\rtimes\sigma$*. Note that* τ *is in the discrete series.*

Proof The proofs of (i) and (ii) are similar to the proof of Theorem 3.1. We leave the straightforward verification to the reader.

The proofs of (iii) and (iv) are also similar to the proof of Theorem 3.1, but there are some differences. We sketch the proof for (iii); the proof of (iv) is similar. The kernel of the last intertwining operator in (3.2) is isomorphic to

$$\delta([\nu^{-l_2}\rho,\nu^{b}\rho])\times\delta([\nu^{-b_-}\rho,\nu^{b_-}\rho])\rtimes\sigma'' \cong \delta([\nu^{-b_-}\rho,\nu^{b_-}\rho])\times\delta([\nu^{-l_2}\rho,\nu^{b}\rho])\rtimes\sigma''. \tag{3.14}$$

By [MT], the induced representation $\delta([\nu^{-l_2}\rho, \nu^b\rho]) \rtimes \sigma''$ has exactly two irreducible subrepresentations. They are in the discrete series and non-isomorphic, say σ_i, $i = 1, 2$. Also, [MT] shows that $\delta([\nu^{-b_-}\rho, \nu^{b_-}\rho]) \rtimes \sigma_i$ is direct sum of two non-equivalent tempered irreducible subrepresentations. Thus, (3.14) has at least four irreducible subrepresentations.

Now, to complete the proof, it is enough to show that the multiplicity of the irreducible representation $\delta([\nu^{-b_-}\rho, \nu^{b_-}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^b\rho]) \otimes \sigma''$ is one in both $\mu^*(\delta \rtimes \sigma)$ and $\mu^*(\delta \rtimes \hat{\sigma})$, and four in

$$\mu^* \left(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-b_-}\rho, \nu^b\rho]) \rtimes \sigma''\right).$$

The last multiplicity can be calculated as in the proof of Theorem 3.1. The other two multiplicities are similar, but we need to use the fact that $\mu^*(\sigma) \geq \delta([\nu^{-b_-}\rho, \nu^b\rho]) \otimes \sigma_3$ for some irreducible representation σ_3 implies that $\sigma_3 \cong \sigma''$ [M3, Theorem 2.3], and similarly for $\hat{\sigma}$. ∎

4 General Case for $l_1 \geq 0$, I

In this section we determine the reducibility of the generalized principal series $\delta \rtimes \sigma$ in the case of $l_1 \geq 0$ and

$$]2l_1 + 1,\ 2l_2 + 1[\, \cap \operatorname{Jord}_\rho \neq \varnothing. \tag{4.1}$$

The main result of this section is the following theorem.

Theorem 4.1 *Assume $l_1 \geq 0$, (2.3), and (4.1) hold. Then $\delta \rtimes \sigma$ is irreducible if and only if there is an alternated triple dominated by that of σ containing all pairs $(2a+1, \rho)$, $2a + 1 \in\,]2l_1 + 1, 2l_2 + 1[\, \cap \operatorname{Jord}_\rho$, $(2l_1 + 1, \rho)$ and $(2l_2 + 1, \rho)$.*

The theorem will be proved in several steps. First, we prove a lemma that extends Theorem 3.1 and Theorem 3.2.

Lemma 4.1 *Assume $l_1 \geq 0$, (2.3), and (4.1) hold. Let us write $2a_0 + 1$ for the smallest element and $2b_0 + 1$ for the largest element in $]2l_1 + 1, 2l_2 + 1[\, \cap \operatorname{Jord}_\rho$, respectively. Then $\delta \rtimes \sigma$ reduces if one of the following holds:*

(i) *There exists $2a_0^- + 1 \in \operatorname{Jord}_\rho$ such that $a_0^- < a_0$, $\epsilon(2a_0+1, \rho)\epsilon(2a_0^- + 1, \rho)^{-1} = 1$, and $\left]2a_0^- + 1, 2a_0 + 1\right[\cap \operatorname{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2a_j^- + 1, 2a_j + 1\}$, $j = 1, \dots, k$ (this defines k), such that $a_j^- < a_j$, $\epsilon(2a_j + 1, \rho)\epsilon(2a_j^- + 1, \rho)^{-1} = 1$ and we can have neither $a_j^- < a_i^- < a_j < a_i$ nor $a_i^- < a_j^- < a_i < a_j$, for $i \neq j$.*

(ii) *There exists $2b_0^+ + 1 \in \operatorname{Jord}_\rho$ such that $b_0 < b_0^+$, $\epsilon(2b_0 + 1, \rho)\epsilon(2b_0^+ + 1, \rho)^{-1} = 1$, and $\left]2b_0 + 1, 2b_0^+ + 1\right[\cap \operatorname{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2b_j^- + 1, 2b_j + 1\}$, $j = 1, \dots, k$ (this defines k), such that $b_j^- < b_j$, $\epsilon(2b_j + 1, \rho)\epsilon(2b_j^- + 1, \rho)^{-1} = 1$ and we can have neither $b_j^- < b_i^- < b_j < b_i$ nor $b_i^- < b_j^- < b_i < b_j$, for $i \neq j$.*

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Proof We prove (i). The proof of (ii) is analogous. We change the indices of the pairs $\{2a_j^- + 1, 2a_j + 1\}$, $j = 1, \dots, k$, so that $a_1 > a_2 > \cdots > a_k$. Note that $a_1 \le l_1 < a_0$.

We also write σ_0 for the discrete series having the triple $(\mathrm{Jord}(\sigma_0), \sigma', \epsilon_{\sigma_0})$, where $\mathrm{Jord}(\sigma_0)$ is the subset of Jord obtained by removing all pairs $(2a_j+1, \rho)$, $(2a_j^- + 1, \rho)$, $1 \le j \le k$, and ϵ_{σ_0} is the restriction of ϵ to that set. Next, the classification of discrete series (see Theorem 1.1) shows that there exists a (unique) sequence of discrete series σ_j, $1 \le j \le k$, such that

$$\begin{cases} \sigma_j \hookrightarrow \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \rtimes \sigma_{j-1}, 1 \le j \le k \\ \sigma_k = \sigma. \end{cases} \tag{4.2}$$

Now, we use Theorem 3.2(i) to show that $\delta \rtimes \sigma_0$ is reducible and identify an irreducible subquotient. First, since $\left]2a_0^- + 1, 2a_0 + 1\right[\cap \mathrm{Jord}_\rho(\sigma_0) = \varnothing$ and

$$\epsilon(2a_0 + 1, \rho)\epsilon(2a_0^- + 1, \rho)^{-1} = 1,$$

there exists a discrete series $\sigma_0{}''$ obtained from σ_0 by removing $(2a_0^- + 1, \rho)$, and $(2a_0 + 1, \rho)$ from the triple of σ_0 and then restricting ϵ to that new set. We have

$$\sigma_0 \hookrightarrow \delta([\nu^{-a_0^-}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0{}''. \tag{4.3}$$

Then there exists a unique irreducible (discrete series) subrepresentation

$$\tau_0 \hookrightarrow \delta([\nu^{-a_0^-}\rho, \nu^{l_1}\rho]) \rtimes \sigma_0{}'', \tag{4.4}$$

such that the Langlands quotient $\mathrm{Lang}(\delta([\nu^{-a_0}\rho, \nu^{l_2}\rho]) \rtimes \tau_0)$ is a subquotient of $\delta \rtimes \sigma_0$, and it occurs with multiplicity one. In fact, $\delta([\nu^{-a_0}\rho, \nu^{l_2}\rho]) \otimes \tau_0$ occurs with multiplicity one in $\mu^*(\delta \rtimes \sigma_0)$, and $\sigma_0 \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \tau_0$.

To complete the proof, we prove the following claim.

Claim 1 *Under the above assumptions, there exist discrete series representations τ_j, $1 \le j \le k$, and a tempered representation τ_{k+1} such that the following hold:*

(1) $\tau_j \hookrightarrow \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \rtimes \tau_{j-1}$, $1 \le j \le k$.
(2) $\mathrm{Lang}(\delta([\nu^{-a_0}\rho, \nu^{l_2}\rho]) \rtimes \tau_j) \le \delta \rtimes \sigma_j$ *and* $\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \otimes \tau_j$ *appears with multiplicity one in* $\mu^*(\delta \rtimes \sigma_j)$, $j = 1, \dots, k$.
(3) $\delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \tau_j$, $j = 1, \dots, k$.

To prove Claim 1, we use induction. In fact we may consider $j = 0$ also, if we adjust the notation appropriately using (4.3) and (4.4) and the discussion immediately before the claim. Now, the proof of the base of induction and the proof of the inductive step are the same. Therefore, we prove the inductive step only. Thus, we assume that the claim holds for $j - 1$, $1 \le j \le k$, and we prove it for j. We start with the following claim.

Claim 2 *Assume that Claim 1 holds for $j - 1$, $1 \le j \le k$. Then we have the following:*

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(i) *$\mu^*(\sigma_{j-1})$ contains $\delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \otimes \tau_{j-1}$ with multiplicity one.*

(ii) *The irreducible representation $\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \otimes \tau_{j-1}$ appears with multiplicity two in $\mu^*(\delta \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \rtimes \sigma_{j-1})$.*

(iii) *The irreducible representation $\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \otimes \tau_{j-1}$ appears with multiplicity two in $\mu^*(\delta \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \tau_{j-1})$.*

(iv) *The irreducible representation $\delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \otimes \tau_{j-1}$ appears with multiplicity two in in $\mu^*(\delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \rtimes \tau_{j-1})$.*

Proof To prove (i), we observe that the assumption that Claim 1(2) holds for $j-1$, implies that $\mu^*(\delta \rtimes \sigma_{j-1})$ contains $\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \otimes \tau_{j-1}$ with multiplicity one. We unfold this using Theorem 1.2. Thus, there are indices $0 \le j' \le i' \le l_1 + l_2 + 1$, and an irreducible constituent $\delta' \otimes \sigma_1'$ of $\mu^*(\sigma_{j-1})$ such that

$$\begin{aligned} &\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \le \delta([\nu^{i'-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j'}\rho, \nu^{l_2}\rho]) \times \delta' \\ &\tau_{j-1} \le \delta([\nu^{l_2+1-i'}\rho, \nu^{l_2-j'}\rho]) \rtimes \sigma_1'. \end{aligned} \tag{4.5}$$

The first formula in (4.5) shows that δ' is non-degenerate. In particular, it is fully induced from the tensor product of essentially square-integrable representations [Ze]. Now, if $i' > 0$, then the first formula in (4.5) shows that one of those essentially square-integrable representations must be attached to the segment of the form $[\nu^{-l_2}\rho, \nu^k\rho]$, for some $k < l_2$. Since $\mu^*(\sigma_{j-1}) \ge \delta' \otimes \sigma_1'$, we obtain $\sigma_{j-1} \hookrightarrow \nu^k\rho \times \cdots \times \nu^{-l_2}\rho \rtimes \sigma''_1$, for some irreducible representation σ''_1. This contradicts the square-integrability criterion for σ_{j-1}. Thus, $i' = 0$, and since $0 \le j' \le i'$, we obtain $j' = 0$. We also obtain $\delta' = \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho])$ and $\sigma_1' = \tau_{j-1}$. Now, (i) is clear.

We prove (ii). Again we use Theorem 1.2. Thus, we take indices $0 \le j' \le i' \le l_1 + l_2 + 1$, $0 \le j'' \le i'' \le a_j^- + a_j + 1$, and an irreducible constituent $\delta' \otimes \sigma_1'$ of $\mu^*(\sigma_{j-1})$, and we obtain

$$\begin{aligned} &\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \\ &\qquad \le \delta([\nu^{i'-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j'}\rho, \nu^{l_2}\rho]) \\ &\qquad\quad \times \delta([\nu^{i''-a_j}\rho, \nu^{a_j^-}\rho]) \times \delta([\nu^{a_j+1-j''}\rho, \nu^{a_j}\rho]) \times \delta' \\ &\tau_{j-1} \le \delta([\nu^{l_2+1-i'}\rho, \nu^{l_2-j'}\rho]) \times \delta([\nu^{a_j+1-i''}\rho, \nu^{a_j-j''}\rho]) \rtimes \sigma_1'. \end{aligned} \tag{4.6}$$

As above, we conclude that $i' = 0$. This implies $j' = 0$. Since $a_j^- < a_j \le l_1 < a_0$, we conclude that δ' must contain $\nu^{l_1+1}, \dots, \nu^{a_0}\rho$ in its support. We note that δ_1 must be non-degenerate and thus [Ze] it must be isomorphic to the representation induced from the product of essentially square integrable representations attached to non-linked segments. We analyze several cases.

- $i'' = a_j^- + a_j + 1$, $j'' = 0$. Then $\delta' \cong \delta([\nu^{-a_j^-}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho])$ if $a_j < l_1$ (and this is a contradiction since then $2a_j + 1 \in \mathrm{Jord}_\rho(\sigma_{j-1})$), and $\delta' \cong \delta([\nu^{-a_j^-}\rho, \nu^{a_0}\rho])$ if $a_j = l_1$. In the last case we find that for some irreducible representation σ''_1 we have

$$\sigma_{j-1} \hookrightarrow \nu^{a_0}\rho \times \cdots \times \nu^{-a_j^-}\rho \rtimes \sigma''_1.$$

Now, since $\mathrm{Jord}_\rho(\sigma_{j-1}) \cap [2a_j^- + 1, 2a_0 + 1] = \{2a_0 + 1\}$, arguing as in [M4, Lemma 4.1], we conclude that

$$\sigma_{j-1} \hookrightarrow \delta([\nu^{-a_j^-}\rho, \nu^{l_1}\rho]) \rtimes \sigma_2', \tag{4.7}$$

or

$$\sigma_{j-1} \hookrightarrow \delta([\nu^{-a_j^-}\rho, \nu^{a_0}\rho]) \rtimes \sigma_2', \tag{4.8}$$

for some irreducible representation σ_2'. Equation (4.7) implies $2l_1+1 \in \mathrm{Jord}_\rho(\sigma_{j-1})$. This is a contradiction. If (4.8) holds, then let $a_0 > \alpha \geq a_j^-$ be the largest such that there exists an irreducible representation $\sigma_2{}''$

$$\sigma_{j-1} \hookrightarrow \delta([\nu^{-\alpha}\rho, \nu^{a_0}\rho]) \rtimes \sigma_2{}'', \tag{4.9}$$

Using [Mœ, Remark 3.2], $\sigma_2{}''$ is in the discrete series. Next, considering Jordan blocks and using (4.9), we see $2\alpha + 1 \in \mathrm{Jord}_\rho(\sigma_{j-1})$. This is impossible since $\mathrm{Jord}_\rho(\sigma_{j-1}) \cap [2a_j^- + 1, 2a_0 + 1] = \{2a_0 + 1\}$.

• $i'' = a_j^- + a_j + 1$ and $j'' > 0$. If $j'' < a_j^- + a_j + 1$, then

$$\delta' \cong \delta([\nu^{-a_j^-}\rho, \nu^{a_j - j''}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]).$$

This either contradicts the square-integrability criterion for σ_{j-1} (if $a_j - j'' < a_j^-$) or implies $\mathrm{Jord}_\rho(\sigma_{j-1}) \cap [2a_j^- + 1, 2a_j + 1] \neq \varnothing$ (if $a_j - j'' \geq a_j^-$) which is also a contradiction. Thus, $j'' = a_j^- + a_j + 1$. Now, the second formula in (4.6) holds using already proved (i). We have produced the desired term once.

• $i'' < a_j^- + a_j + 1$ and $j'' > 0$. An analysis similar to the previous case shows that $\delta' \cong \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho])$ and $i'' = j'' = a_j - a_j^-$. Now, the second formula in (4.6) holds using already proved (i). We have produced the desired term once.

• $i'' < a_j^- + a_j + 1$ and $j'' = 0$. We must have

$$\delta' \cong \begin{cases} \delta([\nu^{-a_j^-}\rho, \nu^{i''-a_j-1}\rho]) \times \delta([\nu^{a_j^-+1}\rho, \nu^{a_j}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]), & a_j < l_1 \\ \delta([\nu^{-a_j^-}\rho, \nu^{i''-a_j-1}\rho]) \times \delta([\nu^{a_j^-+1}\rho, \nu^{a_0}\rho]), & a_j = l_1. \end{cases} \tag{4.10}$$

If the first formula in (4.10) holds then $2a_j + 1 \in \mathrm{Jord}_\rho(\sigma_{j-1})$. This is a contradiction. If the second formula holds in (4.10), $i'' = 0$, or as in the proof for $i' = 0$ we would violate the square-integrability criterion for σ_{j-1}. Thus, $\delta' \cong \delta([\nu^{a_j^-+1}\rho, \nu^{a_0}\rho])$. The second formula in (4.6) shows $\sigma_1' \cong \tau_{j-1}$. This implies

$$\mu^*(\sigma_{j-1}) \geq \delta([\nu^{a_j^-+1}\rho, \nu^{a_0}\rho]) \otimes \tau_{j-1}.$$

This clearly violates (i) since $l_1 > a_j^-$. This completes proof of Claim 2(ii). (iii) and (iv) have similar proofs. We leave the details to the reader. ∎

Now, we complete the proof of the inductive step in Claim 1.

Let us write σ_j' for the other discrete series subrepresentation in the induced representation (4.2). We also write τ and τ' for the irreducible subrepresentations of (i). We have

$$(4.11)\quad \delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \rtimes \tau \oplus \delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \rtimes \tau' \hookrightarrow \delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \times \delta([\nu^{-b_j^-}\rho,\nu^{b_j}\rho]) \rtimes \tau_{j-1}.$$

Now, Frobenius reciprocity and Claim 2(iii) imply the following claim.

Claim 3 *The Langlands quotients*

$$\mathrm{Lang}(\delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \rtimes \tau) \textit{ and } \mathrm{Lang}(\delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \rtimes \tau')$$

are the only irreducible subquotients of $\delta \times \delta([\nu^{-a_j^-}\rho,\nu^{a_j}\rho]) \times \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \rtimes \tau_{j-1}$ *having* $\delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \times \delta([\nu^{-b_j^-}\rho,\nu^{b_j}\rho]) \otimes \tau_{j-1}$ *in the appropriate Jacquet modules.*

Next, we have

$$(4.12)\quad \sigma_j \oplus \sigma_j' \hookrightarrow \delta([\nu^{-a_j^-}\rho,\nu^{a_j}\rho]) \rtimes \sigma_{j-1} \hookrightarrow \delta([\nu^{-a_j^-}\rho,\nu^{a_j}\rho]) \times \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \rtimes \tau_{j-1} \cong \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho,\nu^{a_j}\rho]) \rtimes \tau_{j-1},$$

and

$$(4.13)\quad \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \rtimes \tau \oplus \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \rtimes \tau' \hookrightarrow \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \times \delta([\nu^{-a_j^-}\rho,\nu^{a_j}\rho]) \rtimes \tau_{j-1}.$$

Combining Claim 2(iv), (4.12), and (4.13), we may assume

$$(4.14)\quad \begin{aligned} \sigma_j &\hookrightarrow \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \rtimes \tau,\\ \sigma_j' &\hookrightarrow \delta([\nu^{l_1+1}\rho,\nu^{a_0}\rho]) \rtimes \tau'. \end{aligned}$$

We use (4.14) to define $\tau_j = \tau'$ to show that Claim 1(3) holds. Also, combining (4.14) with Theorem 1.2 and (4.13) implies that

$$(4.15)\quad \begin{aligned} \mu^*\left(\delta \rtimes \sigma_j\right) &\geq \delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \times \delta([\nu^{-b_j^-}\rho,\nu^{b_j}\rho]) \otimes \tau_{j-1},\\ \mu^*\left(\delta \rtimes \sigma_j'\right) &\geq \delta([\nu^{-l_2}\rho,\nu^{a_0}\rho]) \times \delta([\nu^{-b_j^-}\rho,\nu^{b_j}\rho]) \otimes \tau_{j-1}. \end{aligned}$$

Now, Claim 1(1) and (2) also hold. ■

The next combinatorial lemma generalizes Theorem 3.1 and Lemma 4.1.

Lemma 4.2 *Assume that* (2.3) *holds and that there is a pair* $2a'+1 < 2a+1$, $2a+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$ *or* $2a'+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$, $\epsilon(2a+1,\rho)\epsilon(2a'+1,\rho)^{-1} = 1$, *and* $]2a'+1, 2a+1[\cap \mathrm{Jord}_\rho$ *is either empty or can be divided into disjoint sets of pairs* $\{2a_j^- + 1, 2a_j + 1\}$, $j = 1, \dots, k$ *(this defines k), such that* $a_j^- < a_j$, $\epsilon(2a_j+1,\rho)\epsilon(2a_j^- + 1,\rho)^{-1} = 1$ *and we can have neither* $a_j^- < a_i^- < a_j < a_i$ *nor* $a_i^- < a_j^- < a_i < a_j$, *for* $i \neq j$. *Then* $\delta \rtimes \sigma$ *is reducible.*

Proof This follows from Theorem 3.1 and Lemma 4.1. First, if $2a+1, 2a'+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$, then we have two cases.

If $]2a+1, 2a'+1[\cap \mathrm{Jord}_\rho$ is empty, we are done by Theorem 3.1. If not, we can find a pair $\{2a_j^- + 1, 2a_j + 1\}$ such $]2a_j^- + 1, 2a_j + 1[\cap \mathrm{Jord}_\rho$ is empty and we are again done by Theorem 3.1.

Next, assume $2a'+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$ and $2a+1 \notin]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$. Then we take the largest say $2b+1$ in $]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$. By the assumption there must exist j such that $2b+1 = 2a_j+1$ or $2b+1 = 2a_j^- + 1$. If $2b+1 = 2a_j+1$, then $]2a_j^- + 1, 2a_j+1[\subseteq]2l_1+1, 2l_2+1[$. Thus, if $]2a_j^- + 1, 2a_j+1[\cap \mathrm{Jord}_\rho$ is empty, we are done by Theorem 3.1. If not, we can find i so that $]2a_i^- + 1, 2a_i+1[\subseteq]2a_j^- + 1, 2a_j+1[$, $]2a_i^- + 1, 2a_i+1[\cap \mathrm{Jord}_\rho$ is empty, and we are done by Theorem 3.1. If $2b+1 = 2a_j^- + 1$, we apply Lemma 4.1(ii).

We treat the cases

$$2a+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$$

and

$$2a'+1 \notin]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$$

similarly. ■

In view of combinatorics of admissible triples recalled in Section 1, Lemma 4.2 enables us to assume that there exists an alternated triple subordinated to that of σ that contains all $]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho \neq \emptyset$.

We leave to the reader that this assumption excludes the assumption of Lemma 4.2 that there is a pair $2a'+1 < 2a+1$, $2a+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$ or $2a'+1 \in]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho$, $\epsilon(2a+1,\rho)\epsilon(2a'+1,\rho)^{-1} = 1$, and $]2a'+1, 2a+1[\cap \mathrm{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2a_j^- + 1, 2a_j+1\}$, $j = 1, \dots, k$ (this defines k), such that $a_j^- < a_j$, $\epsilon(2a_j+1,\rho)\epsilon(2a_j^- + 1,\rho)^{-1} = 1$ and we can have neither $a_j^- < a_i^- < a_j < a_i$ nor $a_i^- < a_j^- < a_i < a_j$, for $i \neq j$.

It is also easy to verify that they are the only options that we have. The next lemma completes the proof of Theorem 4.1.

Lemma 4.3 *Assume that* (2.3) *holds and that there exists a triple of alternated type* $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$ *subordinated to that of* σ *that contains all* $]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho \neq \emptyset$. *Then* $\delta \rtimes \sigma$ *is irreducible if and only if* $2l_1+1, 2l_2+1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$.

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Proof We write σ_{alt} for a strongly positive discrete series attached to

$$(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}}).$$

We also write $2a_0 + 1$ (resp., $2b_0 + 1$) for the largest (resp., smallest) of

$$[2l_1 + 1, 2l_2 + 1] \cap (\mathrm{Jord}_{\mathrm{alt}})_\rho.$$

Then we have the following result, which follows from [M4, Theorem 4.1] and its proof:

Lemma 4.4

(i) *If $2l_1 + 1, 2l_2 + 1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$, then $\delta \rtimes \sigma_{\mathrm{alt}}$ is irreducible.*

(ii) *Assume $2l_2 + 1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$. This means $a_0 < l_2$, and we define a new strongly positive discrete series σ'_{alt}, replacing $2a_0 + 1$ with $2l_2 + 1$ in all relevant formulas that define the triple of σ_{alt}. Then $\sigma'_{\mathrm{alt}} \hookrightarrow \delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\mathrm{alt}}$. Finally,*

$$\mathrm{Lang}\left(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma'_{\mathrm{alt}}\right) \leq \delta \rtimes \sigma,$$

and $\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \otimes \sigma'_{\mathrm{alt}}$ appears in $\mu^*(\delta \rtimes \sigma_{\mathrm{alt}})$ with multiplicity one.

(iii) *Assume $2l_1 + 1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$. This means $b_0 > l_1$, and we define a new strongly positive discrete series σ''_{alt}, replacing $2b_0 + 1$ with $2l_1 + 1$ in all relevant formulas that define the triple of σ_{alt}. Then $\sigma_{\mathrm{alt}} \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{b_0}\rho]) \rtimes \sigma''_{\mathrm{alt}}$. Finally,*

$$\mathrm{Lang}\left(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma''_{\mathrm{alt}}\right) \leq \delta \rtimes \sigma,$$

and $\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \otimes \sigma''_{\mathrm{alt}}$ appears in $\mu^*(\delta \rtimes \sigma_{\mathrm{alt}})$ with multiplicity one.

Now, we are ready to begin the proof. First, by definition of admissible triple, $\mathrm{Jord} \setminus \mathrm{Jord}_{\mathrm{alt}}$ is either empty or can be divided into disjoint sets of pairs $(2a_j^- + 1, \rho_j)$, $(2a_j + 1, \rho_j)$, $j = 1, \dots, k$ (this defines k), such that $a_j^- < a_j$, $\epsilon(2a_j + 1, \rho_j)\epsilon(2a_j^- + 1, \rho_j)^{-1} = 1$ and we can have neither $a_j^- < a_i^- < a_j < a_i$ nor $a_i^- < a_j^- < a_i < a_j$, for $i \neq j$ and $\rho_i \cong \rho_j$. There exists a (unique) sequence of discrete series σ_j, $1 \leq j \leq k$, such that

$$\begin{aligned} &\sigma_j \hookrightarrow \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1}, \quad 1 \leq j \leq k \\ &\sigma_0 = \sigma_{\mathrm{alt}} \qquad \sigma_k = \sigma. \end{aligned} \tag{4.16}$$

Next, we consider the case $2l_2 + 1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$. Then, as in the proof of Lemma 4.1, the next claim that clearly completes the proof of Lemma 4.3 in that case.

Claim 4 *Under the above assumptions, there exist discrete series representations τ_j, $1 \leq j \leq k$ such that the following hold:*

(1) $\tau_j \hookrightarrow \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \tau_{j-1}$, $1 \leq j \leq k$, $\tau_0 = \sigma'_{\mathrm{alt}}$.

(2) $\mathrm{Lang}(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \tau_j) \leq \delta \rtimes \sigma_j$ *and* $\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \otimes \tau_j$ *appears with multiplicity one in* $\mu^*\left(\delta \rtimes \sigma_j\right)$, $j = 1, \dots, k$.
(3) $\tau_j \hookrightarrow \delta([\nu^{a_0}\rho, \nu^{l_2+1}\rho]) \rtimes \sigma_j$, $j = 1, \dots, k$.

Now, we consider the case $2l_1 + 1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$. Then, as in the proof of Lemma 4.1, the next claim that clearly completes the proof of Lemma 4.3 in that case.

Claim 5 *Under the above assumptions, there exist discrete series representations* τ_j, $1 \leq j \leq k$ *such that the following hold:*

(1) $\tau_j \hookrightarrow \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \tau_{j-1},\ 1 \leq j \leq k, \tau_0 = \sigma''_{\mathrm{alt}}$.
(2) $\mathrm{Lang}(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \tau_j) \leq \delta \rtimes \sigma_j$ *and* $\delta([\nu^{-l_2}\rho, \nu^{b_0}\rho]) \otimes \tau_j$ *appears with multiplicity one in* $\mu^*\left(\delta \rtimes \sigma_j\right)$, $j = 1, \dots, k$.
(3) $\sigma_j \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{b_0}\rho]) \rtimes \tau_j$, $j = 1, \dots, k$.

The details of the verification of both claims are left to the reader.

To complete the proof of Lemma 4.3 we prove the following claim:

Claim 6 *If* $2l_1 + 1, 2l_2 + 1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$, *then* $\delta \rtimes \sigma_j$, $0 \leq j \leq k$ *is irreducible.*

Proof We use induction on j. If $j = 0$, then the claim follows from Lemma 4.4(i) using the definition of σ_0 given in (4.15).

Next, we have the following intertwining operators:

(4.17)
$$\begin{aligned}\delta \rtimes \sigma_j \hookrightarrow \delta \times \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1} &\cong \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \times \delta \rtimes \sigma_{j-1}\\ &\cong \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \times \widetilde{\delta} \rtimes \sigma_{j-1}\\ &\cong \widetilde{\delta} \times \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1}.\end{aligned}$$

The first and third isomorphisms follow form [Ze] using the definition of $\mathrm{Jord}_{\mathrm{alt}}$. (Compare with the discussion immediately after (3.2).) The second intertwining operator is an isomorphism since $\delta \rtimes \sigma_{j-1}$ is irreducible. Now, (4.16) implies the first below formula (and the second is obvious)

$$\begin{aligned}\delta \rtimes \sigma_j &\hookrightarrow \widetilde{\delta} \times \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1}\\ \widetilde{\delta} \rtimes \sigma_j &\hookrightarrow \widetilde{\delta} \times \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1}.\end{aligned} \tag{4.18}$$

Now, if we show that $\mathrm{Lang}(\delta \rtimes \sigma_j)$ appears with multiplicity one in

$$\widetilde{\delta} \times \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1},$$

then clearly $\delta \rtimes \sigma_j$ is irreducible. To establish that it is enough to check that $\widetilde{\delta} \otimes \sigma_{j-1}$ appears in $\mu^*(\widetilde{\delta} \times \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1})$ with multiplicity one. Since this is completely analogous to the computation of multiplicities given in the proof of Claim 2 but much simpler, we leave the verification to the reader. ■

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5 General Case for $l_1 \geq 0$, II

To complete the study of reducibility of the generalized principal series $\delta \rtimes \sigma$ (in the case of $l_1 \geq 0$), we analyze the case

$$]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho = \varnothing,$$

in this section. The main result of this section is the following theorem:

Theorem 5.1 *Assume that* (2.3) *and* $]2l_1+1, 2l_2+1[\cap \mathrm{Jord}_\rho = \varnothing$ *hold. Then* $\delta \rtimes \sigma$ *is irreducible if and only if* $2l_1+1, 2l_2+1 \in \mathrm{Jord}_\rho$ *and* $\epsilon(2l_1+1,\rho)\cdot\epsilon(2l_2+1,\rho)^{-1} = -1$.

First, Theorem 2.1 enables us to assume $2l_1+1 \in \mathrm{Jord}_\rho$ or $2l_2+1 \in \mathrm{Jord}_\rho$. Now, we start the proof of Theorem 5.1 with the next lemma.

Lemma 5.1 *Suppose* $2l_1+1 \in \mathrm{Jord}_\rho$ *or* $2l_2+1 \in \mathrm{Jord}_\rho$ *but not both. Then* $\delta \rtimes \sigma$ *reduces.*

Proof This has the same proof as [M4, Lemma 4.4]. Nowhere in the proof of that lemma did we use the assumption that σ is positive. ∎

The next lemma completes the proof of the theorem.

Lemma 5.2 *Assume* $2l_1+1 \in \mathrm{Jord}_\rho$ *and* $2l_2+1 \in \mathrm{Jord}_\rho$. *Then* $\delta\rtimes\sigma$ *reduces if and only if* $\epsilon(2l_2+1,\rho)\cdot\epsilon(2l_1+1,\rho)^{-1} = 1$. *Moreover, assume that* $\epsilon(2l_2+1,\rho)\cdot\epsilon(2l_1+1,\rho)^{-1} = 1$, *and define a discrete series* σ'' *by removing* $(2l_1+1,\rho)$ *and* $(2l_2+1,\rho)$ *from the triple of* σ *and restricting* ϵ *to that set of Jordan blocks. (See Theorem* 1.1(iii).*) Then in the appropriate Grothendieck group*

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + \sigma_{\mathrm{temp}},$$

where σ_{temp} *is the common irreducible subquotient of*

$$\begin{cases} \delta \rtimes \sigma \\ \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma''. \end{cases}$$

Proof If $\delta \rtimes \sigma$ reduces then it must have a tempered irreducible subquotient. (See Lemma 2.1.) For any such tempered irreducible subquotient π, Lemma 2.1 implies that there must exist a discrete series π' such that

$$\pi \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \pi'.$$

In particular, Frobenius reciprocity implies that we have $\mu^*(\pi) \geq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \pi'$. Hence

$$\mu^*(\delta \rtimes \sigma) \geq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \pi'.$$

This can be analyzed using Theorem 1.2. Thus, for some irreducible constituents $\mu^*(\sigma) \geq \delta' \otimes \sigma_1'$, and indices $0 \leq j \leq i \leq l_1 + l_2 + 1$, we must have

$$\begin{aligned} \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \leq \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \\ \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta' \\ \pi' \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1'. \end{aligned} \tag{5.1}$$

The first formula implies that δ' must be non-degenerate, and, by [Ze], induced from the product of essentially-square integrable representations. Any such essentially-square integrable representation (say δ_1) must have members in its segment between $\nu^{-l_2}\rho$ and $\nu^{l_2}\rho$. On the other hand, the square-integrability criterion applied to σ shows that the sum over all exponents α, where $\nu^\alpha\rho$ ranges over a segment attached to δ_1 must be strictly positive. This implies $\alpha > -l_2$. Now the first formula in (5.1) shows $i = 0$. Since $0 \leq j \leq i$, we obtain $j = 0$. Now, $\delta' \cong \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$. Since $\mu^*(\sigma) \geq \delta' \otimes \sigma_1'$, there exists an irreducible representation ${\sigma''}_1$ such that

$$\sigma \hookrightarrow \nu^{l_2}\rho \times \nu^{l_2-1}\rho \times \cdots \times \nu^{-l_1}\rho \rtimes {\sigma''}_1.$$

Using $]2l_1 + 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho = \varnothing$ and arguing as in the proof of [M4, Lemma 4.1], we see that

$$\sigma \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes {\sigma''}_1.$$

This, using [Mœ, Remark 3.1], implies that $\epsilon(2l_1+1, \rho) \cdot \epsilon(2l_2+1, \rho)^{-1} = 1$, proving the necessary condition for reducibility.

Now, assume that this necessary condition for reducibility holds. Then the discussion above shows that $\pi' \cong \sigma_1'$ (since $i = j = 0$ applying the second formula in (5.1)). We remind the reader that $\delta' \cong \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$. Now, [M3, Theorem 2.3], applied to $\mu^*(\sigma) \geq \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1'$ shows that $\sigma'' \cong \sigma_1'$. Moreover, [M3, Theorem 2.3] shows that $\mu^*(\sigma)$ contains $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1'$ with multiplicity one. This implies that $\mu^*(\delta \rtimes \sigma)$ contains $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \otimes \sigma''$ with multiplicity one.

To prove the existence of the unique tempered subquotient mentioned in the statement of the lemma, we simply show that the multiplicity of

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \otimes \sigma''$$

in the appropriate Jacquet modules of the induced representations on the right-hand and left-hand sides of

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma'' \geq \begin{cases} \delta \rtimes \sigma \\ \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma'' \end{cases} \tag{5.2}$$

is exactly two, at least one, and exactly two, respectively.

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Since Theorem 1.1(iii) implies $\sigma \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma''$, we see that

$$\delta \rtimes \sigma \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma'',$$

and hence the claim for $\delta \rtimes \sigma$.

Next, we apply Theorem 1.2, three times to compute

$$\mu^* \left(\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma''\right).$$

Thus, for some irreducible constituent $\mu^*(\sigma'') \geq \delta' \otimes \sigma'_1$, and indices $0 \leq j \leq i \leq 2l_1 + 1$, $0 \leq j_1 \leq i_1 \leq l_2 - l_1$, and $0 \leq j_2 \leq i_2 \leq l_1 + l_2 + 1$, we must have

$$\begin{aligned}(5.3)\qquad \delta([\nu^{-l_1}\rho,&\nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \\ &\leq \delta([\nu^{i-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1-j}\rho, \nu^{l_1}\rho]) \\ &\quad \times \delta([\nu^{i_1-l_2}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{l_2+1-j_1}\rho, \nu^{l_2}\rho]) \\ &\quad \times \delta([\nu^{i_2-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j_2}\rho, \nu^{l_2}\rho]) \times \delta',\end{aligned}$$

and

$$\begin{aligned}(5.4)\qquad \sigma'' \leq \delta([\nu^{l_1+1-i}\rho, \nu^{l_1-j}\rho]) &\times \delta([\nu^{l_2+1-i_1}\rho, \nu^{l_2-j_1}\rho]) \\ &\times \delta([\nu^{l_2+1-i_2}\rho, \nu^{l_2-j_2}\rho]) \rtimes \sigma'_1.\end{aligned}$$

Now the third term on the right-hand side of inequality of (5.3) must not exist. Thus, $i_1 = l_2 - l_1$. Also, there are no terms in $\mathrm{Jord}(\sigma'')$ that are between $2l_1 + 1$ and $2l_2 + 1$, by the construction. Hence the fourth term on the right-hand side of inequality of (5.3) shows us $j_1 = l_2 - l_1$. Similarly, we conclude $j_2 \geq l_2 - l_1$. As we just remarked, none of the segments can start with $\nu^{l_1}\rho$, since otherwise $(2l_1 + 1, \rho) \in \mathrm{Jord}(\sigma'')$ [Mœ]. Also, all possible terms in the supercuspidal support of σ'_1 are between (perhaps including) $\nu^{-l_1}\rho$ and $\nu^{l_1}\rho$. This violates the square-integrability criterion for σ''. Thus δ' must be trivial, and consequently $\sigma'_1 \cong \sigma''$. Since the first two terms on the right-hand side of (5.3) can not produce $\nu^{-l_1}\rho$ simultaneously, we conclude that $i_2 - l_2 = -l_1$. Now, $l_2 - l_1 \leq j_2 \leq i_2 = l_2 - l_1$ and hence $j_2 = l_2 - l_1$. It is now easy to see that $i = j = 0$ or $i = j = 2l+1$. Now, (5.4) trivially holds. Thus, we have produced the desired term exactly twice. It remains to check the claim for $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma''$. This computation is similar to the one we have just completed. We leave the straightforward verification to the reader. ■

6 The Case $l_1 \leq -1/2$

In this section we investigate the reducibility in the case $l_1 \leq -1/2$. The main result of this section is the following theorem:

Theorem 6.1 *Assume $l_1 \leq -1/2$ and (2.3) holds. Then we have the following:*

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(i) *Suppose* $[-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho = \emptyset$. *Then* $\delta \rtimes \sigma$ *is reducible if and only if*

$$\begin{cases} l_1 = -1/2 \\ \text{if } 2l_2 + 1 \in \mathrm{Jord}_\rho, \text{ then } \epsilon(2l_2 + 1, \rho) = 1. \end{cases}$$

(ii) *Assume* $[-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho \neq \emptyset$. *Then* $\delta \rtimes \sigma$ *is reducible unless there exists an alternated triple* $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$ *dominated by* $(\mathrm{Jord}, \sigma', \epsilon)$ *that contains* $[-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho$ *and* $2l_2 + 1$. *Moreover, if such an alternated triple exists, the induced representation* $\delta \rtimes \sigma$ *is reducible if and only if* $l_1 = -1/2$ *and* $\epsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = 1$

The next few lemmas will complete the proof of this theorem.

Lemma 6.1 *Suppose* $[-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho = \emptyset$. *Then* $\delta \rtimes \sigma$ *is reducible if and only if*

$$\begin{cases} l_1 = -1/2 \\ \text{if } 2l_2 + 1 \in \mathrm{Jord}_\rho, \text{ then } \epsilon(2l_2 + 1, \rho) = 1. \end{cases}$$

Proof Assume $l_1 \leq -1$. Then Lemmas 2.1 and 2.2 show that there are no non-discrete series subquotients of $\delta \rtimes \sigma$. Also, a consideration of Jordan blocks as in [MT, Section 8] shows that there are also no discrete series subquotients if $l_1 \leq -1$. Thus, $\delta \rtimes \sigma$ is irreducible in this case.

Similarly, if $l = -1/2$, we see that only possible irreducible subquotients are tempered if $2l_2 + 1 \in \mathrm{Jord}_\rho$, and in the discrete series if $2l_2 + 1 \notin \mathrm{Jord}_\rho$.

We first consider $2l_2 + 1 \notin \mathrm{Jord}_\rho$. In this case a discrete series subquotient must have $\mathrm{Jord} \cup \{(2l_2 + 1, \rho)\}$ as its set of Jordan blocks.

Let $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$ be any alternated triple dominated by $(\mathrm{Jord}, \sigma', \epsilon)$ obtained removing a set of disjoint pairs $(2a_j^- + 1, \rho_j), (2a_j + 1, \rho_j)$, $1 \leq j \leq k$, such that $a_j^- < a_j, \epsilon(2a_j{+}1, \rho)\epsilon(2a_j^- {+}1, \rho)^{-1} = 1$ and we can have neither $a_j^- < a_i^- < a_j < a_i$ nor $a_i^- < a_j^- < a_i < a_j$, for $i \neq j$ and $\rho_i \cong \rho_j$. In particular, by Theorem 1.1, there exists a unique sequence of discrete series σ_j, $1 \leq j \leq k - 1$, such that

$$\sigma_j \hookrightarrow \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \sigma_{j-1},$$

$1 \leq j \leq k$, where $\sigma_0 = \sigma_{\mathrm{alt}}, \sigma_k = \sigma$.

We write $2l_0 + 1 = \min(\mathrm{Jord}_{\mathrm{alt}})_\rho$. We have two cases.

- $\epsilon(2l_0 + 1, \rho) = -1$, then we write $\sigma_{\mathrm{alt,new}}$ for the discrete series whose triple is alternated and is obtained from that of σ_{alt} by adding $(2l_2 + 1, \rho)$ and extending ϵ so that $\epsilon(2l_2 + 1, \rho) = 1$. It is clear from Theorem 1.1(ii) that $\sigma_{\mathrm{alt,new}}$ is a subrepresentation of $\delta \rtimes \sigma_{\mathrm{alt}}$ and it appears with multiplicity one as an irreducible subquotient [MT, Lemma 4.1]. We put $\tau_0 = \sigma_{\mathrm{alt,new}}$.
- $\epsilon(2l_0 + 1, \rho) = 1$, then we write $\sigma_{\mathrm{alt,new}}$ for the discrete series whose triple is alternated and is obtained from that of σ_{alt} by removing $(2l_0 + 1, \rho)$ from $\mathrm{Jord}_{\mathrm{alt}}$ and restricting ϵ. Then $\delta \rtimes \sigma_{\mathrm{alt}}$ has a unique irreducible subquotient that is different from its Langlands quotient, appears in the composition series with multiplicity one and it is a discrete series subrepresentation of

$$\delta([\nu^{-l_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\mathrm{alt,new}}.$$

(See ([M4, Theorem 5.1(ii)].) We denote that representation by τ_0.

Now, as in the proof of Lemma 4.1, there exists a unique sequence of discrete series τ_j, $1 \le j \le k$, such that

$$\begin{cases} \tau_j \hookrightarrow \delta([\nu^{-a_j^-}\rho_j, \nu^{a_j}\rho_j]) \rtimes \tau_{j-1}, \\ \tau_j \le \delta \rtimes \sigma_j \text{ (appears with multiplicity one).} \end{cases}$$

We omit the details since they are entirely analogous to these of Lemma 4.1. This completes the proof in the case $2l_2 + 1 \notin \mathrm{Jord}_\rho$.

Next, we consider the case $2l_2 + 1 \in \mathrm{Jord}_\rho$.

Claim 1 *Assume that there exists a discrete series σ_1 such that*

$$\mu^*(\sigma) \ge \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \otimes \sigma_1. \tag{6.1}$$

Then

$$\sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1. \tag{6.2}$$

Proof First, arguing as in [M4, Lemma 4.1] we see that there exists an irreducible representation σ_1' such that

$$\sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1'. \tag{6.3}$$

Now, we show σ_1' is in discrete series. If not, we can find an irreducible representation σ''_1 and a segment $[\nu^\alpha\rho', \nu^\beta\rho']$ (ρ' unitarizable, $\alpha, \beta \in \mathbb{R}$, $-\alpha + \beta \in \mathbb{Z}_{\ge 0}$) such that $\alpha + \beta \le 0$ and

$$\sigma_1' \hookrightarrow \delta([\nu^\alpha\rho', \nu^\beta\rho']) \rtimes \sigma''_1. \tag{6.4}$$

Combining this with (6.3), we obtain the following sequence of intertwining maps

$$\begin{aligned} \sigma \hookrightarrow &\delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^\alpha\rho', \nu^\beta\rho']) \rtimes \sigma''_1 \\ &\to \delta([\nu^\alpha\rho', \nu^\beta\rho']) \times \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \rtimes \sigma''_1. \end{aligned}$$

Their composition must be zero, since otherwise the assumption $\alpha + \beta \le 0$ violates the square-integrability criterion for σ. In particular, the segments $[\nu^{1/2}\rho, \nu^{l_2}\rho]$ and $[\nu^\alpha\rho', \nu^\beta\rho']$ are linked. Therefore, $\rho \cong \rho'$, and since $-\alpha + \beta \ge 0$ and $\alpha + \beta \le 0$ implies $\alpha \le 0$, we obtain

$$\sigma \hookrightarrow \delta([\nu^\alpha\rho, \nu^{l_2}\rho]) \times \delta([\nu^{1/2}\rho, \nu^\beta\rho]) \rtimes \sigma''_1.$$

This implies

$$\sigma \hookrightarrow \delta([\nu^\alpha\rho, \nu^{l_2}\rho]) \rtimes \sigma''_2,$$

for some irreducible representation ${\sigma''}_2$. Now, [Mœ, Remark 3.1] implies that $[-2l_1-1, 2l_2+1[\cap \mathrm{Jord}_\rho \neq \emptyset$. This is a contradiction. Thus, σ_1' is in discrete series.

Now, we show $\sigma_1 \cong \sigma_1'$. First, if $l_2 > 1/2$ we may apply [M3, Theorem 2.3] to show that $\sigma_1' \cong \sigma_1$. If $l_1 = 1/2$, then we will show the same. First, (6.1) and (6.3) imply

$$\nu^{1/2}\rho \otimes \sigma_1 \leq \mu^*(\nu^{1/2}\rho \rtimes \sigma_1').$$

This can be easily analyzed using Theorem 1.2. In particular, there are indices $0 \leq j \leq i \leq 1$, and an irreducible constituent $\delta' \otimes \sigma_2'$ of $\mu^*(\sigma_1')$ such that

$$\begin{aligned} \nu^{1/2}\rho &\leq \delta([\nu^{i-1/2}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{3/2-j}\rho, \nu^{1/2}\rho]) \times \delta' \\ \sigma_1 &\leq \delta([\nu^{3/2-i}\rho, \nu^{1/2-j}\rho]) \rtimes \sigma_2'. \end{aligned} \tag{6.5}$$

Since (6.3) implies $2 \notin \mathrm{Jord}_\rho(\sigma_1')$, we see that δ' must be trivial. Hence $\sigma_1' \cong \sigma_2'$. Now, the first inequality in (6.5) shows $i = j = 1$. The second implies $\sigma_1 \cong \sigma_1'$. ■

Now, we show the following claim:

Claim 2 *Keeping the assumptions of Claim* 1*, there exists an alternated triple*

$$(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$$

dominated by $(\mathrm{Jord}(\sigma_1), \sigma', \epsilon_{\sigma_1})$ *such that the triple* $(\mathrm{Jord}_2, \sigma', \epsilon_2)$*, where* $\mathrm{Jord}_2 = \mathrm{Jord}_2 \cup \{(2l_2+1, \rho)\}$ *and* ϵ_2 *extends* ϵ_{alt} *such that*

$$\begin{cases} \epsilon_2(2l_2+1,\rho)\epsilon_2(\min \mathrm{Jord}_{\mathrm{alt}}, \rho)^{-1} = -1, & \text{if } \epsilon_2(\min \mathrm{Jord}_{\mathrm{alt}}, \rho) = -1 \\ \epsilon_2(2l_2+1,\rho)\epsilon_2(\min \mathrm{Jord}_{\mathrm{alt}}, \rho)^{-1} = 1, & \text{if } \epsilon_2(\min \mathrm{Jord}_{\mathrm{alt}}, \rho) = 1, \end{cases}$$

is an admissible triple and subordinated to $(\mathrm{Jord}, \sigma', \epsilon)$.

Proof To accomplish this, let $(2a+1, \rho') \in \mathrm{Jord}(\sigma_1)$ such that $(2a+1)_- := 2a_- + 1$ is defined and $\epsilon(2a+1, \rho')\epsilon(2a_- + 1, \rho')^{-1} = 1$. Let ${\sigma''}_1$ be the discrete series obtained from σ_1 removing pairs $(2a+1, \rho')$ and $(2a_- + 1, \rho')$ from $\mathrm{Jord}(\sigma_1)$ and restricting ϵ_{σ_1}. Then Theorem 1.1 implies

$$\sigma_1 \hookrightarrow \delta([\nu^{-a_-}\rho', \nu^a\rho']) \rtimes {\sigma''}_1. \tag{6.6}$$

Now, by Claim 1, we obtain

$$\begin{aligned} \sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-a_-}\rho', \nu^a\rho']) \rtimes {\sigma''}_1 \cong \\ \delta([\nu^{-a_-}\rho', \nu^a\rho']) \times \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \rtimes {\sigma''}_1. \end{aligned} \tag{6.7}$$

The last isomorphism follows from $2l_2 + 1 \notin \mathrm{Jord}(\sigma_1)$.

Next, (6.7) implies that there exists an irreducible representation σ'' such that

$$\sigma \hookrightarrow \delta([\nu^{-a_-}\rho', \nu^{a}\rho']) \rtimes \sigma''.$$

Theorem 1.1(iii) and [M3, Theorem 2.3] imply ${\sigma''}_1$ is in the discrete series and is obtained from σ by removing pairs $(2a+1,\rho')$ and $(2a_-+1,\rho')$ from Jord and restricting ϵ. Now, we show that an analogue of (6.1) holds for σ'':

$$\mu^*(\sigma'') \geq \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \otimes {\sigma''}_1.$$

First, (6.1) implies that

$$\delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \otimes \sigma_1 \leq \mu^*\left(\delta([\nu^{-a_-}\rho', \nu^{a}\rho']) \rtimes \sigma''\right).$$

Thus, by Theorem 1.2, for some irreducible constituents $\delta' \otimes \sigma_2' \leq \mu^*(\sigma'')$, and indices $0 \leq j \leq i \leq a_- + a + 1$, we must have

$$\begin{aligned} \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) &\leq \delta([\nu^{i-a}\rho', \nu^{a_-}\rho']) \times \delta([\nu^{a+1-j}\rho', \nu^{a}\rho']) \times \delta' \\ \sigma_1 &\leq \delta([\nu^{a+1-i}\rho', \nu^{a-j}\rho']) \rtimes \sigma_2'. \end{aligned} \tag{6.8}$$

Since $l_2 < a_- < a$, by the definition of σ_1, the first inequality in (6.8) implies $i = a_- + a + 1$ and $j = 0$. We also obtain $\delta' \cong \delta([\nu^{1/2}\rho, \nu^{l_2}\rho])$, and

$$\mu^*(\sigma'') \geq \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \otimes \sigma_2'.$$

We must show $\sigma_2' \cong {\sigma''}_1$. First, Claim 1 applied to σ'' implies that σ_2' is in the discrete series. Now, the second inequality in (6.8) and Theorem 2.1(iv) imply

$$\sigma_1 \hookrightarrow \delta([\nu^{-a_-}\rho', \nu^{a}\rho']) \rtimes \sigma_2'.$$

Then, [M3, Theorem 2.3] implies $\sigma_2' \cong {\sigma''}_1$.

We may now repeat this procedure (of removing the pairs) until we get that the triple of ${\sigma''}_1$ is of alternated type. The corresponding σ'' has its triple as in the statement of Claim 2 according to [M4, Theorem 5.1]. Thus, we have proved the claim. ∎

Now, if $\delta \rtimes \sigma$ reduces, it must have a tempered irreducible subquotient such that Lemma 2.1(a) holds with $l_1 = -1/2$. Therefore, the discrete series representation σ_1 obtained there satisfies the conclusion of Claim 1. Now, Claim 2 implies $\epsilon(2l_2+1,\rho) = 1$. This proves the necessary condition for reducibility. Going in the opposite direction of Claim 2, we may use induction to construct a discrete series σ_1 such that

$$\sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1, \tag{6.9}$$

when $\epsilon(2l_2+1,\rho) = 1$ holds. This follows by combining [M4, Theorem 5.1] and Theorem 1.1 (as in Lemma 4.1). We omit the details. Now, we show that $\delta \rtimes \sigma$ has a common irreducible subquotient with the induced representation

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1.$$

This will complete the proof of the lemma. First, we observe the following:

$$\begin{aligned}\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\rtimes\sigma &\leq \delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\rtimes\sigma_1,\\ \delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\rtimes\sigma_1 &\leq \delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\rtimes\sigma_1.\end{aligned}\tag{6.10}$$

We show that $\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\otimes\sigma_1$ appears in

$$\mu^*\big(\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\times\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\rtimes\sigma_1\big)\tag{6.11}$$

with multiplicity exactly two. Then $\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\rtimes\sigma_1$ contains $\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\otimes\sigma_1$ in its appropriate Jacquet module twice. (See [M3, Theorem 2.3]) Now, combining this with (6.11) and the fact (that easily follows from Theorem 1.2 and (6.9))

$$\mu^*(\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\rtimes\sigma)\geq\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\otimes\sigma_1,$$

we see that $\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\rtimes\sigma_1$ and $\delta([\nu^{1/2}\rho,\nu^{l_2}\rho])\rtimes\sigma$ must have a common irreducible subquotient. This proves reducibility.

Let us now compute the multiplicity of $\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\otimes\sigma_1$ in (6.11). We use Theorem 1.2. So, let $\mu^*(\sigma_1)\geq\delta'\otimes\sigma_2'$ irreducible representation, and indices $0\leq j\leq i\leq l_2+1/2$, $0\leq j'\leq i'\leq l_2+1/2$. We have the following formulas

$$\begin{aligned}\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\leq{}&\delta([\nu^{i-l_2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{l_2+1-j}\rho,\nu^{l_2}\rho])\\ &\times\delta([\nu^{i'-l_2}\rho,\nu^{-1/2}\rho])\times\delta([\nu^{l_2+1-j'}\rho,\nu^{l_2}\rho])\times\delta'\end{aligned}\tag{6.12}$$

and

$$\sigma_1\leq\delta([\nu^{l_2+1-i'}\rho,\nu^{l_2-j'}\rho])\times\delta([\nu^{l_2+1-i}\rho,\nu^{l_2-j}\rho])\rtimes\sigma_2'.\tag{6.13}$$

As before, from the first formula in (6.12) we see that δ' must be non-degenerate, and, thus by [Ze], fully induced from the tensor product of essentially square-integrable representations. Since, $l_2+1-j\geq 1/2$, $l_2+1-j'\geq 1/2$. This implies $i=0$ or $i'=0$, since otherwise one of the segments attached to δ' would start at $\nu^{-l_2}\rho$ and end at $\nu^{\alpha}\rho$, $\alpha\leq l_2$. This implies

$$\sigma_1\hookrightarrow\nu^{\alpha}\rho\times\cdots\times\nu^{-l_2}\rho\rtimes\sigma_1',$$

for some irreducible representation σ_1'. This contradicts the square-integrability criterion for σ_1.

First, assume $i=0$. Then since $i\geq j\geq 0$ we also obtain $j=0$. We also see from (6.12) that $i'=l_2+1$, or otherwise the left-hand side of (6.12) would contain $\nu^{-1/2}\rho$ in its supercuspidal support twice. Finally, $l_2+1-j'=1/2$, that is, $j'=l_2+1/2$ or $\delta'\cong\delta([\nu^{1/2}\rho,\nu^{l_2-j'}\rho])$. This implies $2(l_2-j')+1\in\mathrm{Jord}_\rho$, a contradiction. Thus, δ' is trivial and $\sigma'\cong\sigma_1$. Obviously (6.13) holds. This produces $\delta([\nu^{-l_2}\rho,\nu^{l_2}\rho])\otimes\sigma_1$ once. The case $i'=0$ is similar. ■

Now, we analyze the case $[-2l_1-1,2l_2+1[\,\cap\,\mathrm{Jord}_\rho\neq\varnothing$. The remaining results of this section are completely parallel to the ones in Section 4. The first result is an analogue of Lemma 4.1.

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Lemma 6.2 *Assume $l_1 \leq -1/2$, (2.3), and $[-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho \neq \emptyset$. Let us write $2a_0 + 1$ for the smallest element and $2b_0 + 1$ for the largest element in*

$$[-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho .$$

Then $\delta \rtimes \sigma$ reduces if one of the following holds:

(i) *There exists $2a_0^- + 1 \in \mathrm{Jord}_\rho$ such that $a_0^- < a_0$, $\epsilon(2a_0+1, \rho)\epsilon(2a_0^- + 1, \rho)^{-1} = 1$, and $]2a_0^- + 1, 2a_0 + 1[\cap \mathrm{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2a_j^- + 1, 2a_j + 1\}$, $j = 1, \dots, k$ (this defines k), such that $a_j^- < a_j$, $\epsilon(2a_j + 1, \rho)\epsilon(2a_j^- + 1, \rho)^{-1} = 1$ and we can have neither $a_j^- < a_i^- < a_j < a_i$ nor $a_i^- < a_j^- < a_i < a_j$, for $i \neq j$.*

(ii) *There exists $2b_0^+ + 1 \in \mathrm{Jord}_\rho$ such that $b_0 < b_0^+$, $\epsilon(2b_0 + 1, \rho)\epsilon(2b_0^+ + 1, \rho)^{-1} = 1$, and $]2b_0 + 1, 2b_0^+ + 1[\cap \mathrm{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2b_j^- + 1, 2b_j + 1\}$, $j = 1, \dots, k$ (this defines k), such that $b_j^- < b_j$, $\epsilon(2b_j + 1, \rho)\epsilon(2b_j^- + 1, \rho)^{-1} = 1$ and we can have neither $b_j^- < b_i^- < b_j < b_i$ nor $b_i^- < b_j^- < b_i < b_j$, for $i \neq j$.*

Proof As in Lemma 4.1. ■

Lemma 6.3 *Assume that* (2.3) *holds and that there is a pair $2a' + 1 < 2a + 1$, $2a + 1 \in [-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho$ or $2a' + 1 \in [-2l_1 - 1, 2l_2 + 1[\cap \mathrm{Jord}_\rho$, $\epsilon(2a+1, \rho)\epsilon(2a' + 1, \rho)^{-1} = 1$, and $]2a' + 1, 2a + 1[\cap \mathrm{Jord}_\rho$ is either empty or can be divided into disjoint sets of pairs $\{2a_j^- + 1, 2a_j + 1\}$, $j = 1, \dots, k$ (this defines k), such that $a_j^- < a_j$, $\epsilon(2a_j + 1, \rho)\epsilon(2a_j^- + 1, \rho)^{-1} = 1$ and we can have neither $a_j^- < a_i^- < a_j < a_i$ nor $a_i^- < a_j^- < a_i < a_j$, for $i \neq j$. Then $\delta \rtimes \sigma$ is reducible.*

Proof Exactly as in Lemma 4.2. ■

Lemma 6.4 *Assume that* (2.3) *holds and that there exists a triple of alternated type* $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$ *subordinated to that of σ that contains all*

$$[-2l_1 - 1, 2l_2 + 1[\ \cap \ \mathrm{Jord}_\rho \neq \emptyset.$$

Then we have the following cases:

(i) *If $2l_2 + 1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$, then $\delta \rtimes \sigma$ is reducible.*

(ii) *If $2l_2 + 1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$ and $l_1 \leq -1$, or $l_1 = -1/2$ and $\epsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = -1$, then is $\delta \rtimes \sigma$ is irreducible.*

(iii) *If $2l_2 + 1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$, $l_1 = -1/2$, and $\epsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = 1$, then is $\delta \rtimes \sigma$ is reducible.*

Proof Again, this proof is completely analogous to the proof of Lemma 4.3. We just indicate necessary modifications.

We write σ_{alt} for a strongly positive discrete series attached to $(\mathrm{Jord}_{\mathrm{alt}}, \sigma', \epsilon_{\mathrm{alt}})$. We also write $2a_0 + 1$ for the largest of

$$[-2l_1 - 1, 2l_2 + 1[\cap (\mathrm{Jord}_{\mathrm{alt}})_\rho.$$

Instead of Lemma 4.4, here we use the next lemma to complete the proof.

Lemma 6.5 *Under the above assumptions, we have the following:*

(i) *If* $2l_2+1 \notin (\mathrm{Jord}_{\mathrm{alt}})_\rho$*, then we define a discrete series* σ'_{alt} *by replacing* $2a_0+1$ *with* $2l_2+1$ *in all the formulas that define* σ_{alt}*, and we have*

$$\mathrm{Lang}(\delta([\nu^{1/2}\rho, \nu^{a_0}\rho]) \rtimes \sigma'_{\mathrm{alt}}) \leq \delta \rtimes \sigma_{\mathrm{alt}}.$$

(ii) *If* $2l_2+1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$ *and* $l_1 \leq -1$*, or* $l_1 = -1/2$ *and* $\epsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = -1$*, then is* $\delta \rtimes \sigma_{\mathrm{alt}}$ *is irreducible.*

(iii) *If* $2l_2+1 \in (\mathrm{Jord}_{\mathrm{alt}})_\rho$*,* $l_1 = -1/2$*, and* $\epsilon(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho, \rho) = 1$*, then we define a discrete series* σ'_{alt} *by removing* $\min(\mathrm{Jord}_{\mathrm{alt}})_\rho$ *from* $(\mathrm{Jord}_{\mathrm{alt}})_\rho$*, and we have*

$$\mathrm{Lang}(\delta([\nu^{-(\min(\mathrm{Jord}_{\mathrm{alt}})_\rho - 1)/2}\rho, \nu^{l_2}\rho]) \rtimes \sigma'_{\mathrm{alt}}) \leq \delta \rtimes \sigma_{\mathrm{alt}}.$$

Proof This follows from [M4, Proposition 3.1] and [M4, Theorem 5.1]. ■

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