A NOTE ON POSITIVE DEFINITE MATRICES

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1. This note is concerned with an inequality for even order positive definite hermitian matrices together with an application to vector spaces.

The abbreviations p.d. and p.s-d. are used for positive definite and positive semi-definite respectively. An asterisk denotes the conjugate transpose of a matrix.

THEOREM 1. *Let*

$$A = \begin{bmatrix} A_1 & B \\ B^* & A_2 \end{bmatrix} \qquad (1.1)$$

be an hermitian matrix of even order, partitioned in such a way that A_1, A_2 *and* B *are all* $n \times n$ *matrices. If* A *is p.d. or p.s-d. then*

$$|\det B|^2 = \det B \det B^* \leqslant \det A_1 \det A_2, \qquad (1.2)$$

and equality occurs if and only if either (i) A *is p.s-d.*, $\det A_1 > 0$, $\det A_2 > 0$ *and* $A_2 = B^* A_1^{-1} B$ *or* (ii) $\det A_1 \det A_2 = 0$.

This result leads to

THEOREM 2. *Let* V *be a vector space over the field of complex numbers and denote by* (,) *an inner product defined on* V. *If* $f_1, f_2, \dots, f_n$ *and* $g_1, g_2, \dots, g_n$ *all belong to* V, *then*†

$$|\det[(f_r, g_s)]|^2 \leqslant \det[(f_r, f_s)] \det[(g_r, g_s)] \quad (1 \leqslant r,\ s \leqslant n). \qquad (1.3)$$

A sufficient, but not necessary, condition for this inequality to be strict is that the set $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n$ *should be linearly independent.*

2. To prove Theorem 1 we use the fact that if an hermitian matrix A is p.d. or p.s-d. and T is any non-singular matrix (of suitable order), then T^*AT is p.d. or p.s-d. respectively. Also if the hermitian matrix A_1 is p.d. then the hermitian matrix A_1^{-1} is p.d.

The following inequality for $n \times n$ ($n > 1$) hermitian matrices is required [**1**, 420].

$$\det(A + B) \geqslant \det A + \det B, \qquad (2.1)$$

provided that A and B are p.d. or p.s-d. If A is p.d., there is equality in (2.1) if and only if B is the null matrix.

3. The case of Theorem 1 when $n = 1$ can be dismissed since the results follow at once from the definition of p.d. or p.s-d. hermitian matrices.

The proof of Theorem 1 when $n > 1$‡ follows from an idea in the book by Mirsky [see **1**, 426, example 37]. We assume throughout this section that

$$\det A_r > 0 \quad (r = 1, 2), \qquad (3.1)$$

i.e. that both A_1 and A_2 are p.d.

Define the matrix T by

$$T \equiv \begin{bmatrix} I_n & -A_1^{-1}B \\ 0 & I_n \end{bmatrix}, \quad I_n = [\delta_{rs}] \quad (1 \leqslant r,\ s \leqslant n).$$

† The factors on the right-hand side of (1.3) are real valued and non-negative. Cf. § 6.

‡ With this restriction (2.1) can be applied where necessary.

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Clearly $\det T = 1$, and it is at once verified that

$$T^*AT = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 - B^*A_1^{-1}B \end{bmatrix} \qquad \text{(3.2)}$$

From this it follows that if A is p.d. or p.s-d. then $A_2 - B^*A_1^{-1}B$ is p.d. or p.s-d. respectively. Also $B^*A_1^{-1}B$ is p.d. or p.s-d. according to whether B is non-singular or otherwise.

Writing

$$A_2 = A_2 - B^*A_1^{-1}B + B^*A_1^{-1}B,$$

we now have, from (2.1),

$$\det A_2 \geqslant \det (A_2 - B^*A_1^{-1}B) + \det B^*A_1^{-1}B\,; \qquad \text{(3.3)}$$

hence

$$\det A_2 \geqslant \det B^*A_1^{-1}B, \qquad \text{(3.4)}$$

i.e.

$$\det B \det B^* \leqslant \det A_1 \det A_2. \qquad \text{(3.5)}$$

If now A is p.d., then $A_2 - B^*A_1^{-1}B$ is p.d. and, whether B is singular or otherwise, there is strict inequality in (3.4) and hence in (3.5).

If on the other hand A is p.s-d. and B is singular† then, from (3.1) there is strict inequality in (3.5). Alternatively, if A is p.s-d. and B is non-singular, then $B^*A_1^{-1}B$ is p.d. and, from (2.1), there is equality in (3.3) and hence in (3.5), if and only if $A_2 - B^*A_1^{-1}B$ is the null matrix.

4. Contrary to (3.1), we suppose in this section that

$$\det A_1 \det A_2 = 0. \qquad \text{(4.1)}$$

Define the matrix A' by

$$A' \equiv A + \mu I_{2n} \quad (\mu > 0).$$

Clearly A' is p.d., so that (3.5) holds for this matrix. Letting $\mu \to 0$, we obtain

$$\det B = 0,$$

so that there is equality in (1.2) if (4.1) holds. This completes the proof of Theorem 1.

5. It seems worth mentioning that a sharper result than (1.2) holds.‡ If (3.1) is satisfied, then, from (3.2), we have

$$\det A = \det A_1 \det (A_2 - B^*A_1^{-1}B),$$

and using (3.3) we obtain the inequality

$$\det A + |\det B|^2 \leqslant \det A_1 \det A_2. \qquad \text{(5.1)}$$

It is clear that this result also holds when (4.1) is satisfied.

If $n = 1$, equality always holds in (5.1). Suppose then that $n > 1$. If A is p.s-d., then (5.1) reduces to (1.2), and the same conditions for equality hold. If A is p.d., then $A_2 - B^*A_1^{-1}B$ is p.d., from (3.2), and there is equality in (5.1) if and only if there is equality in (3.3), i.e. if and only if $B^*A_1^{-1}B$ is null; that is, if and only if B is null.

† With (3.1) satisfied this case is impossible for $n = 1$ but can occur if $n > 1$.

‡ The author is greatly indebted to a referee for this section and several other constructive comments.

6. To prove Theorem 2 we notice that if $f_1, f_2, \dots, f_n$ all belong to the vector space V, and if α_r $(1 \leqslant r \leqslant n)$ are n arbitrary complex numbers, then

$$\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r, f_s)\alpha_r \bar{\alpha}_s = \left(\sum_{r=1}^{n} \alpha_r f_r, \sum_{s=1}^{n} \alpha_s f_s \right) \geqslant 0, \qquad \text{.....................(6.1)}$$

by the well known properties of the inner product. Moreover there is equality in (6.1), for a non-trivial set $\{\alpha_r\}$, if and only if $\Sigma \alpha_r f_r = 0$, i.e. if and only if the set $\{f_r\}$ is linearly dependent.

Clearly then the hermitian matrix $[(f_r, f_s)]$ $(1 \leqslant r, s \leqslant n)$ is either p.d. or p.s-d. according as to whether the set $\{f_r\}$ is linearly independent or not.

With the hypothesis of Theorem 2 define the augmented set $\{F_r\,;\, 1 \leqslant r \leqslant 2n\}$ by

$$F_r = f_r, \quad F_{r+n} = g_r \quad (1 \leqslant r \leqslant n).$$

The matrix $[(F_r, F_s)]$ $(1 \leqslant r, s \leqslant 2n)$ then clearly satisfies all the conditions of Theorem 1, and (1.3) now follows by application of (1.2) to this matrix. If the set $\{F_r\}$ is linearly independent it also follows that there is strict inequality in (1.3).

7. It seems difficult to find a straightforward condition on the f_r and g_r of Theorem 2 which is both necessary and sufficient for strict inequality. That the condition given is not necessary, for vector spaces of both finite and infinite dimensions, is discussed in the following examples.

(*a*) Let V be the Hilbert space of L^2 functions over a measurable linear set E with the usual inner product.† For example, let $n = 2$ with $E = [0, 1]$ and

$$f_1(x) = 1, \quad f_2(x) = x, \quad g_1(x) = 1, \quad g_2(x) = x^2.$$

It may then be verified that strict inequality holds in (1.3), and yet the augmented set $\{F_r\}$ is clearly linearly dependent over E.

(*b*) Let V be a finite dimensional vector space of dimension d. It can be verified that the following results hold : (i) If $n > d$, both sides of (1.3) are zero. (ii) If $n = d$, then, whether the sets $\{f_r\}$ and $\{g_r\}$ are linearly dependent or not, there is equality in (1.3). (iii) If $n < d$, it is possible to have the augmented set $\{F_r\}$ linearly dependent and obtain strict inequality in (1.3). For example, if $n = 2$ and $d = 3$, put $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $g_1 = (1, 0, 0)$ and $g_2 = (0, 0, 1)$.

† In this case the inequality (1.3) is an extension of the Cauchy inequality for integrals.

REFERENCE

1. L. Mirsky, *An introduction to linear algebra* (Oxford, 1955).

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