THE CATEGORY OF GRAPHS WITH A GIVEN SUBGRAPH—WITH APPLICATIONS TO TOPOLOGY AND ALGEBRA

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1. Introduction. By a graph we mean a pair (X, R) where X is a non-void set and $R \subset X \times X$. A mapping $f: X \to Y$ is called a compatible map (or morphism) from (X, R) into (Y, S) if ${}^2f(R) \subset S$, where ${}^2f: X^2 \to Y^2$ is defined by ${}^2f((x_1, x_2)) = (f(x_1), f(x_2))$. The set of all compatible maps from (X, R) into itself forms a monoid (semigroup with a unit element) under composition, which is denoted by $M(X, R)$. A graph (X_1, R_1) is said to be a full subgraph of (X, R) if $X_1 \subset X$ and $R_1 = R \cap (X_1 \times X_1)$. A graph (X, R) is said to be without loops if $(x, x) \notin R$ for all $x \in X$. Our aim in this paper is to prove a generalization and to show applications of the following theorem.

THEOREM 1. *Let M be any monoid, (X_1, R_1) any graph without loops. Then there exists a graph (X, R) such that (X_1, R_1) is a full subgraph of (X, R) and $M(X, R)$ is isomorphic to M.*

Using the methods introduced in (**2**), it is easy to show that this theorem implies the following result.

THEOREM 2. *For any pair of monoids M_1, M_2 there exists a semigroup S_1 with a subsemigroup S_2 such that the monoid of all endomorphisms of S_i is isomorphic to M_i for $i = 1, 2$.*

Roughly speaking, the theorem states that in general there is no relationship between the monoid of endomorphisms of a semigroup and the monoids of endomorphisms of its subsemigroups. Using (**4**), one obtains a similar result for universal algebras. A similar result also holds for topological spaces. The following theorem is a generalization of a theorem of de Groot (**1**).

THEOREM 3. *For any pair of groups G_1, G_2 there is a complete metric space T_1 with a complete subspace T_2 such that the group of all autohomeomorphisms of T_i is isomorphic to G_i, $i = 1, 2$.*

In the proofs of the theorems we will use some graph-theoretic ideas and

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some methods and language of category theory. In the next two sections we will state the definitions, notation, and results which are used in the proofs.

2. Graph-theoretic notions. If (X, R) and (Y, S) are graphs, then $M((X, R), (Y, S))$ denotes the set of all morphisms from (X, R) into (Y, S). We shall use $M(X, R)$ as an abreviation for $M((X, R), (X, R))$. This agrees with $M(X, R)$ as defined in the introduction.

$f \in M((X, R), (Y, S))$ is called an isomorphism if $f: X \to Y$ is one-to-one and onto and $f^{-1} \in ((Y, S), (X, R))$. (X, R) and (Y, S) are called isomorphic if there exists an isomorphism between (X, R) and (Y, S). If there is no danger of misunderstanding we shall not distinguish isomorphic graphs.

If (X, R) is a graph, we let $l(X, R)$ be the graph $(X, l(R))$, where $(x_1, x_2) \in l(R)$ if and only if $(x_1, x_2) \in R$ or $(x_2, x_1) \in R$. Similarly, $r(X, R)$ is the graph $(X, r(R))$, where $(x_1, x_2) \in r(R)$ if and only if $(x_1, x_2) \in R$ and $(x_2, x_1) \in R$.

A graph (X, R) is called undirected (symmetric), if $R = l(R)$. (We use here l and r to indicate the relationship with the left and right adjoint functor to the inclusion functor from the category of undirected graphs into the category of (directed) graphs.)

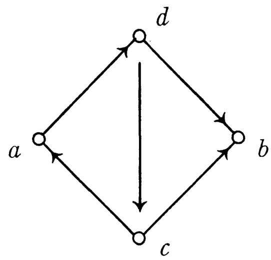
If (X, R) is a graph, $x, x' \in X$, then x and x' are said to be in the same component if either $x = x'$ or there is a finite sequence $x_1 = x, x_2, \ldots, x_n = x'$ such that $(x_i, x_{i+1}) \in l(R)$ for every $i = 1, 2, \ldots, n - 1$. The relation "to be in the same component" is evidently an equivalence on X. The equivalence classes are called components. (X, R) is called connected, if X is a component. In the proofs we shall use the fact that if $f \in M((X, R), (Y, S))$ then every component of (X, R) is mapped by f into a component of (Y, S). (X, R) is said to have no isolated points, if it has no one-point components which are not loops.

A graph (X, R) is called rigid, if $M(X, R) = 1_X$, where 1_X denotes the identity mapping on X. We shall make use of the following assertion, proof of which is found in (**2**): *If X is any set, then there exists a connected rigid graph (X, R).*

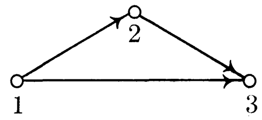
A concept of strongly rigid graphs introduced in (**3**) plays an important role in this paper. To define it we need some auxiliary notions.

Let (X, R) and (Y, S) be graphs, $a, b \in X$, $(a, b) \notin l(R)$, $a \neq b$, (Y, S) being without loops or isolated points. We define a graph, denoted by $(X, R, a, b) * (Y, S)$, which has the following intuitive meaning: every edge s of (Y, S) (i.e. $s \in S$) is replaced by a copy of (X, R) in such a way that a is the beginning and b the end point of s, alternatively for each $(y_1, y_2) \in S$ we take a copy of (X, R); these copies are taken to be disjoint except for the points a, b which are chosen to be y_1 and y_2, respectively, if the copy is associated with the pair (y_1, y_2). To clarify the idea we shall show an easy example, where we draw an arrow from a to b, where $(a, b) \in R$. Let

$$(X, R) = (\{a, b, c, d\}, \{(a, d), (d, b), (c, b), (c, a), (d, c)\}),$$
$$(Y, S) = (\{1, 2, 3\}, \{(1, 2), (2, 3), (1, 3)\}).$$

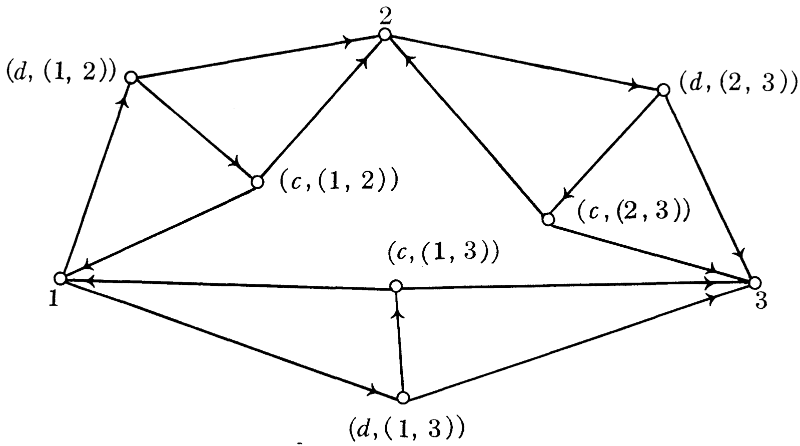


(X, R)



(Y, S)

Then $(X, R, a, b) * (Y, S)$ is pictorially the following graph:



To define $(X, R, a, b) * (Y, S)$ formally, we introduce an auxiliary notation. Put $T = X - \{a, b\}$ and $\bar{Y} = Y \cup (S \times T)$. If $s = (y_1, y_2) \in S$, define a mapping $f_s\colon X \to Y$ by $f_s(a) = y_1, f_s(b) = y_2, f_s(t) = (t, s)$ for $t \in T$.

Now, we are able to define $(X, R, a, b) * (Y, S) = (\bar{Y}, \bar{S})$ by:

$$\bar{Y} = Y \cup (S \times T),$$

$(\bar{y}_1, \bar{y}_2) \in \bar{S}$ if and only if there exists $(x_1, x_2) \in R$ and $s = (y_1, y_2)$, $s \in S$, such that $f_s(x_i) = \bar{y}_i$, $i = 1, 2$.

Observe that $s \neq 1$, $s, t, \in S$, implies $\operatorname{card}(f_s(X) \cap f_t(X)) \leqq 2$. Moreover, the only possible elements of the last intersection are $f_s(a)$, $f_s(b)$, $f_t(a)$, and $f_t(b)$. Since we assumed that $(a, b) \notin R$, $(b, a) \notin R$, we have

$$^2f_s(R) \cap {}^2f_t(R) = \emptyset.$$

It follows that for every $s \in S$, the graph $(f_s(X), \bar{S} \cap (f_s(X) \times f_s(X)))$ is isomorphic to (X, R) under the isomorphism f_s if f_s is considered to be a mapping from X into $f_s(X)$. We shall need the following easy lemma.

LEMMA 1. *If (X, R) is undirected, then $(\bar{Y}, \bar{S})$ is undirected. If (X, R) and (Y, S) are connected, then $(X, R, a, b) * (Y, S)$ is connected.*

A graph (X, R) is called strongly rigid with respect to a, b $(a, b \in X)$ if for every (Y, S) without loops or isolated points, any morphism

$$f\colon (X, R) \to (X, R, a, b) * (Y, S)$$

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can be written in the form $f = f_s$, for some $s \in S$. Sometimes we say that (X, R, a, b) is strongly rigid.

If k is a cardinal number, let $W_k = (Z_k, T_k)$ be the following graph: Z_k is the set of all ordinals less than k and $(x, y) \in T_k$ if and only if $x < y$ in the usual sense of well-ordering of ordinals. Let C_k be the graph $l(W_k) = (Z_k, l(T_k))$, called the complete k-point graph.

A graph $G = (X, R)$ is said to be k-colourable if $M(G, C_k) \neq \emptyset$. Let $c(G)$ be the least cardinal such that $M(G, C_k) \neq \emptyset$. $c(G)$ is called the chromatic number of G. It is easy to see that these definitions coincide with those given in classical graph theory.

LEMMA 2. $c(W_k) = k$.

Proof. By definition, $T_k \subset l(T_k)$, and hence ${}^2 1_Z(T_k) = T_k \subset l(T_k)$, where 1_Z denotes the identity transformation of Z_k. Hence, $1_Z \in M(W_k, C_k)$ and $c(W_k) \leqq k$.

If j is a cardinal, $j < k$, then for every $f: Z_k \to Z_j$ there are $z_1, z_2 \in Z_k$ such that $f(z_1) = f(z_2)$, $z_1 < z_2$. If $f \in M(W_k, C_j)$, then $(f(z_1), f(z_2)) = (f(z_1), f(z_1)) \in l(T_j)$ since $(z_1, z_2) \in T_k$. We obtain a contradiction with the statement C_j has no loops.

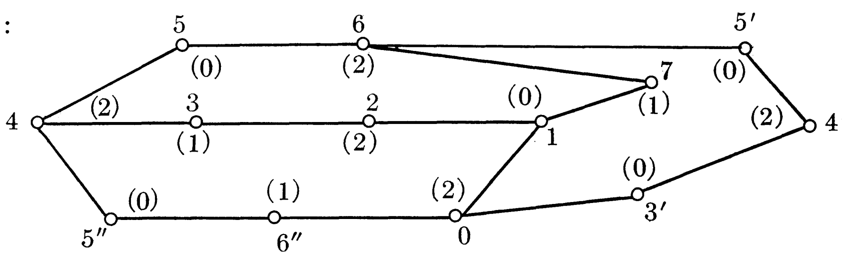
LEMMA 3. *Let (Y, S) be a graph without loops or isolated points, $c((X, R)) = k$, $a, b \in X$, $a \neq b$. If there is a $g \in M((X, R), C_k)$ such that $g(a) = g(b)$, then $c((X, R, a, b) * (Y, S)) = k$.*

Proof. It is easy to see that the following g^* is a morphism from $(X, R, a, b) * (Y, S)$ into C_k: $g^*(y) = g(a)$ for all $y \in Y$, $g^*((t, s)) = g(t)$ for all $t \in T$, $s \in S$.

Since $g(a) = g(b)$, we have $(a, b) \notin R$ and $(b, a) \notin R$. Hence, by our remark concerning the definition $(X, R, a, b) * (Y, S)$, this graph contains an isomorphic copy of (X, R) as a full subgraph. Hence, $c((X, R, a, b) * (Y, S)) \geqq k$, which completes the proof.

In (**3**) it was shown that the following undirected graph (P, V) is strongly rigid with respect to points $5'$ and $5''$.

(P,V):



$$P = \{0, 1, 2, 3, 4, 5, 6, 7, 3', 4', 5', 5'', 6''\}, \qquad V = l(R),$$

where

$$R = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 1), (0, 3'), (3', 4'), (4', 5'), (5', 6), (4, 5''), (5'', 6''), (6'', 0)\}.$$

It is easy to see that the mapping $g\colon P \to Z_3$, defined by $g(i) = 0$ for $i = 1, 3', 5, 5', 5''$, $g(j) = 1$ for $j = 3, 6'', 7$, and $g(k) = 2$ for $k = 0, 2, 4, 4', 6$, is a morphism from (P, V) into Z_3 such that $g(5') = g(5'')$. (In the picture, the values of g are in the brackets.) Evidently, $c((P, V)) = 3$. Using Lemma 3 we obtain that if (Y, S) is any graph without loops or isolated points, then $c((P, V, 5', 5'') * (Y, S)) = 3$. Since (P, V) is undirected, $(P, V, 5, 5') * (Y, S)$ is undirected by Lemma 1.

3. Categorical notions. A category $\mathscr{A}$ is determined by a class of objects $\mathscr{A}^{\mathrm{o}}$ and class of morphisms $\mathscr{A}^{\mathrm{m}}$ subject to the following conditions.

With every pair of objects $A_1, A_2 \in \mathscr{A}^{\mathrm{o}}$ a set of morphisms $M(A_1, A_2) \subset \mathscr{A}^{\mathrm{m}}$ is associated ($M(A_1, A_2)$ is usually described as the set of morphisms from A_1 into A_2); and with every $f \in A^{\mathrm{m}}$ there exists a unique pair A_1, A_2 such that $f \in M(A_1, A_2)$.

A composition of morphisms is defined so that:

(1) if $f \in M(A_1, A_2)$, $g \in M(A_2, A_3)$, then the composition of g and f denoted by gf is defined and $gf \in M(A_1, A_3)$,

(2) for all $f, g, h \in \mathscr{A}^{\mathrm{m}}$, we have $f(gh) = (fg)h$ whenever defined,

(3) for every $A \in \mathscr{A}^{\mathrm{o}}$, there is $1_A \in M(A, A)$ such that $f1_A = f$, $1_A g = g$ whenever the composition is defined.

If we wish to stress the category in which we are working, we sometimes write $M_{\mathscr{A}}(A_1, A_2)$ instead of $M(A_1, A_2)$.

Considering graphs as objects, compatible mappings as morphisms, and usual composition as composition, we obtain a category $\mathscr{R}$. (1) is satisfied since composition of compatible mappings is a compatible mapping; (2) is the usual associativity of mappings; and (3) is satisfied since every identity mapping is compatible.

If $\mathscr{A}$ and $\mathscr{B}$ are categories, a functor $F\colon \mathscr{A} \to \mathscr{B}$ is an assignment of an object $F(A) \in \mathscr{B}^{\mathrm{o}}$ to each $A \in \mathscr{A}^{\mathrm{o}}$, and a morphism $F(f)\colon F(A) \to F(A')$ to each morphism $f\colon A \to A'$, $A' \in \mathscr{A}^{\mathrm{o}}$, subject to the following conditions:

(1) if gf is defined in $\mathscr{A}$, then $F(gf) = F(g)F(f)$;

(2) for every $A \in \mathscr{A}^{\mathrm{o}}$, $F(1_A) = 1_{F(A)}$.

A category $\mathscr{A}$ is called a full subcategory of $\mathscr{B}$ if $\mathscr{A}^{\mathrm{o}} \subset \mathscr{B}^{\mathrm{o}}$ and $M_{\mathscr{A}}(A_1, A_2) = M_{\mathscr{B}}(A_1, A_2)$ for every $A_1, A_2 \in \mathscr{A}^{\mathrm{o}}$.

A functor $F\colon \mathscr{A} \to \mathscr{B}$ is called full embedding if the association f to $F(f)$ is a one-to-one onto function from $M_{\mathscr{A}}(A_1, A_2)$ onto $M_{\mathscr{B}}(F(A_1), F(A_2))$ for all objects $A_1, A_2 \in \mathscr{A}^{\mathrm{o}}$, and F is one-to-one on objects.

Denote by $\mathscr{R}_e$ the full subcategory of $\mathscr{R}$ whose objects are all graphs without loops and isolated points. We shall make use of the following proposition proved in (**2**).

PROPOSITION 1. *There exists a full embedding of $\mathscr{R}$ into $\mathscr{R}_e$.*

4. Full embeddings. The following proposition is a categorial formulation of (**2**, Theorem 1).

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PROPOSITION 2. *Let* (X, R, a, b) *be a strongly rigid graph. Then the mapping which associates with every* $(Y, S) \in \mathscr{R}_e{}^o$ *a graph* $(X, R, a, b) * (Y, S) \in \mathscr{R}^o$ *can be extended to a functor which is a full embedding of* $\mathscr{R}_e$ *into* $\mathscr{R}$.

Proof. Denote this functor by F. Then $F(Y, S) = (X, R, a, b) * (Y, S)$. Let g be any morphism from (Y, S) into (Y', S'). Define $F(g)\colon F(Y, S) \rightarrow F(Y', S')$ by:

$$F(g)(y) = g(y) \quad \text{for } y \in Y,$$

$$F(g)(x, s) = (x, {}^2g(s)) \quad \text{for } (x, s) \in T \times S.$$

Since ${}^2g(s) \in S'$, this map is well-defined. We must prove that it is a morphism in $\mathscr{R}$. Let $F(Y, S) = (\bar{Y}, \bar{S})$, $F(Y', S') = (\bar{Y}', \bar{S}')$. If $(\bar{y}_1, \bar{y}_2) \in \bar{S}$, then there are $(x_1, x_2) \in R$ and $s \in S$ such that $f_s(x_i) = \bar{y}_1$, $i = 1, 2$. Now, by the definition of $F(g)$, $F(g)(\bar{y}_i) = f_t(x_i)$, $i = 1, 2$, where $t = {}^2g(s)$. Thus, ${}^2(F(g))(\bar{y}_1, \bar{y}_2) \in \bar{S}'$ by the definition of $\bar{S}'$. Thus, $F(g)$ is a morphism from $F(Y, S)$ into $F(Y', S')$. Since $Y \subset F(Y, S)$ and $F(g)$ restricted to Y is equal to g, $g_1 \neq g_2$ implies $F(g_1) \neq F(g_2)$. It remains to prove that F maps $\mathscr{R}_e$ onto a full subcategory of $\mathscr{R}$.

Let h be a morphism from $F(Y, S)$ into $F(Y', S')$. Let $y_1 \in Y$. Since (Y, S) has no isolated points, there exists $y_2 \in Y$ such that $(y_1, y_2) \in S$ or $(y_2, y_1) \in S$. We may assume without loss of generality that $(y_1, y_2) \in S$. Consider the graph $(f_s(X), {}^2f_s(R))$, where $s = (y_1, y_2)$. It is isomorphic to (X, R) and is a full subgraph of $F(Y, S)$. Since (X, R, a, b) is strongly rigid, h restricted to $f_s(X)$ must be equal to f_t for some $t \in S'$. Let $t = (y_1', y_2')$. Then $h(y_1) = y_1'$, $h(y_2) = y_2'$, and $h(x, s) = (x, t)$, $x \in T$. Thus h restricted to Y is a morphism, called g^*, from (Y, S) into (Y', S').

Let (x, s) be any point in $\bar{Y} - Y$. If $s = (y_1, y_2)$, then we know that $h(y_1) = y_1'$, $h(y_2) = y_2'$ for some $(y'_1, y_2') \in S'$. Furthermore, $(f_s(X), {}^2f_s(R))$ is an isomorphic copy of (X, R) in $(\bar{Y}, \bar{S})$. Since (X, R, a, b) is strongly rigid, $hf_s = f_t$ for some $t \in S'$. However, $h(y_1) = y_1'$, $h(y_2) = y_2'$. Hence $h((x, s)) = (x, t) = (x, {}^2g^*(s))$. Since $F(g^*)(y) = g^*(y)$ and $F(g^*)((x, s)) = (x, {}^2g^*(s))$, we have $F(g^*) = h$ and F is a full embedding.

COROLLARY 1. *For any cardinal* k *there exists a rigid undirected connected graph,* (Y, S), *such that* $c((Y, S)) = 3$ *and* $\operatorname{card} Y > k$.

Proof. It was proved in (**2**) that for every cardinal k there exists a rigid connected graph (X_1, R_1) such that $\operatorname{card} X_1 > k$. Since (X_1, R_1) is rigid, it has no loops if $\operatorname{card} X_1 > 1$. Let $(P, V, 5, 5'')$ denote the strongly rigid undirected graph introduced above. Using Lemma 1, the graph

$$(Y, S) = (P, V, 5', 5'') * (X_1, R_1)$$

is undirected and connected. By Lemma 3, $c((Y, S)) = 3$. Evidently, $\operatorname{card} Y > k$. Now, using Proposition 2, we conclude that (Y, S) is rigid.

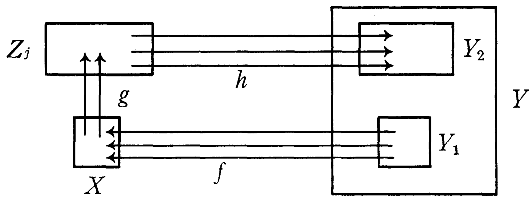
A point $x \in X$ is called a dead end of (X, R), if $(y, x) \notin R$ for all $y \in X$.

PROPOSITION 3. *Every graph (X, R) without loops is a full subgraph of a rigid connected graph (X^*, R^*) without dead ends.*

Proof. Let (X, R) be the graph without loops, card $X = i$. Choose a cardinal j such that $j > i + 3$. We shall make use of the graph $W_j = (Z_j, T_j)$, where Z_j is the set of all ordinals smaller than j and T_j is the strict well-ordering relation. Further, choose a cardinal $k > i + j$ and let (Y, S) be a rigid undirected connected graph such that card $Y > k$ and $c((Y, S)) = 3$.

Now, we introduce three auxiliary mappings which will enable us to define the graph (X^*, R^*).

Choose two disjoint subsets Y_1, Y_2 of Y such that card $Y_1 = i$, card $Y_2 = j$. Let $f: Y_1 \to X$ be any one-to-one onto mapping, $g: X \to Z_j$ any one-to-one mapping, and $h: Z_j \to Y_2$ any one-to-one onto mapping. By the assumptions on the cardinals, such mappings exist. The situation is illustrated by the following figure.



We may assume that X, Y, and Z_j are mutually disjoint sets (otherwise we can achieve this by proper indexing).

Put

$$X^* = X \cup Y \cup Z_j;$$

$$R^* = R \cup S \cup T_j \cup \{(y, f(y)) \mid y \in Y_1\}$$
$$\cup \{(x, g(x)) \mid x \in X\} \cup \{(z, h(z)) \mid z \in Z_j\}.$$

We claim that (X^*, R^*) is the required graph.

First it is clear that (X^*, R^*) is without dead ends, as dead end cannot appear in Y since S is symmetric, cannot appear in X since $(x, g(x)) \in R^*$, $x \in X$, and cannot appear in Z_j since $(z, h(z)) \in R'$, $z \in Z_j$.

It is easy to see that (X^*, R^*) is connected since (Y, S) and (Z_j, T_j) are connected graphs.

From the definition of R^*, it follows that (X, R) is a full subgraph of (X^*, R^*).

It remains to prove that (X^*, R^*) is rigid. Take any $f: X^* \to X^*$ such that $f \in M(X^*, R^*)$. Observe that $f \in M(X^*, l(R^*))$ and $f \in M(X^*, r(R^*))$.

We shall use the fact that $f \in M(X^*, l(R^*))$ to show that $f(Z_j) \subset Z_j$. The graph (Z_j, T_j) is a full subgraph of (X^*, R^*) and $(Z_j, l(T_j))$ is a full subgraph of $(X^*, l(R^*))$. The graph $(Z_j, l(T_j))$ is a complete j-point graph, $j > 3$. Since $(X^*, l(R^*))$ has no loops, the mapping f restricted to Z_j must be one-to-one. Since card $X < j$, we have $f(Z_j) \not\subset X$. Since $c((Y, S)) = 3$, we have $f(Z_j) \not\subset Y$; otherwise $c((Z_j, T_j)) \leqq 3$, which is a contradiction to $c((Z_j, T_j)) = j$.

Hence, $f(Z_j) \cap Z_j \neq \emptyset$. Assume that $f(Z_j) \cap X \neq \emptyset$. Since the image of $(Z_j, l(T_j))$ is a complete graph, it follows that card$(f(Z_j) \cap X) = 1$, since for each point in X there is exactly one $z \in Z$ such that $(x, z) \in R^*$. It follows that card $Z_j \leqq 2$, a contradiction. A similar argument shows that

$$f(Z_j) \cap Y = \emptyset.$$

We conclude that $f(Z_j) \subset Z_j$.

Now, using the fact that $f \in M(X^*, r(R^*))$ we shall show that $f(Y) \subset Y$. Consider the components of $(X^*, r(R^*))$. By definition, $(X^*, r(R^*))$ is obtained from (X^*, R^*), omitting all edges which are oriented in only one way. It is easy to see that each component of $(X^*, r(R^*))$ is a subset of one of the sets X, Y, Z_j, the set Y forms a component since (Y, S) is connected undirected and each point in Z_j is a component. Since card $Y > 1$, we have $f(Y) \cap Z_j = \emptyset$. Since a component is mapped by a morphism into a component, there are two possibilities; either $f(Y) \subset X$ or $f(Y) \subset Y$. Assume that $f(Y) \subset X$. If $z \in Z_j$, then $(z, h(z)) \in R^*$ and $h(z) \in Y$. Hence, $(f(z), f(h(z))) \in R^*$, $f(z) \in Z_j$, $f(h(z)) \in X$. However this is impossible since there is no pair $(z_1, x_1) \in R^*$ such that $z_1 \in Z_j$ and $x_1 \in X$. We have $f(Y) \subset Y$. Since (Y, S) is rigid, $f(y) = y$ for all $y \in Y$.

Let $x \in X$. There is exactly one $y_1 \in Y_1$ and $z \in Z_j$ such that $(y_1, x) \in R^*$, $(x, z) \in R^*$. Hence $(f(y_1), f(x)) = (y_1, f(x)) \in R^*$ and we have $f(x) \in Y \cup X$. Furthermore, $(f(x), f(z)) \in R^*$ and since $f(z) \in Z_j$, we have $f(z) \in X \cup Z_j$. We conclude that $f(x) \in X$ and $f(x) = x$ for $x \in X$. Since there is exactly one y_2 for every $z \in Z_j$ such that $(z, y_2) \in R^*$ and $(f(z), f(y_2)) = (f(z), y_2) \in R^*$, we conclude that $f(z) = z$ and (X^*, R^*) is rigid. The proof is complete.

If (X, R) is a graph, put $I(x) = \{y \mid (y, x) \in R\}$ for $x \in X$.

LEMMA 4. *Let (X', R') be a rigid connected graph without dead ends. Assume that there exist sets $A, B \subset X'$ such that $A \not\subset B$, $B \not\subset A$ and $A \not\subset I(x)$, $B \not\subset I(x)$ for all $x \in X'$. If* 0, 1 $\notin X'$, *then the graph*

$$(X' \cup \{0\} \cup \{1\}, R' \cup \{A \times \{0\}\} \cup \{B \times \{1\}\})$$

is strongly rigid with respect to the points 0 *and* 1.

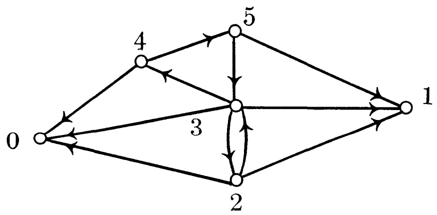
Proof. Put $X = X' \cup \{0\} \cup \{1\}$, $R = R' \cup \{A \times \{0\}\} \cup \{B \times \{1\}\}$. If (Y, S) is a graph without loops and isolated points, consider a morphism $f: (X, R) \to (X, R, 0, 1) * (Y, S) = (\bar{Y}, \bar{S})$. It follows immediately from the definition that all points in Y are dead ends of the graph $(\bar{Y}, \bar{S})$. Hence, f must map X' into the set $\bar{Y} - Y$. However, (X', R') is connected; hence X' must be mapped into a component of the graph

$$(\bar{Y} - Y', \bar{S} \cap (\bar{Y} - Y') \times (\bar{Y} - Y')).$$

It is easy to prove that these components are all isomorphic to (X', R'). Since (X', R') is rigid, it follows that f restricted to X' is equal to f_s restricted to X' for some $s \in S$. Consider $f(0)$ and $f(1)$. Since $(x_0, 0) \in R$ for every $x_0 \in A$, $(f(x_0), f(0)) = (x_0, f(0)) \in \bar{S}$. Similarly, $(x_1, 1) \in R$ for every $x_1 \in B$, hence $(f(x_1), f(1)) = (x_1, f(1)) \in \bar{S}$. If $f(0) \in f_s(X')$ say $f(0) = (x, s)$, then $A \subset I(x)$, which contradicts our assumption. Similarly, $f(1) \notin f_s(X')$.

It remains to prove that if $s = (y_1, y_2)$, then $f(0) = y_1$ and $f(1) = y_2$. However, it follows easily from the fact that only arrows from $f(X')$ in (Y, S) go to the points y_1, y_2. Since $A \not\subset B$ and $B \not\subset A$, we conclude that $f(0) = y_1$ and $f(1) = y_2$.

Example. $(X', R') = (\{2, 3, 4, 5\}, \{(2, 3), (3, 2), (3, 4), (4, 5), (5, 3)\})$. (X', R') is rigid and without dead ends. The sets $A = \{2, 3, 4\}$ and $B = \{2, 3, 5\}$ meet the requirements of Lemma 4. Hence, the following graph (X', R') is strongly rigid with respect to points 0 and 1.



If (X, R) is a graph, let $\mathscr{R}(X, R)$ be the full subcategory of $\mathscr{R}$ whose objects are all the graphs containing (X, R) as a full subgraph.

MAIN THEOREM. *Let (X, R) be a graph without loops. Then there exists a full embedding of $\mathscr{R}$ into the category $\mathscr{R}(X, R)$.*

Proof. Using Proposition 3, the graph (X, R) is a full subgraph of a rigid connected graph (X^*, R^*) without dead ends, where $X^* = X \cup Y \cup Z_j$. If we put $A = Z_j \cup Y$, $B = Y \cup X$, then it is easy to show that $A, B \not\subset I(x)$ in (X^*, R^*) for any $x \in X^*$. Hence, using Lemma 4, we obtain a graph which is strongly rigid with respect to points 0 and 1 and contains (X, R) as a full subgraph. Let this graph be (X, R).

It has been found in (**2**) that $\mathscr{R}$ can be fully embedded into the category $\mathscr{R}_e$. By Proposition 2, choosing for the strongly rigid graph the graph (X, R) with respect to points 0, 1, we obtain a full embedding of $\mathscr{R}_e$ into $\mathscr{R}(X, R)$. Composing the two full embeddings we obtain the required full embedding.

5. Applications. Let $\mathscr{O}$ be a one-object category. Then, evidently $\mathscr{O}^{\mathrm{m}}$ forms a monoid under composition; and conversely, every monoid can be considered as $\mathscr{O}^{\mathrm{m}}$ for a one-object category. In (**2**) it was shown that $\mathscr{O}$ can be fully embedded into $\mathscr{R}_e$. Combining this assertion with the main theorem, we can present a proof of Theorem 1.

Proof of Theorem 1. Let M be a monoid, $\mathscr{O}$ a one-object category such that $\mathscr{O}^{\mathrm{m}}$ is isomorphic to M. Let $F: \mathscr{O} \to \mathscr{R}_e$ be the full embedding the existence of which was shown in (**2**). Let $G: \mathscr{R}_e \to \mathscr{R}(X_1, R_1)$ be the full embedding given by the main theorem. Then the image of the object in $\mathscr{O}$ under GF is a graph (X, R) with a full subgraph (X_1, R_1) such that $M(X, R)$ is isomorphic to M.

COROLLARY 2. *If M_1 and M_2 are any two monoids, then there exist graphs (X_1, R_1) and (X_2, R_2) such that (X_1, R_1) is a full subgraph of (X_2, R_2) and $M(X_i, R_i)$ is isomorphic to M_i, $i = 1, 2$.*

Proof. Given M_1, there is a graph (X_1, R_1) without loops such that $M(X_1, R_1)$ is isomorphic to M_1. Putting $M = M_2$ and applying Theorem 1 we obtain the required graphs.

Let $\mathscr{Z}$ be the class of all ordinals, $\mathscr{M}$ the category with monoids as objects, and their homomorphisms as morphisms. Let $f: \mathscr{Z} \to \mathscr{M}^{\mathrm{o}}$ be any function from $\mathscr{Z}$ into $\mathscr{M}^{\mathrm{o}}$. (We must, of course, regard f as a subclass of $\mathscr{Z} \times \mathscr{M}^{\mathrm{o}}$ rather than a subset. This is to be understood for all functions whose domain is a class.) We can generalize Corollary 2 as follows.

PROPOSITION 4. *Let f be any function from $\mathscr{Z}$ to $\mathscr{M}^{\mathrm{o}}$. Then there exists a function $g: \mathscr{Z} \to \mathscr{R}_e^{\mathrm{o}}$ such that:*

(i) *$g(i)$ is isomorphic to a full subgraph of $g(i')$ for $i < i'$, $i, i' \in \mathscr{Z}$;*

(ii) *$M(g(i))$ is isomorphic to $f(i)$ for all $i \in \mathscr{Z}$.*

Proof. Define g inductively as follows:

(a) $g(0) = (X_0^*, R_0^*)$, where (X_0^*, R_0^*) is obtained from a full embedding of $f(0)$ as one-object category into $\mathscr{R}_e$;

(b) assume that $i > 0$ and $g(j)$ is defined for all $j < i$;

(c) if $i = k + 1$, put $g(i) = (X_i^*, R_i^*)$, where (X_i^*, R_i^*) is the graph (X, R) obtained from Theorem 1 by putting $M = f(i)$, $(X_1, R_1) = g(k)$;

(d) if i is a limit ordinal, put $g(i) = (X_1^*, R_1^*)$, where (X_1^*, R_1^*) is the graph (X, R) obtained from Theorem 1 by putting $M = f(i)$, and $(X_1, R_1) = (\bigcup Y_j \mid j < i, \bigcup S_j \mid j < i)$, where (Y_j, S_j) is isomorphic to (X_j^*, R_j^*) and Y_j are mutually disjoint.

To be able to apply the results to semigroups we need a slight modification of Theorem 1. First, a notation: Let $\mathscr{R}_i$ be the full subcategory of $\mathscr{R}$ whose objects are all those graphs (X, R) such that:

(i) for any $x \in X$, $(x, x) \notin R$,

(ii) for any $x \in X$, there is $y \in X$ such that $(y, x) \in R$.

THEOREM 1′. *Let M be any monoid, (X_1, R_1) any graph without loops. Then there exists a graph (X, R) such that (X_1, R_1) is a full subgraph of (X, R), $M(X, R)$ is isomorphic to M, and (X, R) is an object of $\mathscr{R}_i$.*

Proof. First, we use Proposition 3 for a graph (X_1, R_1) and obtain a rigid connected graph (X^*, R^*) without dead ends. It is easy to see that if we chose the function g in the proof such that $g(x) = 0$ for some $x \in X$, we have $(X^*, R^*) \in \mathscr{R}_i^\circ$.

Applying Lemma 4 to the graph (X^*, R^*) we obtain a strongly rigid graph $(X^{**}, R^{**}) \in \mathscr{R}_i^\circ$. Using this graph as a strongly rigid graph in Proposition 2 we easily see that the main theorem can be modified as follows: *Let (X, R) be a graph without loops. Then there exists a full embedding of R into the category $\mathscr{R}(X, R) \cap \mathscr{R}_i$.* Theorem 1′ then easily follows.

Let $\mathscr{R}_u$ be the full subcategory of $\mathscr{R}$ such that (X, R) is an object of $\mathscr{R}_u$ if and only if:

(i) for any $x \in X$, we have $(x, x) \notin R$,

(ii) for any $x \in X$, there is $y \in X$ such that $(x, y) \in R$.

If $G = (X, R)$ is a graph, put $i(G) = (X, i(R))$, where $(x, y) \in i(R)$ if and only if $(y, x) \in R$.

The following functor $I: \mathscr{R} \to \mathscr{R}$ is an isomorphism:

$$I(G) = i(G), \qquad I(f) = f.$$

Using this functor, it follows immediately that Theorem 1′ is equivalent to the following theorem.

THEOREM 1″. *Let M be any monoid, (X_1, R_1) any graph without loops. Then there exists a graph (X, R) such that (X_1, R_1) is a full subgraph of (X, R), $M(X, R)$ is isomorphic to M, and (X, R) is an object of $\mathscr{R}_u$.*

In (**2**) it was also proved that $\mathscr{R}_u$ (see **2**, the remark before Proposition 2) is fully embeddable in the category $\mathscr{S}$, whose objects are semigroups and whose morphisms homomorphisms, by a functor F which fulfills the following condition: If G_1 is a full subgraph of G, then $F(G_1)$ is a subsemigroup of $F(G)$. Using this full embedding and Proposition 4, we obtain a generalization of Theorem 2.

PROPOSITION 5. *Let f be any function from $\mathscr{Z}$ into $\mathscr{M}^\circ$. Then there exists $g: \mathscr{Z} \to \mathscr{S}^\circ$ such that*

(i) *$g(i)$ is isomorphic to a subsemigroup of $g(i')$ for $i < i'$, $i, i' \in \mathscr{Z}$,*

(ii) *the monoid of all endomorphisms of $g(i)$ is isomorphic to $f(i)$ for all $i \in \mathscr{Z}$.*

Let $\Delta = \{n_i\}$, $i \in I$, be a type and $A(\Delta)$ the category of algebras of type Δ. Δ is said to be non-trivial if and only if at least one of the $n_i \geqq 2$ or at least two of the $n_i = 1$. In (**4**) it was shown that there are full embeddings of $\mathscr{R}$ into $\mathscr{A}(2)$, $\mathscr{A}(2, 0)$, $\mathscr{A}(1, 1)$, $\mathscr{A}(1, 1, 0)$ which send subgraphs to subalgebras If Δ is any non-trivial type, there is an obvious full embedding of at least one of these four categories of algebras in $\mathscr{A}(\Delta)$. (All "unnecessary" operations are projections and/or all constants equal.) Thus we have the following result.

PROPOSITION 6. *Let Δ be a non-trivial type, and f any function from $\mathscr{Z}$ to M°. Then there exists $g: \mathscr{Z} \to A(\Delta)^\circ$ such that:*

(i) *$g(i)$ is isomorphic to a subalgebra of $g(i')$ for $i < i'$, $i, i' \in \mathscr{Z}$,*

(ii) *the monoid of all endomorphisms of $g(i)$ is isomorphic to $f(i)$ for all $i \in \mathscr{Z}$.*

In (**1**) the existence of a "strongly rigid" complete metric space is proved and the construction is given which associates with every graph (X, R) a complete metric space $C(X, R)$ in such a way that

(1) the group of all automorphisms of (X, R) is isomorphic to the group of all autohomeomorphisms of $C(X, R)$,

(2) if (X_1, R_1) is a full subgraph of (X, R), then $C(X_1, R_1)$ is a complete subspace of $C(X, R)$.

Let $\mathscr{G}$ be the category of groups. Using this construction and Proposition 4, we obtain a generalization of Theorem 3.

PROPOSITION 7. *Let f be any function from $\mathscr{Z}$ into $\mathscr{G}^o$. Then there exists a function g from $\mathscr{Z}$ into the class of all complete metric spaces such that:*

(i) *$g(i)$ is homeomorphic to a complete subspace of $g(i')$ for $i < i'$, $i, i' \in \mathscr{Z}$,*

(ii) *the group of all autohomeomorphisms of $g(i)$ is isomorphic to $f(i)$ for all $i \in \mathscr{Z}$.*

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