Appendix C
Weizsäcker–Williams approximation

A useful approximation for the electron scattering cross section at low q^2 follows from the results in chapter 11; it is due to Weizsäcker and Williams (WWA). This approximation gives the dominant part of the inelastic cross section whenever the electron is undetected as it passes through matter, for then one has to sum over all possible momentum transfers, and the WWA cross section increases as $1/q^2$ for small q^2. Furthermore, the form of the WWA result derived below provides a stepping stone into the renormalization group evolution equations for quantum field theory [Al77], as discussed, for example, in [Ro90, Wa95].

We relate the electron scattering process in Fig. 11.1 to the corresponding *real photon* process illustrated in Fig. 11.4. This will allow us to express the electron scattering cross section as $q^2 \to 0$ in terms of a cross section measured in photoabsorption. In the course of the analysis, we will be able to identify a *probability of finding a photon* in the field of the electron. The classical basis for the WWA is described, for example, in [Ja62]. The Coulomb field of a relativistic electron Lorentz contracts and becomes predominantly transverse; the electron current produces a transverse magnetic field of comparable magnitude (Fig. C.1). This transverse field configuration is equivalent to a collection of *real* photons with a certain, specified momentum distribution.

The QED analysis here follows [Dr64, Wa84]. Recall the structure of the response tensor in Eqs. (11.20) and (11.27) for a target of mass m

$$\begin{aligned}
W_{\mu\nu} &= (2\pi)^3 \overline{\sum_i} \sum_f \delta^{(4)}(q + p' - p)\langle p|J_\nu(0)|p'\rangle\langle p'|J_\mu(0)|p\rangle(\Omega E) \\
&= W_1(q^2, q\cdot p)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \\
&\quad + W_2(q^2, q\cdot p)\frac{1}{m^2}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) \qquad \text{(C.1)}
\end{aligned}$$

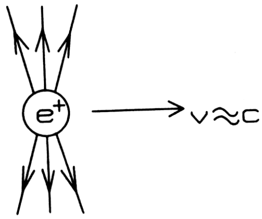

Fig. C.1. Lorentz contracted electric field of relativistic electron; basis for Weizsäcker–Williams approximation.

The (unpolarized) cross section for real photon processes follows directly from this response tensor. The relationship is derived in chapter 11, and the photoabsorption cross section is given by Eq. (11.39)

$$\begin{aligned}\sigma_\gamma &= \frac{(2\pi)^2\alpha}{\sqrt{(k\cdot p)^2}}\frac{1}{2}W_{\mu\mu} \\ &= \frac{(2\pi)^2\alpha}{\sqrt{(k\cdot p)^2}}W_1(0,-k\cdot p)\end{aligned} \tag{C.2}$$

The first line follows from the covariant polarization sum, and the second from a change to incoming photon momentum. Note that the real photon limit ($q^2 \to 0$) of Eq. (C.1) is perfectly *finite*; there are no singularities of the r.h.s. in this limit. Hence one establishes the following relations as $q^2 \to 0$ (chapter 11)

$$\begin{aligned}W_2(q^2, q\cdot p) &= O(q^2) \qquad ; \; q^2 \to 0 \\ W_1(q^2, q\cdot p) &= \frac{(p\cdot q)^2}{m^2 q^2}W_2(q^2, q\cdot p)\end{aligned} \tag{C.3}$$

These equations can be inverted to give for $q^2 \to 0$

$$\begin{aligned}W_1 &\doteq \frac{\sqrt{(q\cdot p)^2}}{(2\pi)^2\alpha}\sigma_\gamma\left(\frac{q\cdot p}{m}\right) \\ W_2 &\doteq \frac{m^2 q^2}{(p\cdot q)^2}W_1\end{aligned} \tag{C.4}$$

The electron scattering cross section can be written in terms of the variables in Fig. 11.1 as (chapter 11)

$$d\sigma_e = \frac{4\alpha^2}{q^4}\frac{d^3k_2}{2\varepsilon_2}\frac{1}{\sqrt{(k_1\cdot p)^2}}\left\{q^2 W_1 + \left[\frac{2(k_1\cdot p)(k_2\cdot p)}{m^2} - \frac{1}{2}q^2\right]W_2\right\} \tag{C.5}$$

The overall dependence of $1/q^4$ coming from the square of the virtual photon propagator implies that in the integrated cross section, most of

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the contribution arises from the region where $q^2 \to 0$. In this case, one can replace the structure functions by their limiting forms in Eqs. (C.4)[1]

$$\begin{aligned} d\sigma_e \doteq & \frac{4\alpha^2}{q^4}\frac{d^3k_2}{2\varepsilon_2}\frac{\sqrt{(q\cdot p)^2}}{\sqrt{(k_1\cdot p)^2}}\frac{1}{(2\pi)^2\alpha}\sigma_\gamma\left(\frac{q\cdot p}{m}\right) \\ & \times\left\{q^2+\frac{m^2q^2}{(p\cdot q)^2}\left[\frac{2(k_1\cdot p)(k_2\cdot p)}{m^2}-\frac{1}{2}q^2\right]\right\} \end{aligned} \tag{C.6}$$

This expression is Lorentz invariant. It is exact in the limit $q^2 \to 0$; at finite, but small q^2, it forms the Weizsäcker–Williams approximation. Equation (C.6) is the principal result of this appendix.

Let us, however, further develop this expression by using some kinematics. From Fig. 11.1 one has in the lab frame

$$\begin{aligned} q\cdot p &= m(\varepsilon_1-\varepsilon_2) = m\omega \\ k_1\cdot p &= -m\varepsilon_1 \end{aligned} \tag{C.7}$$

Also, the expression in brackets in Eq. (C.6) can be rewritten as

$$\begin{aligned} \{\cdots\} &= q^2+\frac{4\varepsilon_1\varepsilon_2\sin^2\theta/2}{\omega^2}\left[2\varepsilon_1\varepsilon_2-2\varepsilon_1\varepsilon_2\sin^2\frac{\theta}{2}\right] \\ &= q^2+\frac{8\varepsilon_1^2\varepsilon_2^2\sin^2\theta/2\cos^2\theta/2}{\omega^2}=q^2+\frac{2\varepsilon_1^2\varepsilon_2^2\sin^2\theta}{\omega^2} \end{aligned} \tag{C.8}$$

Hence the result in Eq. (C.6) becomes

$$d\sigma_e \doteq \frac{8\alpha^2}{q^4}\frac{d^3k_2}{2\varepsilon_2}\frac{\omega}{\varepsilon_1}\frac{1}{(2\pi)^2\alpha}\sigma_\gamma(\omega)\left[\frac{\varepsilon_1^2\varepsilon_2^2\sin^2\theta}{\omega^2}+\frac{1}{2}q^2\right] \tag{C.9}$$

Now change variables using

$$\begin{aligned} \omega &= \varepsilon_1-\varepsilon_2 \\ q^2 &= 2\varepsilon_1\varepsilon_2(1-\cos\theta) \end{aligned} \tag{C.10}$$

Hence (after an immediate integration over $d\phi$)

$$\frac{d^3k_2}{2\varepsilon_2} = \frac{\varepsilon_2\varepsilon_2 d\omega}{2\varepsilon_2}2\pi\frac{dq^2}{2\varepsilon_1\varepsilon_2}=\frac{\pi}{2\varepsilon_1}d\omega dq^2 \tag{C.11}$$

The limit $q^2 \to 0$ is achieved at finite ε_2 by going to small angles where $\theta \to 0$. In this case one has

$$\varepsilon_1^2\varepsilon_2^2\sin^2\theta \doteq \varepsilon_1^2\varepsilon_2^2\theta^2 \doteq q^2\varepsilon_1\varepsilon_2 \tag{C.12}$$

[1] Here the symbol $\doteq$ implies an approximate relation that is exact in the limit $q^2 \to 0$.

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Hence

$$d\sigma_e \doteq \frac{4\pi\alpha}{q^2}\frac{d\omega}{\varepsilon_1}\frac{\omega}{\varepsilon_1}\frac{1}{(2\pi)^2}\left[\frac{\varepsilon_1\varepsilon_2}{\omega^2}+\frac{1}{2}\right]dq^2\sigma_\gamma(\omega) \tag{C.13}$$

Now introduce the *momentum fraction* of the virtual photon

$$\frac{\omega}{\varepsilon_1} \equiv z \qquad\qquad \frac{\varepsilon_2}{\varepsilon_1} = 1 - z \tag{C.14}$$

Also introduce the differential of the so-called *resolution* in the electron scattering process defined here by

$$d\tau \equiv d\ln\left(\frac{q^2}{q_0^2}\right) = \frac{dq^2}{q^2} \tag{C.15}$$

The electron scattering cross section in Eq. (C.13) can then be rewritten as

$$d\sigma_e \doteq \frac{\alpha}{2\pi}d\tau\, z\, dz\left[\frac{2(1-z)}{z^2}+1\right]\sigma_\gamma(z) \tag{C.16}$$

We are now in a position to provide a more detailed interpretation of this result [Al77]. The contribution from the accompanying photon field to the electron scattering cross section for a beam of N electrons can be written as the following product: [number of photons $d\gamma(z,\tau)dz$ viewed with resolution between τ and $\tau+d\tau$ carrying a momentum fraction between z and $z+dz$ of the beam]$\times$ (photoabsorption cross section at that z). The first factor can in turn be related to the probability that at that τ, a photon carrying momentum fraction z is produced by an electron; we define that differential probability by $(\alpha/2\pi)P_{\gamma\leftarrow e}(z)d\tau dz$. It follows that

$$\begin{aligned} N d\sigma_e &\equiv [d\gamma(z,\tau)dz]\,\sigma_\gamma(z) \\ &\equiv \left[N\frac{\alpha}{2\pi}P_{\gamma\leftarrow e}(z)d\tau dz\right]\sigma_\gamma(z) \end{aligned} \tag{C.17}$$

One is now in a position to identify the *splitting function* $P_{\gamma\leftarrow e}(z)$ which forms the heart of the analysis of the evolution equations of QED and QCD. A comparison of Eqs. (C.16) and (C.17) gives

$$P_{\gamma\leftarrow e}(z) = z\left[\frac{2(1-z)}{z^2}+1\right] = \frac{1}{z}[(z-1)^2+1] \tag{C.18}$$

Note that the splitting function as calculated here is independent of τ. For the additional splitting functions in QED and QCD, see for example [Qu83, Wa95].