A GENERATING FUNCTION FOR JACOBI POLYNOMIALS

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1. Introduction. The following notations will be employed throughout this note.

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)},$$

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\ \Gamma(a-b+1)}.$$

The object of the present note is to obtain a new generating function for the Jacobi polynomials defined by [4, page 268]

$$(1.1)\qquad P_n^{(a,b)}(x) = \binom{a+n}{n}\ {}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-x}{2}\right),$$

where $|1-x| < 2$.

The generating function developed in § 2 is

$$(1.2)\qquad \sum_{n=o}^{\infty} \frac{(-1)^n (\gamma)_n (\delta)_n}{(\alpha+1)_n\ (\alpha+\beta+n+1)_n}\ P_n^{(\alpha,\beta)}(x)$$

$$\times\ F_4\ (\gamma+n, \delta+n; \alpha+\beta+2n+2, \nu+1; y, z)\ y^n$$

$$= F_4\ [\gamma, \delta; \alpha+1, \nu+1; \tfrac{1}{2} y(1-x), z],$$

where $|y|^{1/2} + |z|^{1/2} < 1$, $|1-x| < 2$ and F_4 is Appell's hypergeometric function of two variables defined by

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n,$$

where $|x|^{1/2} + |y|^{1/2} < 1$. Some of its interesting particular cases will be given in § 3.

Proof. To prove (1.2) we start with Watson's formula [5, page 140]

$$\sum_{n=o}^{\infty} \frac{(2n+\alpha)\Gamma(n+\alpha)}{n!} {}_2F_1(n+\alpha, -n; \mu+1; x^2)\, J_{2n+\alpha}(z)$$

(2.1)

$$= \frac{\Gamma(\mu+1)}{x^{\mu}} \left(\frac{1}{2} z\right)^{\alpha-\mu} J_{\mu}(xz) .$$

On multiplying both sides of (2.1) by $z^{\rho-1} J_{\nu}(yz) J_{\lambda}(uz)$, and integrating with respect to z between the limits 0 and ∞, with the help of the integral ([1], page 38 and [5], § 13.46)

$$\int_o^{\infty} x^{\rho-1} J_{\lambda}(ax) J_{\mu}(bx) J_{\nu}(cx)\, dx$$

(2.2)
$$= \frac{2^{\rho-1} a^{\lambda} b^{\mu} c^{-\lambda-\mu-\rho}\, \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\rho)}{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(1 - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\rho)}$$

$$\times F_4\left(\frac{\lambda+\mu-\nu+\rho}{2}, \frac{\lambda+\mu+\nu+\rho}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right),$$

where $R(\lambda+\mu+\nu+\rho) > 0$, $R(\rho) < \frac{5}{2}$; $a, b, c > 0$; $c > a+b$, it is found that

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$$\sum_{n=0}^{\infty} \frac{\binom{\frac{1}{2}\lambda - \frac{1}{2}\alpha - \frac{1}{2}\rho - \frac{1}{2}\nu}{n} \left(\frac{1}{2}\nu + \frac{1}{2}\lambda + \frac{1}{2}\alpha + \frac{1}{2}\rho\right)_n}{(\alpha + n)_n}$$

$$\times\ {}_2F_1\,(n + \alpha,\ -n;\ \mu + 1;\ x^2)\ u^{-2n}$$

$$\times\ F_4\left[\frac{\alpha+\rho+\nu-\lambda}{2} + n,\ \frac{\alpha+\rho+\nu+\lambda}{2} + n;\ 2n+\alpha+1, \nu+1,\ u^{-2},\ \frac{y^2}{u^2}\right]$$

$$= F_4\left[\frac{\alpha+\rho+\nu-\lambda}{2},\ \frac{\alpha+\rho+\nu+\lambda}{2};\ \mu+1, \nu+1;\ \frac{x^2}{u^2},\ \frac{y^2}{u^2}\right].$$

If we now replace $\frac{1}{2}\nu + \frac{1}{2}\alpha + \frac{1}{2}\rho - \frac{1}{2}\lambda$ by γ, $\frac{1}{2}\nu + \frac{1}{2}\alpha + \frac{1}{2}\rho + \frac{1}{2}\lambda$ by δ, x^2 by x, u^{-2} by y and $\frac{y^2}{u^2}$ by z, we see that

$$\sum_{n=0}^{\infty} \frac{\binom{-\gamma}{n} (\delta)_n}{(\alpha+n)_n}\ {}_2F_1\,(\alpha+n,\ -n;\ \mu+1;\ x)$$

(2.3)

$$\times\ F_4\,(\gamma+n,\ \delta+n;\ 2n+\alpha+1,\ \nu+1;\ y,\ z)\ y^n$$

$$= F_4\,(\gamma, \delta;\ \mu+1,\ \nu+1;\ xy,\ z),$$

where $|y|^{1/2} + |z|^{1/2} < 1$ and $|x| < 1$.

(1.2) now immediately follows from (2.3) on using (1.1).

The change of the order of integration and summation in the above process is justified due to the absolute convergence of the series and the integral under consideration [1, page 500].

2. Particular cases. (i) If we write $2y$ for y, $\nu+\alpha-\gamma+1$ for δ and $x(1-y)$ for z in (1.2) and make use of the following property of Appell's function F_4 [3, page 238]

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$$\text{(3.1)}\quad \begin{aligned} &F_4[\alpha, \gamma+\gamma'-\alpha-1; \gamma, \gamma'; x(1-y), y(1-x)] \\ &= {}_2F_1(\alpha, \gamma+\gamma'-\alpha-1; \gamma; x)\ {}_2F_1(\alpha, \gamma+\gamma'-\alpha-1; \gamma'; y) \end{aligned}$$

we obtain

$$\text{(3.1')}\quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n (\alpha-\gamma+\nu+1)_n}{(\alpha+1)_n (\alpha+\beta+n+1)_n} P_n^{(\alpha,\beta)}(x) \\ &\times F_4[\gamma+n, \nu+\alpha-\gamma+n+1; 2n+\alpha+\beta+2, \nu+1; 2y, x(1-y)](2y)^n \\ &= {}_2F_1(\gamma, \nu+\alpha-\gamma+1; \nu+1; x) \\ &\times {}_2F_1(\gamma, \nu+\alpha-\gamma+1; \alpha+1; y), \end{aligned}$$

where $|1-x| < 2$ and $|y| < 1$.

(ii) In a similar way it can be easily seen from (1.2) and (3.1) that

$$\text{(3.2)}\quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n (\alpha+\beta-\gamma+\nu+2)_n}{(\alpha+1)_n (\alpha+\beta+n+1)_n} P_n^{(\alpha,\beta)}(x) \\ &\times {}_2F_1(n+\gamma, n+\alpha+\beta-\gamma+\nu+2; 2n+\alpha+\beta+2; y) \\ &\times {}_2F_1(n+\gamma, n+\alpha+\beta-\gamma+\nu+2; \nu+1; z)\, y^n \\ &= F_4[\gamma, \alpha+\beta+\nu-\gamma+2; \alpha+1, \nu+1; \tfrac{1}{2} y(1-x)(1-z), z(1-y)], \end{aligned}$$

where $|y| < 1$, $|z| < 1$ and $|1-x| < 2$.

(iii) Lastly if we set $\alpha = \beta = \nu - \frac{1}{2}$ then by virtue of the formula

$$\text{(3.3)}\qquad C_n^{\nu}(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2}, \nu-\frac{1}{2})}(x),$$

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we find that

$$\sum_{n=o}^{\infty} \frac{(-1)^n (\gamma)_n (\delta)_n}{(2\nu)_{2n}} C_n^{\nu}(x)$$

(3.4) $$\times F_4(\gamma+n, \delta+n; 2n+2\nu+1, \nu+1; y, z)\, y^n$$

$$= F_4[\gamma, \delta; \nu+\tfrac{1}{2}, \nu+1; \tfrac{1}{2}y(1-x), z],$$

where $|y|^{1/2} + |z|^{1/2} < 1$, $|1-x| < 2$ and $C_n^{\nu}(x)$ is the Gegenbauer polynomial.

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