A NOTE ON POLAR TOPOLOGIES

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Let (E, E') be a dual pair and $\mathfrak{A}$ be a collection of $\sigma(E, E')$ bounded sets in E with the following properties:

i) $\mathfrak{A}$ contains all singletons.

ii) If A, B are in $\mathfrak{A}$ there is a C in $\mathfrak{A}$ with $A \cup B \subseteq C$.

iii) If A is in $\mathfrak{A}$ then $\lambda A \in \mathfrak{A}$ for any scaler λ.

Then sets of the form $\bigcap_{i=1}^{n} A_i^o$, $A_i \in \mathfrak{A}$, A_i^o the polar of A_i, form a base of neighbourhoods in E' for the polar topology $T_{\mathfrak{A}}$ of $\mathfrak{A}$ convergence. Such polar topologies are always separated. Different collections $\mathfrak{A}$ can generate the same polar topology but there is always a maximal collection $\mathfrak{A}$ corresponding to a given polar topology. Such collections are called saturated collections; given a collection $\mathfrak{A}$ the corresponding saturated collection is obtained by successively forming all finite unions, taking weakly-closed circled convex extensions, and finally taking all subsets. The saturated collection corresponding to finite sets in E generates $\sigma(E', E)$.

The following lemma is useful when working with polar topologies. Examples of its use follow and topics are suggested which might use the lemma.

LEMMA. <u>For saturated covers,</u> $\mathfrak{A} \subset \mathfrak{B}$ <u>if and only if</u> $T_{\mathfrak{A}} \subset T_{\mathfrak{B}}$. See [3, page 117].

PROPOSITION 1. <u>If</u> $\sigma(E', E) \subset T_{\mathfrak{A}}$ <u>properly then there is a polar topology</u> $T_{\mathfrak{B}}$ <u>with</u> $\sigma(E', E) \subset T_{\mathfrak{B}} \subset T_{\mathfrak{A}}$ <u>with all inclusions proper</u>.

<u>Proof</u>. Since $T_{\mathfrak{A}} \nsubseteq \sigma(E', E)$, there is an $A \in \mathfrak{A}$ which is not generated by saturating the collection of finite sets in E. Assume A is closed convex and circled and let A_1 be a maximal linearly independent subset of A. Suppose $A_1 = \{x_1, \ldots, x_n\}$, then any $x \in A$ has the form $\alpha_1 x_1 + \ldots + \alpha_n x_n$. Also there are weakly continuous linear forms x_i' such that $x_i'(x_j) = \delta_{ij}$, $1 \leq i, j \leq n$. Since $|\langle A, x_i' \rangle|$ is bounded there is a λ such that if $x = \alpha_1 x_1 + \ldots + \alpha_n x_n \in A$

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then $|\alpha_j| < \lambda$ for all j. Then A is a subset of the closed convex circled hull of $\{\lambda n x_1, \ldots, \lambda n x_n\}$ since $x = \alpha_1 x_1 + \ldots + \alpha_n x_n = \frac{\alpha_1}{n\lambda}(\lambda n x_1) + \ldots + \frac{\alpha_n}{n\lambda}(\lambda n x_n)$ and $\sum_{i=1}^{n} \left| \frac{\alpha_i}{n\lambda} \right| \leq 1$. This contradicts the way A was chosen. So A_1 must be infinite.

As in Proposition 2.2 of [1], choose a sequence $\{y_i\}$ from A_1 and let $\{x_i\}$, $\{x'_i\}$ be biorthogonal sequences with the span of $\{x_1, \ldots, x_n\}$ equal to the span of $\{y_1, \ldots, y_n\}$ for every n and $\{x'_i\}$ a sequence from E'. Since $x_n = \alpha_1 y_1 + \ldots + \alpha_n y_n$, for sufficiently small γ_n, $\gamma_n x_n \in A$. Let $B = \bigcap_{n=0}^{\infty} (x'_{2n+1})^{-1}(0)$. Then $x_{2n} \in B$, $x_{2n+1} \notin B$ for each n. Also B is a closed subspace of E. Set $B_1 = B \cap A$ and suppose $\mathfrak{B}$ is the saturated collection generated by the collection of singletons together with scalar multiples of B_1.

Since finite sets only generate finite dimensional sets under the process of saturation, $\sigma(E', E) \subset T_{\mathfrak{B}}$ properly since $B_1 \in \mathfrak{B}$ is infinite dimensional. To show $T_{\mathfrak{B}} \subset T_{\mathfrak{A}}$ properly let C be the closed convex circled hull of a finite set. Then the closed convex circled hull F of $B_1 \cup C$ is the closure of the set of elements of the form $\sum_{i=1}^{n} \alpha_i b_i + \sum_{i=1}^{m} \beta_i c_i$, $b_i \in B_1$, $c_i \in C$, and $\sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{m} |\beta_i| \leq 1$. Each of these elements is contained in $B_1 + C$ which is closed and bounded since C is compact. It follows that $F \subset B_1 + C \subset X_1 \oplus X_2$ where X_1 is the span of B_1 and X_2 is a finite dimensional subspace containing C. So $A \notin \mathfrak{B}$ and $T_{\mathfrak{B}} \subset T_{\mathfrak{A}}$ properly.

COROLLARY 1. <u>The only case in which the dual pair generates a finite number of polar topologies on E' is when all polar topologies are equal to $\sigma(E', E)$.</u>

COROLLARY 2. <u>If there is more than one topology for the dual pair (E', E) there are infinitely many. In fact there is a decreasing sequence of topologies for the dual pair in this case.</u>

COROLLARY 3. A non-trivial lattice of polar topologies for the dual pair (E', E) is without atoms.

The polar topologies under inclusion form a complete lattice. If $\{\mathfrak{A}_\alpha\}$ is a set of saturated collections of weakly bounded sets, $\vee \mathfrak{A}_\alpha$ is the saturated collection generated by $\cup \mathfrak{A}_\alpha$ and $\wedge \mathfrak{A}_\alpha = \cap \mathfrak{A}_\alpha$

In [2, p. 64] it is shown that if E is a linear topological space with continuous dual $E' = E^*$ then $\sigma(E', E) = \beta(E', E)$, that is, all polar topologies are the same. The lemma leads to the following partial converse. Recall that bounded mappings are continuous in bornological spaces.

PROPOSITION 2. If E is a bornological linear topological space with topology ξ and continuous dual E' and if all polar topologies are the same $(\sigma(E', E) = \beta(E', E))$ then $E' = E^*$.

Proof. We know β is generated by the saturated collection $\mathfrak{B}$ of all $\sigma(E, E')$ bounded sets in E while σ is generated by the collection $\mathfrak{A}$ which is the saturated family corresponding to finite sets in E. Since $\sigma = \beta$ by the lemma $\mathfrak{A} = \mathfrak{B}$, that is, the only $\sigma(E, E')$ bounded sets are obtained by the saturating operations on finite sets. Thus all ξ bounded sets are finite dimensional since E has the same bounded sets in the ξ and $\sigma(E, E')$ topologies. It follows that any $x' \in E^*$ is bounded and thus continuous since E is a bornological space. Thus $E' = E^*$.

The question of whether or not Proposition 2 holds for non-bornological spaces might be answered with the use of the lemma. Also the lemma appears to be the proper tool for investigating lattice properties in the lattice of polar topologies.

REFERENCES

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