CHARACTERIZING f-RINGS

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Birkhoff and Pierce [2] introduced the class of *f-rings*—those lattice-ordered rings R which satisfy the additional condition that if a, b, and c are positive elements of R and if $a \wedge b = 0$, then $ac \wedge b = 0 = ca \wedge b$. They showed that f-rings may be characterized as lattice-ordered rings which are subdirect products of totally-ordered rings.

Since then various characterizations of f-rings have appeared in the literature. See, for instance, Bernau [1] and Fuchs [3]. Also f-rings with no non-zero nilpotent elements have been characterized in Bernau [1], Goffman [4], and Hayes [5].

In the present paper, Theorem 1 gives a further characterization of f-rings, and Theorem 2 gives some characterizations of *l-simple f-rings*.

The following results will be used:

(A) If a, b, c, are elements of an f-ring R, then:

(i) if $a \geqq 0$, $a(b \wedge c) = ab \wedge ac$; $(b \wedge c)a = ba \wedge ca$.
(ii) $|ab| = |a||b|$.
(iii) if $a \wedge b = 0$, then $ab = 0$.

(B) If R is an l-ring, a non-empty subset A of R is said to be *solid* if $|x| \leqq |y|$ and $y \in A$ together imply that $x \in A$. An algebraic ideal which is solid is called an *l-ideal.* The smallest l-ideal containing $a \in E$ is denoted by $\langle a \rangle$. Clearly $\langle a \rangle$ is the set of all elements x in R for which

$$|x| \leqq n|a| + r|a| + |a|s + u|a|v$$

for some natural number n and some r, s, u, $v \in \mathbf{R}^+$.

(C) If r is any element of an f-ring R, then $r^\perp$ denotes the set $\{x \in R\colon |x| \wedge |r| = 0\}$. Clearly, $r^\perp$ is an l-ideal of R.

THEOREM 1. *An l-ring R is an f-ring if and only if for all* $a, b \in \mathbf{R}^+$, $\langle a \wedge b \rangle = \langle a \rangle \cap \langle b \rangle$.

Proof. Clearly for any l-ring R and any a, $b \in \mathbf{R}^+$, $\langle a \wedge b \rangle \subset \langle a \rangle \cap \langle b \rangle$, since $0 \leqq a \wedge b \leqq a, b$. Now suppose that R is an f-ring, and is in fact a subdirect product of the totally-ordered rings $\{R_\alpha : \alpha \in \Lambda\}$. Suppose that x is a positive element of $\langle a \rangle \cap \langle b \rangle$ where a, $b \in \mathbf{R}^+$. Then for $i = 1, 2$ there exist natural numbers n_i and elements r_i, s_i, u_i, v_i of $\mathbf{R}^+$ such that $0 \leqq x \leqq n_1a + r_1a + as_1 + u_1av_1$ and $0 \leqq x \leqq n_2b + r_2b + bs_2 + u_2bv_2$. If we put

$$n = \max(n_1, n_2), \quad r = r_1 \vee r_2, \quad s = s_1 \vee s_2, \quad u = u_1 \vee u_2, \quad \text{and} \quad v = v_1 \vee v_2,$$

then

$$0 \leqq x \leqq na + ra + as + uav \quad \text{and} \quad 0 \leqq x \leqq nb + rb + bs + ubv.$$

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If we denote the elements of R by $\{c_\alpha\}$ where $c_\alpha \in R_\alpha$ for all $\alpha \in \Lambda$, then, given $\alpha \in \Lambda$, either $a_\alpha \leqq b_\alpha$ or $b_\alpha \leqq a_\alpha$. If $a_\alpha \leqq b_\alpha$, then (see Johnson [6]) $a_\alpha = (a \wedge b)_\alpha$ and so

$$0 \leqq x_\alpha \leqq (n(a \wedge b)+r(a \wedge b)+(a \wedge b)s+u(a \wedge b)v)_\alpha.$$

The same inequality holds if $b_\alpha \leqq a_\alpha$, and so it follows that

$$0 \leqq x \leqq n(a \wedge b)+r(a \wedge b)+(a \wedge b)s+u(a \wedge b)v,$$

and this implies that $x \in \langle a \wedge b \rangle$. Since this is an l-ideal, $\langle a \rangle \cap \langle b \rangle \subseteq \langle a \wedge b \rangle$.

Conversely, suppose that the given condition holds for R and that $a, b, c \in \mathbf{R}^+$ and $a \wedge b = 0$. Then $ac \in \langle a \rangle$ and $b \in \langle b \rangle$ together imply that $ac \wedge b \in \langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle = (0)$. Thus $ac \wedge b = 0$. Similarly $ca \wedge b = 0$, so R is an f-ring.

We now show how Theorem 1 may be used to give a characterization of l-simple f-rings.

Definitions. (i) An l-ideal P of an l-ring R is said to be *l-prime* if for $a, b \in R$, $a \wedge b \in P$ implies that $a \in P$ or $b \in P$.

(ii) An l-ideal J of an l-ring R is said to be *regular* if there exists $r \in R$ such that J is a maximal element in the set of l-ideals of R which do not contain r. In such a case we call J a *value* of r. (Clearly any regular l-ideal is l-prime.)

It is clear that if R is an f-ring, a Zorn's Lemma argument shows that each non-zero element of R has at least one value. If M is a value of $x \in R$ then, in general, $M \cap \langle x \rangle$ contains non-zero elements of R. It seems interesting to consider which f-rings satisfy the condition that any non-zero element x has at least one value M for which $M \cap \langle x \rangle = (0)$. In fact, this property characterizes the l-simple f-rings.

Definition. An f-ring is said to be *l-simple* if $R^2 \neq (0)$ and R contains no proper l-ideals.

Theorem 2. *If R is an f-ring then the following conditions are equivalent:*

(1) *If r is any non-zero element of R and M is any value of r, then $M \cap \langle r \rangle = (0)$.*
(2) *If r is any non-zero element of R, then r has at least one value M satisfying $M \cap \langle r \rangle = (0)$.*
(3) *Each non-zero element has the unique value* (0).
(4) *R is l-simple.*

Proof. (1) *implies* (2). Since any non-zero element has at least one value, it is obvious that (1) implies (2).

(2) *implies* (3). We show, firstly, that if $r = 0$ then $r^\perp$ is the unique value of r. It should be noted that if M is any value of r, then, since M is l-prime, $r^\perp \subseteq M$. Also, if M is any value of r satisfying $M \cap \langle r \rangle$, then for any $a \in M$, $\langle a \rangle \cap \langle r \rangle = (0)$ and Theorem 1 implies that $M \subseteq r^\perp$. Thus condition (2) implies that $r^\perp$ is a value of r. Now if N is any value of r, $r^\perp \subseteq N$ and since values cannot be comparable, equality must hold in this inclusion. Hence each $r \neq 0$ has a unique value, namely $r^\perp$, and it follows from this, that R is totally ordered. Hence, if $r \neq 0$, $r^\perp = (0)$.

(3) *implies* (4). Clearly (3) implies that R is totally ordered and has no proper l-ideals. Hence R has no non-zero divisors of zero, and in particular no non-zero nilpotents; so $R^2 \neq (0)$.

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(4) *implies* (1). If R satisfies (4), then (0) is the unique value of each $r \neq 0$ and so (1) is obvious.

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