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POSITIVE POLYNOMIALS AND TIME DEPENDENT INTEGER-VALUED RANDOM VARIABLES

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Let $\{P_i\}$ be a sequence of real (Laurent) polynomials each of which has no negative coefficients, and suppose that f is a real polynomial. Consider the problem of deciding whether

(∗) for all integers k, there exists N such that the product of polynomials $P_{k+1} \cdot P_{k+2} \cdot \cdots \cdot P_{k+N} \cdot f$ has no negative coefficients.

This corresponds to a random walk problem for time dependent integer-valued random variables; as we shall see, it leads to questions involving strong *local flatness* on the sums of the random variables. Very special cases of this problem were studied by Poincaré [Po] in 1883, Meissner [Me] in 1911, Pòlya [Pl] in 1928, and Hardy, Littlewood and Pòlya [HLP] in 1934. Poincaré [Po] was interested in minimizing the number of sign changes—given f with k real negative roots, find a polynomial P with no negative coefficients such that the product $P \cdot f$ has exactly k sign changes among its coefficients, the minimum number possible. Meissner and Pólya dealt with special cases of the problem in several variables, where $P_i = 1 + x + y + xy$ (Meissner) or $P_i = 1 + x + y$ (Pólya) for all i. Our interest in this class of problems arose from the study of actions of rotation groups on a class of C*-algebras (e.g., [HR1, HR2, P]). In fact, this problem is equivalent to that of determining the positive cone of the equivariant K_0-group associated with this class of operator algebras.

An obvious necessary condition on f in order that (∗) hold is that its restriction to the positive reals, $f|(0,\infty)$, be strictly positive. We say the sequence $\{P_i\}$ is *strongly positive* if this condition is also sufficient for (∗) to have an affirmative solution. There is an equivalent statement for the corresponding random walk. The correspondence between P in the set of nonzero Laurent polynomials having no negative coefficients, $\mathbf{R}[x, x^{-1}]^+ \setminus \{0\}$, and the random variable X is as follows. Let (P, x^j) denote the coefficient of x^j in P. Then

$$(P, x^j)/P(1) = \Pr(X = j).$$

The polynomial P or its corresponding random variable X is said to be ε*-convex* (for $0 \le \varepsilon \le 2$) if

$$\text{for all integers } j, \quad \Pr(X = j+1) + \Pr(X = j-1) \ge (2-\varepsilon)\Pr(X = j).$$

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Let $\{X_i\}$ be a sequence of (independent) integer-valued random variables of finite support with corresponding Laurent polynomials $\{P_i\}$. For integers N and k, define $S_{N,k} = \sum_{t=1}^{N} X_{k+t}$, and $P^{N,k} = \prod_{t=1}^{N} P_{k+t}$. Then $\{P_i\}$ is strongly positive if and only if

for all $\varepsilon > 0$ and all integers k, there exists N so that $S_{N,k}$ is ε-convex.

One direction is easy to see—just set $f = x^{-1} - (2-\varepsilon) + x$ and work out the conditions on the coefficients of P which guarantee that the product Pf has no negative coefficients. The other implication follows from Corollary 1.13. This *eventual ε-convexity* is a particularly strong form of local flatness in the distributions of the sums of the random variables. In any event, we say that $\{X_i\}$ is *strongly positive* when it occurs.

Our results concern necessary and sufficient conditions for strong positivity of sequences $\{P_i\}$. As a consequence, we determine all pure (extremal) $[0,\infty]$-valued harmonic functions on the random walk in the strongly positive case, as well some other representative instances. There is a natural one-parameter family of these harmonic functions arising as evaluations at points of $[0,\infty]$; we investigate purity of such evaluations. We show (2.5) that evaluation at 1 (the most natural of the harmonic functions, and usually the only bounded one) is pure if and only if the conclusion of Mineka's theorem [Mi] holds:

$$(**) \qquad \lim_{N\to\infty} \sum_{j\in\mathbf{Z}} \left|\Pr(S_{N,k} = j) - \Pr(S_{N,k} = j+1)\right| = 0 \qquad \text{for all integers } k.$$

We say P (or X) is *unimodal* if there is only one sign change in the coefficients of the polynomial $(1-x)P$; in other words, the coefficients are non-decreasing starting from the left, then become non-increasing. The *ratio* of a unimodal P (or X) is defined via:

$$r(P) = \max\left\{\frac{(P,x^{i_0+1})}{(P,x^{i_0})}, \frac{(P,x^{i_0-1})}{(P,x^{i_0})}\right\},$$

where i_0 is a point at which the maximum occurs. We show that for unimodal $\{P_i\}$, if $\sum r(P_i)$ diverges, then $\{P_i\}$ is strongly positive (1.9). The converse holds when each P_i is strongly unimodal or if there is a bound on the total degree (2.3).

In the non-unimodal situation, there is a more elaborate criterion. If P is a Laurent polynomial with no negative coefficients, let $\mathcal{M}(P)$ denote the set of local maxima, i.e., the integer i_0 belongs to $\mathcal{M}(P)$ if $0 \neq (P,x^{i_0}) \geq (P,x^{i_0+1}), (P,x^{i_0-1})$. (It is easy to construct unimodal P for which $\mathcal{M}(P)$ contains non-contiguous points.) Set

$$\mathcal{C}(P) = \min_{i_0\in\mathcal{M}(P)} \max\left\{\frac{(P,x^{i_0+1})}{(P,x^{i_0})}, \frac{(P,x^{i_0-1})}{(P,x^{i_0})}\right\}.$$

Then $\{P_i\}$ is strongly positive if $\sum \mathcal{C}(P_i) = \infty$ (1.11). This immediately yields the following analogue of Mineka's criterion. Suppose $\{X_i\}$ is a sequence of independent, finitely supported integer-valued random variables, and X_i is ε_i-convex. Then $\{X_i\}$ is strongly positive if

$$\sum(2-\varepsilon_i) \qquad \text{diverges (1.13).}$$

The converse is true if telescoping (blocking) is permitted (precisely as in Mineka's Theorem).

There are classes of strongly positive sequences to which these numerical criteria cannot be applied directly. To construct examples illustrating this phenomenon, let β be a positive real number, and define the sequence $\{P_i\}$ (depending on β) via

$$(P_i, x^j) = \begin{cases} i^{\beta+1} & \text{if } j = 0 \\ 1 & \text{if } 1 \leq |j| \leq [i^\beta] \\ 0 & \text{if } |j| > [i^\beta]. \end{cases}$$

The bulk of the mass is at 0, and the rest is distributed uniformly over a very wide range. The ratio of the i-th polynomial is $1/i^{\beta+1}$, and their sum *converges*. As a result of this, the aforementioned strong positivity criteria do not apply. Since each P_i is symmetric and unimodal, it will follow (Section 3) that to check for strong positivity, it is sufficient to decide if for all integers k,

$$\frac{\Pr(S_{N,k} = 1)}{\Pr(S_{N,k} = 0)} \longrightarrow 1 \qquad \text{as } N \longrightarrow \infty.$$

For a large class of similar distributions (symmetric and unimodal), we perform the necessary calculations. An unexpected transition phenomenon is observed. For example, if $\beta < 2$, the sequence given above is strongly positive, while if $\beta > 2$, it is not (3.9(c)). If instead, the support grows exponentially, similar results are obtained (3.9(e)). For example, if K is a real number exceeding 1, define $\{P_i\}$ via

$$(P_i, x^j) = \begin{cases} K^i & \text{if } j = 0 \\ 1 & \text{if } 1 \leq |j| \leq [K^i] \\ 0 & \text{if } |j| > [K^i]. \end{cases}$$

Then $\{P_i\}$ is strongly positive if $K < 3$ and is not strongly positive if $K > 3$. The status of the sequence is unknown if $K = 3$. Two formulas (3.4 and 3.7) give sufficient conditions for strong positivity of large classes of sequences of symmetric unimodal polynomials.

Turning now to harmonic functions, for each point r in $\mathbf{R}^{++} = \mathbf{R}^{+} \setminus \{0\}$, we can define a harmonic function $h_r \colon \mathbf{Z} \times \mathbf{N} \longrightarrow \mathbf{R}^{+}$ by the formula $h_r(m, i) = r^m / P_1 \cdot P_2 \cdot \cdots \cdot P_i(r)$. We call this a *point evaluation* at r. We remark that this is unbounded unless $r = 1$. On restriction to appropriate subsets of $\mathbf{Z} \times \mathbf{N}$ (depending on the supports of the X_i), we may consider limit points of these "point evaluations"; these are $[0, \infty]$-valued harmonic functions, and yield evaluation at 0 or ∞. If $\{P_i\}$ is strongly positive, then all pure (extremal) $[0, \infty]$-valued harmonic functions are indeed point evaluations (if we include 0 and ∞), and all such point evaluations are pure. Theorem 2.1 indicates why the converse fails.

This yields two types of obstructions to strong positivity: either a point evaluation is not pure, or there exists a pure harmonic function that is not a point evaluation. We investigate purity of individual point evaluations. For example (2.5), evaluation at $r = 1$

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is pure if and only if mass cancellation occurs (as in the conclusion of Mineka's theorem (**)). Purity of evaluations at other points can be decided by reparameterizing the polynomials. When a point evaluation is not pure, it decomposes into other harmonic functions which are not point evaluations. If the polynomials are all unimodal, then a converse to Mineka's theorem *that does not require blocking* is valid for all reparameterizations of the polynomials (2.6). The outcome is that there is a simple necessary and sufficient test of purity for every point evaluation. If the polynomials are additionally symmetric (that is, $(P, x^j) = (P, x^{-j})$ for all j), then except for evaluation at 1, either the point evaluation is pure or a convergence phenomenon (described below) leads to a discrete decomposition.

An extreme case of impurity occurs when the infinite product of suitably normalized Laurent polynomials $\prod P_i(z)/P_i(1)$ converges on an annulus in the plane. If r is a positive real number within the region of convergence, not only is h_r impure, but it can be decomposed into a (discrete) family of harmonic functions that arise from the Laurent series expansion of the infinite product. Other point evaluations may decompose if the normalization is altered. Unusual examples are constructed; the set $\{ r \in \mathbf{R}^{++} \mid h_r \text{ is pure} \}$ can be the set of all positive reals, the empty set, a finite set, all but a single point, or even a union of open or half-open intervals, among others. On the other hand, if the distributions are symmetric and unimodal, then this set can only be the complement of an open or closed interval (symmetric with respect to $r \mapsto r^{-1}$) (2.7).

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1. **Strong positivity.** In this section, we prove results on strong positivity of the following type. For an appropriate numerical invariant, $z(P)$, of polynomials P, divergence of $\sum z(P_i)$ entails that the sequence $\{P_i\}$ be strongly positive (1.8, 1.9, 1.11, 1.13). If each polynomial in the sequence $\{P_i\}$ is unimodal and $\sum r(P_i)$ diverges, then $\{P_i\}$ is strongly positive. The converse holds if either all the P_i are strongly unimodal (log-concave) or if there is a bound on the degrees of the polynomials. For polynomials which are not necessarily unimodal, divergence of $\sum C(P_i)$ is sufficient for strong positivity. There is a precise analogue to Mineka's theorem (1.13). If each P_i is ε_i-convex (see the Introduction for the definitions), then $\sum(2-\varepsilon_i) = \infty$ is sufficient for strong positivity.

Initially, we wish to show that if each P_i is unimodal and the sum of the ratios (definition given in the Introduction), $\sum r(P_i)$, diverges, then $\{P_i\}$ is strongly positive. We begin with linear polynomials, that is, $P_i = a_i + b_i x$ (equivalently, the random variables X_i have two-point contiguous support). If the P_i are all equal (the "stationary" case), the result can be found in [Me]. If $\inf_i \min\{a_i/b_i, b_i/a_i\} > 0$, a careful convergence argument permits reduction to the stationary case. An argument involving both superposition and blocking (telescoping) allows us to prove the complete result in the linear situation, and another application of superposition yields the most general result of this section (1.11).

The real algebra of Laurent polynomials is defined as

$$\mathbf{R}[x,x^{-1}] = \left\{\sum a_i x^i \mid i \in \mathbf{Z}, a_i \in \mathbf{R}, \text{ and } a_i = 0 \text{ for almost all } i\right\}$$

The set of Laurent polynomials with no negative coefficients will be denoted $\mathbf{R}[x,x^{-1}]^+$. For f in $\mathbf{R}[x,x^{-1}]$, we let (f,x^j) denote the coefficient of x^j appearing in f, and set

$$\operatorname{Log} f = \{j \in \mathbf{Z} \mid (f,x^j) \neq 0\}.$$

Recall that a sequence of Laurent polynomials (having no negative coefficients) $\{P_i\}$ is *strongly positive* if

for any real Laurent polynomial, f, such that the restriction satisfies $f|(0,\infty) > 0$, then for any integer k, there exists an integer N so that the product

$$P_k \cdot P_{k+1} \cdot \cdots \cdot P_{k+N} \cdot f$$

has no negative coefficients.

We remark that the condition on f (as a function on $(0,\infty)$) is obviously necessary.

For P a (nonzero) Laurent polynomial with no negative coefficients, we define its *contiguity coefficient,* $c(P)$, as follows. First, for a linear polynomial, $L = a + bx$ (with both coefficients strictly greater than 0), define $d(L) = ab/(a+b)^2$. Then define

$$c(P) = \sup\left\{\inf_i\left\{d(L_i) \mid P = \sum h_i L_i,\ h_i \in \mathbf{R}[x,x^{-1}]^+\right\}\right\},$$

where the L_i are linear with positive coefficients and the supremum is over all such decompositions, $P = \sum_i h_i L_i$. Clearly, $c(P_2 + P_2) \geq \min\{c(P_1), c(P_2)\}$ if both P_i have no negative coefficients. For example, if L is linear, then $d(L) = c(L)$. If $P = x^2 + \lambda x + \mu$ (where λ and μ are strictly positive real numbers), then $P = x(x + \lambda/2) + (\lambda x/2 + \mu)$, so that

$$c(P) \geq \min\left\{\frac{\lambda/2}{1+\lambda+\lambda^2/4}, \frac{\lambda\mu/2}{\mu^2+\lambda\mu+\lambda^2/4}\right\}.$$

In general, $c(P)$ is awkward to compute. It is a useful tool, but will be replaced later by a much more convenient invariant, namely $C(P)$ (defined in the Introduction.)

If P (or its corresponding random variable X) is a unimodal Laurent polynomial (with no negative coefficients), we define $r(P)$ ($r(X)$) to be the ratio of the second largest coefficient (mass at a point) to the largest. In particular, if there is a tie for largest coefficient, then $r(P) = 1$. Note that the second largest coefficient must occur at one of the two points adjacent to the maximum.

LEMMA 1.1. *If P is unimodal, then*

$$c(P) \geq \frac{4}{25} r(P).$$

PROOF. We may assume that the peak coefficient occurs at 0 and the second largest coefficient occurs at 1 (rather than -1). Write $P = \sum_{-l \leq i \leq m} \lambda_i x^i$, with $m \geq 2$ and assume for now that $l \geq 1$. Consider the following polynomials:

$$\begin{array}{ll}
\lambda_m\left(\frac{1}{4}+\frac{1}{2}x+x^2+x^3+\cdots+x^m\right), & \lambda_{-l}\left(\frac{1}{4}+\frac{1}{2}x^{-1}+\cdots+x^{-l}\right) \\
(\lambda_{m-1}-\lambda_m)\left(\frac{1}{4}+\frac{1}{2}x+x^2+\cdots+x^{m-1}\right), & (\lambda_{-l+1}-\lambda_{-l})\left(\frac{1}{4}+\frac{1}{2}x^{-1}+\cdots+x^{-l+1}\right) \\
\vdots & \vdots \\
(\lambda_1-\lambda_2)\left(\frac{1}{4}+\frac{1}{2}x\right) & (\lambda_{-1}-\lambda_{-2})\left(\frac{1}{4}+\frac{1}{2}x^{-1}\right).
\end{array}$$

Note that $0 \leq \lambda_{-l} \leq \lambda_{-l+1} \leq \cdots \leq \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Form the sum of these polynomials, and subtract this from P. Almost everything telescopes, and the remainder is $Q = \frac{1}{2}\lambda_1 x + \lambda_0 - \frac{1}{4}(\lambda_1 + \lambda_{-1}) + \frac{1}{2}\lambda_{-1}x^{-1}$. The c-values for each of the polynomials in the list is at least that of $(\frac{1}{4} + x)$, i.e., 4/25. It is readily verified that $c(Q) \geq (4/25)(\lambda_1/\lambda_0) = (4/25)r(P)$.

As P is the sum of the listed polynomials together with Q, it follows that $c(P)$ is at least as large as that of the minimum of the c-values of these polynomials, hence the desired result is obtained in this case. If $l = 0$, the same argument will work. ■

Alan Kelm has shown that $4/25$ can be replaced by $2/9$ in Lemma 1.1, and this is sharp. The following is a consequence of [H3; Appendix C], but we provide a short and easy proof for this special case. A Laurent polynomial P is *gapless* if for integers $i < j < k$, $(P, x^i)(P, x^k) \neq 0$ implies that $(P, x^j) \neq 0$.

LEMMA 1.2. *Suppose $\{P_i\}$ is a sequence of gapless Laurent polynomials with no negative coefficients such that*

(o) $|\operatorname{Log} P_i| > 1$;

(i) $\operatorname{Log} P_i = \operatorname{Log} P_1$ *for all i;*

(ii) $\sup\{(P_i, x^j) \mid i \in \mathbf{N}, j \in \operatorname{Log} P_1\} < \infty$;

(iii) $\inf\{(P_i, x^j) \mid i \in \mathbf{N}, j \in \operatorname{Log} P_1\} > 0$.

If f is an element of $\mathbf{R}[x, x^{-1}]$ such that $f|(0, \infty) > 0$, there exists an integer M so that $P_1 P_2 \ldots P_M \cdot f$ has no negative coefficients.

PROOF. We may assume that $P_i(1) = 1$ for all i, without altering the rest of the hypotheses. Regard elements of $\mathbf{R}[x, x^{-1}]$ as elements of $l^1(\mathbf{Z})$; that is, assign the norm $\sum |\lambda_j|$ to the polynomial $g = \sum \lambda_j x^j$. If g has no negative coefficients, then $\|g\| = g(1)$. Clearly $\{P_i\}$ contains a subsequence converging to a Laurent polynomial P in this norm. By (i) and the boundedness below hypotheses, P is gapless and $\operatorname{Log} P = \operatorname{Log} P_1$.

By [H1; V.1(b)], there exists an integer m so that $P^m f$ has no negative coefficients. An easy consequence is the existence of an integer $N \geq m$ so that $P^N f$ is additionally gapless (as P itself is gapless). It follows (by considering the endpoints), that $\operatorname{Log} P^N f =$

$N \operatorname{Log} P + \operatorname{Log} f$ (adopting the convention that the sum of two sets of integers is the set of sums). Define

$$\varepsilon = \inf\{ (P^N f, x^j) \mid j \in \operatorname{Log}(P^N f) \}.$$

There exists a finite subset S (having N elements) of $\mathbf{N}$ such that for all j in S, $\|P_j - P\| < \varepsilon / (N\|f\|)$. Then

$$\begin{aligned} \Big\|P^N f - \big(\prod_S P_j\big) f\Big\| &\leq \|f\| \cdot \Big\|P^N - \prod_S P_j\Big\| \\ &< \|f\| N\varepsilon / \big(N\|f\|\big) = \varepsilon. \end{aligned}$$

Clearly $N \operatorname{Log} P + \operatorname{Log} f$ contains $\operatorname{Log}(\prod_S P_j) \cdot f$, and as $(P^N f, x^k) \geq \varepsilon$ for k in the former set, it follows that $\big((\prod_S P_j) \cdot f, x^k\big) > 0$ for all such k. Now $M = \max\{ i \in S\}$ will do. ∎

The following *Superposition Lemma* is elementary but extremely useful.

LEMMA 1.3 (SUPERPOSITION LEMMA). *Let $\{P_i\}$ be a sequence of Laurent polynomials each having no negative coefficients, and let f be in $\mathbf{R}[x, x^{-1}]$. Suppose that for all i, there exist positive integers $M(i)$, and $\Lambda_{j,i}$ and $T_{j,i}$ in $\mathbf{R}[x, x^{-1}]^+$ for $0 \leq j \leq M(i)$, such that*

$$P_i = \sum_{0 \leq j \leq M(i)} \Lambda_{j,i} T_{j,i}.$$

Suppose in addition that for all sequences $k = (k(1), k(2), \ldots)$ (with $0 \leq k(i) \leq M(i)$), there exists i (depending on the sequence k) so that

$$T_{k(1),1} \cdot T_{k(2),2} \cdot \cdots \cdot T_{k(i),i} \cdot f \quad \textit{has no negative coefficients.}$$

Then there exists N so that $P_1 \cdot P_2 \cdot \cdots \cdot P_N \cdot f$ has no negative coefficients.

PROOF. For each i, define the set $K_i = \{0, 1, 2, \ldots, M(i)\}$, and let K denote the cartesian product of the K_i's, with the product topology. Given k in K and $i \equiv i_k$ (i depending on k) as hypothesized, form the clopen set

$$U_k = \{ k' \in K \mid k'(j) = k(j) \text{ for all } j \leq i_k \}.$$

As k varies over K, we obtain an open covering; by the compactness of K, there exists a finite set $\{ k^{(1)}, k^{(2)}, \ldots, k^{(m)} \}$ of elements of K, so that

$$\bigcup_{j=1}^{m} U_{k^{(j)}} = K.$$

Note that there are integers $\{ i^{(1)}, i^{(2)}, \ldots, i^{(m)} \}$ so that if k' belongs to $U_{k^{(t)}}$ $(1 \leq t \leq m)$, then

$$\left(\prod_{j=1}^{i^{(t)}} T_{k'(j),j} \right) \cdot f$$

has no negative coefficients. Upon setting $N = \max\{ i^{(t)} \}_{t \leq m}$, we find that for all choices $\{ k(0), k(1), \ldots, k(N) \}$ (with $k(i) \in K_i$), the product $T_{k(1),1} \cdot T_{k(2),2} \cdot \cdots \cdot T_{k(N),N} \cdot f$ has no negative coefficients. However, $P_1 \cdot P_2 \cdot \cdots \cdot P_N \cdot f$ is an $\mathbf{R}[x, x^{-1}]^+$-combination of such terms, so has no negative coefficients. ∎

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PROPOSITION 1.4. *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients, such that $\{c(P_i)\}$ is bounded below away from zero. Then given f in $\mathbf{R}[x, x^{-1}]$ satisfying $f|(0,\infty) > 0$, there exists an integer N so that $P_1 \cdot P_2 \cdot \cdots \cdot P_N \cdot f$ has no negative coefficients.*

PROOF. Decompose each P_i as in 1.3 (using the definition of $c(P)$) in order to apply 1.2 along each path. ∎

The polynomial P or its corresponding $\mathbf{Z}$-valued random variable X is *strongly unimodal* (also known as *log concave*—the former term is used in probability, the latter in combinatorics) if it is gapless and for all j, $(P, x^j)^2 \geq (P, x^{j+1}) \cdot (P, x^{j-1})$. By Ibragimov's theorem ([Ib]) applied to discrete random variables, strong unimodality is equivalent to:

For all *unimodal* polynomials Q, the product PQ is unimodal.

It follows that if P and P' are strongly unimodal, then so is their product PP'.

Let λ be a positive real number. We can reparameterize a polynomial P or its random variable X by λ on setting

$$\Pr(X^{(\lambda)} = j) = \frac{\lambda^j \Pr(X = j)}{\sum_k \lambda^k \Pr(X = k)};$$

so P is replaced by $P(\lambda\,\cdot)$, i.e., $P(\lambda x)$. The next result includes a proof of Ibragimov's theorem in this special case of discrete distributions (this is very well known; see the recent survey article by Stanley [S]); by refining the mesh, one can obtain Ibragimov's theorem in general.

PROPOSITION 1.5. *Let P be any Laurent polynomial with no negative coefficients. The following are equivalent:*

(i) P is strongly unimodal;

(ii) $P(r\cdot)$ is unimodal for all strictly positive real r;

(iii) PQ is unimodal for all unimodal Q in $\mathbf{R}[x, x^{-1}]^+$.

PROOF THAT *(i)* IMPLIES *(ii)*. We observe that for a strongly unimodal polynomial, the set of quotients of consecutive coefficients $\{(P, x^j)/(P, x^{j-1})\}$, is monotone decreasing—in fact this characterizes strong unimodality. Hence $P(r\cdot)$ is strongly unimodal and *a fortiori* unimodal.

PROOF THAT *(ii)* IMPLIES *(i)*. Suppose that $(P, x^j)^2 < (P, x^{j+1}) \cdot (P, x^{j-1})$ for some j. This entails that both (P, x^{j+1}) and (P, x^{j-1}) be strictly greater than zero. Select r so that

$$\frac{(P, x^j)}{(P, x^{j-1})} < \frac{1}{r} < \frac{(P, x^{j+1})}{(P, x^j)}$$

Then we quickly see that the coefficients $j-1, j$, and $j+1$ of $P(r\cdot)$ decrease then increase, violating unimodality of $P(r\cdot)$.

PROOF THAT *(i)* IMPLIES *(iii)*. To prove (iii), we show that $(1-x)PQ$ has exactly one sign change among its coefficients. Write (as we may, by shifting)

$$P=\sum_{i=0}^{d} a_i x^i; \qquad a_i \geq 0; \quad a_0, a_d > 0.$$

Then $a_i^2 \geq a_{i+1}a_{i-1}$. As $(1-x)Q$ has just one sign change, we may write (after multiplying Q by a suitable monomial) $(1-x)Q = A_0 + A_1 x + \cdots + A_n x^n - B_1 x^{n+1} - \cdots - B_m x^{n+m}$, where $A_i, B_j \geq 0$; $A_0 > 0$; $B_1 > 0$; and $B_m > 0$. Clearly, if $t \leq n$, the coefficient of x^t in $(1-x)PQ$ is nonnegative, and if $t \geq n+d+1$, its coefficient is not positive. Hence to show exactly one sign change occurs in $(1-x)PQ$, it suffices to prove:

If for $n < t < n+d$, the coefficient of x^t in $(1-x)PQ$ is strictly negative, then so is that of x^{t+1}.

Say $t = n+k$ (with $1 \leq k < d$), and the coefficient of x^t is negative. It is given by the formula:

$$a_d A_{n-d+k} + a_{d-1}A_{n-d+k+1} + \cdots + a_k A_n - a_{k-1}B_1 - a_{k-2}B_2 - \cdots - a_0 B_k.$$

Delete the $a_d A_{n-d+k}$ term, multiply the rest by a_{k+1}/a_k, and subtract the outcome from the coefficient of x^{t+1}. This yields

$$\begin{aligned}
&A_{n-d+k+1}[a_d - (a_{d-1}a_{k+1}/a_k)] + A_{n-d+k+2}[a_{d-1} - (a_{d-2}a_{k+1}/a_k)] \\
&\quad + \cdots + A_{n-1}[a_{k+2} - (a_{k+1}^2/a_k)] + A_n \cdot 0 \\
&\quad - B_1[a_k - (a_{k-1}a_{k+1}/a_k)] - B_2[a_{k-1} - (a_{k-2}a_{k+1}/a_k)] \\
&\quad - \cdots - B_k[a_1 - (a_0 a_{k+1}/a_k)] - B_{k+1}a_0
\end{aligned} \tag{1}$$

Strong unimodality yields

$$a_1/a_0 \geq a_2/a_1 \geq a_3/a_2 \geq \cdots.$$

For $k < d$, this gives $a_d a_k \leq a_{d-1}a_{k+1}$, and also $a_{k-i}a_k \geq a_{k-i-1}a_{k+1}$. Hence every single term in (1) is not positive. Thus if c_i is the coefficient of x^i in $(1-x)PQ$, we have

$$c_{t+1} - (a_{k+1}/a_k)(c_t - a_d A_{n-d+k}) \leq 0.$$

As $c_t < 0$, we deduce $c_{t+1} < 0$ as desired. Thus $(1-x)PQ$ has exactly one sign change among its coefficients, so that PQ is unimodal.

PROOF THAT *(iii)* IMPLIES *(i)*. Let $P = \sum_{i=0}^{d} a_i x^i$ (with $a_i \geq 0$; $a_0, a_d > 0$) be a polynomial with no negative coefficients that is not strongly unimodal. If P has a gap in its coefficients, then it is not unimodal itself; otherwise there exists k so that $a_{k+1}/a_k > a_k/a_{k-1}$. If $a_{k+1}/a_k > 1$, choose a real number $R > 1$ such that $a_{k+1}/a_k > R/(R-1) > a_k/a_{k-1}$. Select $n > d$, and define the unimodal polynomial

$$Q = 1 + x + x^2 + \cdots + x^{n-1} + Rx^n.$$

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Then $(1-x)Q = 1 + (R-1)x^n - Rx^{n+1}$. Computing the coefficients of x^{k+n-1}, x^{k+n} and x^{k+n+1} in $(1-x)QP$, we find that there are at least two sign changes among the coefficients. Hence QP is not unimodal.

If $1 \geq a_{k+1}/a_k$, replace P by $x^d P(x^{-1})$, find Q_1 as in the preceding paragraph, and set $Q(x) = Q_1(x^{-1})$. ∎

One (of several) obstructions to extending our results to several variables is that as far as we know, there is no notion of strongly unimodal (and a corresponding definition of unimodal) for which Proposition 1.5 would hold, even assuming that the same monomials appear in P and Q.

Let P be a normalized Laurent polynomial with positive coefficients, with corresponding random variable, X. We define the *fluctuation* of P or X, via

$$\mathcal{F}(P) = \sum \frac{|\lambda_i - \lambda_{i+1}|}{P(1)},$$

where $P = \sum \lambda_i x^i$. This is also just the l^1-norm of $(1-x)P/P(1)$, and in the unimodal case, it is simply $2 \max \lambda_i / P(1)$.

Recall from the introduction that $P^{N,t} = \prod_{i=t+1}^{N+t} P_i$; $X^{N,t}$ is similarly defined. The following is a translation of Mineka's Theorem to our notation.

THEOREM 1.6 (MINEKA'S THEOREM [MI]). *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients. If*

$$\sum (2 - \mathcal{F}(P_i)) = \infty,$$

then for all integers t, $\mathcal{F}(P^{N,t})$ tends to zero as N tends to infinity.

LEMMA 1.7. *(a) If P is a strongly unimodal Laurent polynomial with $P(1) = 1$ and maximum coefficient a, then*

$$a \geq \big(1 - r(P)\big) / \big(1 + r(P)\big).$$

(b) Suppose that $\{P_i\}$ is a sequence of strongly unimodal polynomials such that $\sum r(P_i) = \infty$. Then for all $t > 0$,

$$r(P_t \cdot P_{t+1} \cdot \cdots \cdot P_{t+N}) \longrightarrow 1 \text{ as } N \longrightarrow \infty.$$

PROOF. (a) Without loss of generality, we may assume $(P, x^0) = a$. From strong unimodality,

$$\frac{(P, x^i)}{(P, x^{i-1})} \geq \frac{(P, x^{i+1})}{(P, x^i)}.$$

Thus the total mass to the right of 0 is bounded above by the geometric series

$$a \cdot \{ r(P) + r(P)^2 + r(P)^3 + \cdots \} = a \cdot \frac{r(P)}{\big(1 - r(P)\big)}.$$

Combining this with the left side of the distribution, we obtain

$$a \cdot \left\{ \frac{\big(1 + r(P)\big)}{\big(1 - r(P)\big)} \right\} \geq 1$$

(b) As $\sum_{i \geq t} r(P_i) = \infty$ for any t, we may assume that t is 1. By part (a), it suffices to show $\max\{ (P^{N,0}, x^i) \mid i \in \mathbf{Z}\}$ tends to zero as N becomes arbitrarily large. By Mineka's theorem, 1.6, it suffices to show $\sum_i (2 - \mathcal{F}(P_i)) = \infty$. Suppose P is strongly unimodal, $P(1) = 1$, and $a = \max\{ (P, x^i)\}$ (so $\mathcal{F}(P) = 2a$). By the preceding,

$$1 - a \geq r(P)/\big(1 + r(P)\big).$$

Now since $0 \leq r(P) \leq 1$,

$$\frac{1}{2}\sum\big(2 - \mathcal{F}(P_i)\big) \geq \sum \frac{r(P_i)}{(1 + r(P_i))} \geq \frac{1}{2}\sum r(P_i).$$

This yields the result. ∎

Now consider the case that $P_i = a_i + b_i x$ ($a_i, b_i > 0$). Then $P^{N,0} = P_1 \cdot P_2 \cdot \cdots \cdot P_N$ is a product of strongly unimodal Laurent polynomials, and by 1.5, is itself strongly unimodal. We may assume $a_i + b_i = 1$; then $c(P_i) = a_i(1 - a_i)$. Moreover, $r(P_i) = \min\{ a_i/(1 - a_i), (1 - a_i)/a_i\}$. Thus if $\sum c(P_i)$ diverges, so does $\sum r(P_i)$. On the other hand, if $\sum r(P_i)$ diverges, we may assume $a_i \leq 1 - a_i$ for all i (by taking a suitable divergent subsequence and interchanging x and x^0 throughout if necessary). Then $c(P_i) \geq \frac{4}{25} r(P_i)$, so that $\sum r(P_i)$ diverges if and only if $\sum c(P_i)$ does. Suppose that $\sum c(P_i)$ diverges. By 1.7, given t, there exists $n(t)$ such that $r(P_{t+1} \cdot P_{t+2} \cdot \cdots \cdot P_{t+n(t)}) \geq \frac{1}{2}$. Thus we may telescope (block) the P_i's, via

$$\begin{aligned} Q_1 &= P_1 \cdot P_2 \cdot \cdots \cdot P_{n(1)}, \\ Q_2 &= P_{n(1)+1} \cdot \cdots \cdot P_{n(2)}, \\ &\ldots \\ Q_t &= P_{n(t-1)+1} \cdot \cdots \cdot P_{n(t)}, \\ &\ldots \end{aligned}$$

to satisfy the following conditions:

(i) $r(Q_t) \geq \frac{1}{2}$ for all t;

(ii) each Q_t is strongly unimodal.

By 1.1, $c(Q_t) \geq 2/25$ for all t. Hence $\{ Q_t\}$ satisfies the hypotheses of 1.4, and thus is strongly positive. This obviously implies that the original sequence $\{ P_i\}$ is also strongly positive. We thereby have deduced:

If $P_i = a_i + b_i x$, with $a_i, b_i > 0$ and

$$(\dagger) \qquad \sum \frac{a_i b_i}{(a_i + b_i)^2} = \infty,$$

then $\{ P_i\}$ is strongly positive.

Now we apply the Superposition Lemma (1.3). Let $\{P_i\}$ satisfy $\sum c(P_i) = \infty$. Decompose P_i according to the definition of c. On choosing any path k (in K, see the proof of 1.3), the sum of the $c(T_{(k(j),j)})$'s along that path—that is, $\sum_{1\le j\le\infty} c(T_{(k(j),j)})$—diverges. Here we have taken the T's as the linear terms arising in decompositions of P_i (from the definition of $c(P_i)$) for example so that

$$\min\{\, c(T_{i,j}) \mid 0 \le j \le M(i)\} \ge \frac{1}{2} c(P_i).$$

Then (†) yields via 1.3:

THEOREM 1.8. *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients. If $\sum c(P_i) = \infty$, then $\{P_i\}$ is strongly positive.*

The converse is not quite true: Set $P_{2i} = x^3 + x + 1$ and $P_{2i+1} = x^3 + x^2 + 1$ for all i. Then $c(P_i) = 0$; however, the sequence is strongly positive, because $Q_i = P_{2i}P_{2i+1}$ is gapless and $\sum c(Q_i) = \infty$.

It is routine to show that if $\{P_i\}$ is strongly positive, then there is a blocking (telescoping) of the P_i's, $\{Q_t\}$, so that $\sum c(Q_t) = \infty$, and the $c(Q_t)$ are arbitrarily close to $1/4$ (in any case, $c(P) \le 1/4$). However, this is not particularly useful, as it is usually quite difficult to find the right blocking.

COROLLARY 1.9. *Suppose $\{P_i\}$ is a sequence of unimodal Laurent polynomials such that $\sum r(P_i) = \infty$. Then $\{P_i\}$ is strongly positive.*

PROOF. By 1.1, $c(P_i) \ge (4/25)r(P_i)$ so $\sum c(P_i) = \infty$; now 1.8 applies. ■

The converse of 1.9 does not hold in general, but will hold if the P_i are either all strongly unimodal or of bounded total degrees, as we shall see in Section 2. Here is an easy example, previewing (rather coarsely) some of the subtle phenomena associated with symmetric unimodal polynomials which will be studied in Section 3.

EXAMPLE 1.10. Let $d\colon \mathbf{N} \longrightarrow \mathbf{N}$ be a function, and define a sequence of polynomials, $\{P_i\}$,

$$P_i = i^2 + \sum_{1\le j\le d(i)} (x^j + x^{-j}).$$

Then $r(P_i) = 1/i^2$, so that $\sum r(P_i)$ converges (as does $\sum c(P_i)$). First consider the case that $d(i) = i^3$. Set $Q_i = P_{2i}\cdot P_{2i+1}$; we see that each Q_i is still unimodal (the product of symmetric unimodal polynomials is also unimodal), and $r(Q_i)$ is approximately $1/2i$. By Corollary 1.9, $\{Q_i\}$ is strongly positive, and therefore so is $\{P_i\}$. In Section 3, we shall see that $d(i) = i^3$ can be replaced by $d(i) = i$, and the sequence of P_i's will still be strongly positive, but this will fail if $d(i)$ is asymptotic to $i/3$. In the case of $d(i) = i$, there is no obvious telescoping (as there is with $d(i) = i^3$) that will yield a sequence which is easily seen to be strongly positive. ■

Corollary 1.9 extends via the Superposition Lemma to sequences of random variables which need not be unimodal. Let $\mathcal{M}(P)$ denote the set of local maxima; that is, the integer

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i_0 belongs to $\mathcal{M}(P)$ if (P, x^{i_0}) is greater than or equal both (P, x^{i_0+1}) and (P, x^{i_0-1}). Set

$$C(P) = \min_{i_0 \in \mathcal{M}(P)} \max \left\{ \frac{(P, x^{i_0+1})}{(P, x^{i_0})}, \frac{(P, x^{i_0-1})}{(P, x^{i_0})} \right\}.$$

For example, if

$$P = 1 + 3x + 2x^2 + 5x^3 + x^4 + x^5 + \frac{1}{2}x^6,$$

then $C(P) = \min\{ \max\{ 1/3, 2/3 \}, \max\{ 2/5, 1/5 \}, \max\{ 1, 1/2 \} \} = 2/5$.

If we apply the definition of $C(P)$ and use the decomposition method occurring in the proof of 1.1, together with 1.3 (the Superposition Lemma), we see that 1.9 yields that $\sum C(P_i) = \infty$ is sufficient for strong positivity. We record this more general criterion.

THEOREM 1.11. *Let $\{P_i\}$ be a sequence of positive Laurent polynomials such that $\sum C(P_i)$ diverges. Then $\{P_i\}$ is strongly positive.*

In general, it is quite difficult to calculate $c(P)$ (although rough estimates are usually sufficient), but $C(P)$ is readily determined. Moreover, $\sum c(P_i) = \infty$ implies $\sum C(P_i) = \infty$. To see this, regard the polynomial as a chain of increasing and decreasing sequences of its coefficients; use Lemma 1.1 for the local monotone sequences and show there is a K for which $KC(P) \geq c(P)$ for all P. The converse, that $\sum C(P_i) = \infty$ implies $\sum c(P_i) = \infty$, is also true, having been proved by Alan Kelm, one of our students (DH).

If each P_i is unimodal, then $\sum C(P_i)$, $\sum c(P_i)$, and $\sum r(P_i)$ simultaneously diverge or converge. In 1.10, we saw some examples of sequences $\{P_i\}$ of unimodal polynomials which are strongly positive, but for which $\sum C(P_i)$ converges. However, those sequences for which the sum diverges possess a peculiar stability property under a wide range of perturbations. Let Δ denote an infinite sequence (indexed by $\mathbf{Z}$), $\Delta = \big(\Delta(j)\big)$ such that $\Delta(j) > 0$ for all j, and $\Delta(j) = 1$ for all but a finite number of j. We define the *transformation induced by* Δ, to be the transformation that sends a Laurent polynomial $P = \sum \alpha_j x^j$ to $P' = \sum \Delta(j)\alpha_j x^j$. Define the *index* of Δ to be $\rho(\Delta) = \inf\{ \Delta(j)/\Delta(k) \mid j, k \in \mathbf{Z}\}$.

COROLLARY 1.12. *Suppose that $\{P_i\}$ is a sequence of positive Laurent polynomials with $\sum C(P_i) = \infty$, and that $\{\Delta_i\}$ is a collection of sequences, each as described above, with $\rho(\Delta_i) = \rho_i$. If $\sum \rho_i C(P_i) = \infty$, and if $\{P'_i\}$ is obtained from the transformation induced by Δ_i on P_i, then $\{P'_i\}$ is also strongly positive (and satisfies $\sum C(P'_i)) = \infty$). In particular, this holds if $\{\rho_i\}$ is bounded below.*

PROOF. It is an immediate consequence of the definitions that $C(P'_i) \geq \rho_i C(P_i)$. ∎

Not all strongly positive sequences are insensitive to this type of perturbation. We shall see in Section 3 that if $P_i = i^2 + \sum_{1 \leq j \leq i}(x^j + x^{-j})$, then $\{P_i\}$ is strongly positive but the sequence $\{P'_i\}$ defined via

$$P'_i = 3i^2 + \sum_{1 \leq j \leq i} (x^j + x^{-j})$$

(obtained from the sequence $\Delta_i = \Delta$, where $\Delta(0) = 3$, and $\Delta(j) = 1$ for all other values of j) is not (3.9(c)). Thus sequences which are strongly positive because $\sum C(P_i) = \infty$

can be perturbed in quite drastic ways without losing strong positivity; sequences which are strongly positive may in general lose this property even under mild perturbations.

In the Introduction, we defined a polynomial P or its corresponding X to be *ε-convex* (for a positive real number $\varepsilon \leq 2$) if

$$\text{for all integers } j, \quad \Pr(X = j+1) + \Pr(X = j-1) \geq (2-\varepsilon)\Pr(X = j).$$

If $\{X_i\}$ is the sequence of random variables corresponding to $\{P_i\}$, then we defined $\{X_i\}$ to be strongly positive if for all integers k and positive real numbers ε, there exists N so that the sum of N consecutive random variables beginning with X_{k+1} (denoted $S_{N,k}$), is ε-convex. This is a particularly strong form of local flatness. Now we prove the analogue of Mineka's criterion in the context of strong positivity. A consequence of the following is that $\{X_i\}$ is strongly positive if and only if the corresponding sequence of polynomials is strongly positive.

COROLLARY 1.13. *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients and $\{\varepsilon_i\}$, such that each P_i is ε_i-convex. Then $\{P_i\}$ is strongly positive if*

$$\sum(2-\varepsilon_i) \quad \textit{diverges.}$$

PROOF. *We show that $C(P_i) \geq \frac{1}{2}(2-\varepsilon_i)$, so that 1.11 will apply. Let j be an element of $\mathcal{M}(P_i)$ (the set of local maxima), and write $P_i = \sum a_{i,k}x^k$. Then $a_{i,j-1} + a_{i,j+1} \geq (2-\varepsilon_i)a_{i,j}$, so at least one of the ratios $a_{i,j-1}/a_{i,j}$, $a_{i,j+1}/a_{i,j}$ must be at least as large as $\frac{1}{2}(2-\varepsilon_i)$.* ∎

2. **Harmonic functions, states, and dimension groups.** In this section, we discuss harmonic functions and their relatives, primarily as obstructions to strong positivity. There is a natural one-parameter family of harmonic functions on the random walk $\{X_i\}$, consisting of "point evaluations", corresponding to points in $\mathbf{R}^+ \cup \{\infty\}$. For strong positivity to occur, it is necessary (but not sufficient) that all of the point evaluations be pure (extreme), and that all pure harmonic functions arise as point evaluations. (We actually work with a slight generalization, called $[0,\infty]$-valued harmonic functions.) The most natural point evaluation is that at 1, and this is pure if and only if the conclusion of Mineka's theorem holds (2.5). By means of reparameterization, Mineka's theorem gives rise to a method to decide whether an individual point evaluation is pure (as in the Examples following 2.5). In general, the criterion only determines sufficient conditions. However, if X_i are all unimodal, Mineka's criterion also gives necessary conditions, even though the reparameterized random variables need not be unimodal themselves.

When a point evaluation is not pure, by definition, it can be decomposed into a combination of other harmonic functions (which are not point evaluations). One extreme situation arises when there exists a sequence of positive real numbers $\{\alpha_i\}$ so that the infinite product $\prod_{i=1}^{\infty} P_i(z)/\alpha_i$ converges (in the usual sense of infinite products) on an annulus or disk in $\mathcal{C}$. Let F denote the limit function. Each coefficient of its Laurent series expansion (about the origin), gives rise to a harmonic function, yielding a family of harmonic functions indexed by $\mathbf{Z}$ or $\mathbf{N}$ (the latter for convergence on a disk). If $F(r)$ exists for a fixed positive real number r, the point evaluation at r decomposes into a convex linear combination of members of this discrete family. In some cases (e.g., if the P_i are strongly unimodal or unimodal of bounded total degree, and $\sum r(P_i) < \infty$)), this single discrete family constitutes all of the pure harmonic functions (4.4). In other cases, the pure harmonic functions include numerous discrete families in addition to all of the point evaluations. Of particular interest is the fairly sharp dichotomy in the symmetric unimodal case (2.7).

For our purposes, a (space-time) *harmonic function* on the random walk arising from $\{X_i\}$ is a nonzero function $h\colon \mathbf{Z} \times \mathbf{N} \longrightarrow \mathbf{R}^+$ satisfying the compatibility conditions:

$$(\ddagger) \qquad h(j,n) = \sum_{k \in \mathbf{Z}} h(j+k, n+1) \Pr(X_{n+1} = k)$$

(formally, $h(j,n) = \int_{\mathbf{Z}} h(j+k, n+1)\, dX_{n+1}(k)$). We usually insist that $h(j,1) > 0$ for some j in the support of X_1, and then normalize (dividing by a scalar) so that $1 = \sum h(k,1) \Pr(X_1 = k)$. For each positive real number r, there is a naturally occurring harmonic function, given via the formula

$$h_r(j,n) = \frac{r^j}{P_1 \cdot P_2 \cdot \cdots \cdot P_n(r)}.$$

This arises from evaluating the rational function $x^j / P_1 \cdot P_2 \cdot \cdots \cdot P_n$ at r, hence the name "point evaluation". These are often known as "space-time" harmonic functions; since we are not assuming the process is stationary (X_i equal to each other), there is little likelihood of confusion with spatial harmonic functions. This interpretation will be more concrete when we define the dimension groups associated to the random walk. Point evaluations are automatically normalized, and we may let r tend to 0 or to ∞. In this case, the limit functions are not everywhere defined, and this motivates the following definition.

We say $h\colon \mathbf{Z} \times \mathbf{N} \longrightarrow \mathbf{R}^+ \cup \{\infty\}$ is a $[0,\infty]$-*valued harmonic function* if (i) it satisfies ($\ddagger$), (ii) for all j in the support of $S_{n,0}$, we have $h(j,n) < \infty$, and (iii) for at least one such j, we have $h(j,n) > 0$. These can be thought of as harmonic functions finite on a suitable cone in $\mathbf{Z} \times \mathbf{N}$. We usually normalize as for harmonic functions. Typical examples (and in the strongly positive case, the *only* pure examples) which are not harmonic functions are "evaluation at 0", and "evaluation at ∞", given respectively by

$$h_0(j,n) = \lim_{r \to 0} \frac{r^j}{P_1 \cdot P_2 \cdot \cdots \cdot P_n(r)} \qquad \text{and}$$
$$h_\infty(j,n) = \lim_{r \to \infty} \frac{r^j}{P_1 \cdot P_2 \cdot \cdots \cdot P_n(r)}.$$

(These pick out the coefficients of the terms of lowest and highest degrees in $P_1 \cdot P_2 \cdot \cdots \cdot P_n$ and are finite on the cone $\{(j,n) \mid j \in \operatorname{Log} P_1 \cdot \cdots \cdot P_n\}$.) Finally, there is the notion of an *extended* $[0,\infty]$-*valued harmonic function:* h satisfies ($\ddagger$), and there exists j in the support of $S_{n,0}$ such that $0 < h(j,n) < \infty$. This is used only in one result, a formal characterization of strongly positive sequences (2.1).

All of these concepts translate to the theory of dimension groups and Choquet theory. Regard the algebra of Laurent polynomials, $\mathbf{R}[x, x^{-1}]$, as a partially ordered (real) vector space with positive cone

$$\mathbf{R}[x, x^{-1}]^{+} = \{ P \in \mathbf{R}[x, x^{-1}] \mid P \text{ has no negative coefficients} \}.$$

Then multiplication by any P in $\mathbf{R}[x, x^{-1}]^{+}$ is a homomorphism between partially ordered vector spaces. If $\{P_i\}$ is a sequence of elements from $\mathbf{R}[x, x^{-1}]^{+}$, we may take the direct limit, as ordered vector spaces,

$$\begin{aligned} S(\{P_i\}) &= \lim \mathbf{R}[x, x^{-1}] \xrightarrow{\times P_1} \mathbf{R}[x, x^{-1}] \xrightarrow{\times P_2} \mathbf{R}[x, x^{-1}] \xrightarrow{\times P_3} \cdots \\ &= \{ f / P_1 \cdot P_2 \cdot \cdots \cdot P_n \mid f \in \mathbf{R}[x, x^{-1}] \}, \end{aligned}$$

and with the latter realization (as a vector space of certain rational functions), the positive cone is given by

$$S(\{P_i\})^{+} = \{ f / P_1 \cdot P_2 \cdot \cdots \cdot P_n \mid f \cdot P_{n+1} \cdot \cdots \cdot P_k \in \mathbf{R}[x, x^{-1}]^{+} \text{ for some } k > n. \}$$

Of critical importance is a certain ordered subspace of $S(\{P_i\})$, called $R(\{P_i\})$. This consists of the rational functions that are bounded by a multiple of the constant function, with respect to the relative ordering:

$$\begin{aligned} R(\{P_i\}) &= \{ f / P_1 \cdot P_2 \cdot \cdots \cdot P_n \mid \operatorname{Log} f \subseteq \operatorname{Log} P_1 \cdot P_2 \cdot \cdots \cdot P_n \} \\ R(\{P_i\})^{+} &= R(\{P_i\}) \cap S(\{P_i\})^{+}. \end{aligned}$$

It is routine to verify that the constant function $u = \mathbf{1}$ satisfies:

For all elements g of $R(\{P_i\})$, there exists an integer M so that $-Mu \leq g \leq Mu$.

An element of a partially ordered vector space (or abelian group) with this property is called an *order unit*. A *state* on a partially ordered vector space is a nonzero real-valued group homomorphism sending the positive cone into $\mathbf{R}^{+}$. If the ordered vector space admits an order unit u, we may normalize the state γ by replacing γ by $\gamma / \gamma(u)$. A state is *pure* (or extremal) if it cannot be expressed as a non-trivial positive linear combination of distinct states; if the state is normalized, "convex linear" replaces "positive linear". Similar definitions apply to harmonic functions, as well as $[0, \infty]$-valued harmonic functions.

As $S(\{P_i\})$ is a limit of lattice ordered abelian groups, and $R(\{P_i\})$ is an order ideal therein, they are *dimension groups* [El], [EHS]. We shall use a characterization of purity for states of dimension groups later (Theorem 2.5). For details, see any of the standard reference works, e.g. [G], [Ef].

It is completely routine (using the Riesz interpolation property) to verify that there is a natural correspondence between the states of $S(\{P_i\})$ and the harmonic functions associated to $\{X_i\}$; they are related via

$$h(j, n) = \gamma(x^j / P_1 \cdot P_2 \cdot \cdots \cdot P_n).$$

The same correspondence holds between (normalized) $[0, \infty]$-valued harmonic functions and states on $R(\{P_i\})$, normalized at $u = \mathbf{1}$. In both cases, purity is preserved by this identification of functions with states.

A convex directed subgroup of a partially ordered abelian group or vector space is called an *order ideal* of the larger object. Then the extended $[0, \infty]$-valued harmonic functions correspond with states on order ideals of $R(\{P_i\})$.

As there is a great deal known about states on vector spaces (see for example, [A], [AE], etc.), we shall deal almost entirely with states rather than harmonic functions and their generalizations. However, the correspondence above is complete, so the reader may freely translate between the two notions.

The states corresponding to point evaluations are given simply by evaluation of the rational function at the indicated point, i.e.,

$$\gamma_r(f/P_1 \cdot P_2 \cdot \cdots \cdot P_n) = (f/P_1 \cdot P_2 \cdot \cdots \cdot P_n)(r),$$

and the states, γ_0 and γ_∞ on $R(\{P_i\})$, are precisely the two limiting states. We note that these two states are always pure (for arbitrary $\{X_i\}$). However, point evaluations need not be pure, and we shall give necessary and sufficient conditions for their purity. Theorem 2.1 below is a straightforward consequence of [EHS, 1.4].

A Laurent polynomial P is *projectively faithful* if the set of differences $\operatorname{Log} P - \operatorname{Log} P$ generates the standard copy of $\mathbf{Z}$ in $\mathbf{R}$ as an abelian group. A sequence $\{P_i\}$ is *projectively faithful* if for all k,

$$\bigcup_n \left(\operatorname{Log} P^{n,k} - \operatorname{Log} P^{n,k}\right) = \mathbf{Z}.$$

As was pointed out by Alan Kelm, this is strictly stronger than requiring only a telescoping so that every term is projectively faithful (which was our original definition).

THEOREM 2.1. *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients. If $\{P_i\}$ is strongly positive, then every pure extended $[0, \infty]$-valued harmonic function is a point evaluation or a limit (restricted to the appropriate domain) of such, and all of these are pure. Conversely, if the sequence is projectively faithful and every pure extended $[0, \infty]$-valued harmonic function is a point evaluation or a limit of such, then $\{P_i\}$ is strongly positive.*

In particular, if $\{P_i\}$ is strongly positive, then all point evaluations are pure, and all pure states (pure $[0, \infty]$-valued harmonic functions) are point evaluations. These conditions are not sufficient for strong positivity: consider the case that $P_i = P = 1 + x + x^3$ for all i; then all pure states on $R(\{P_i\})$ are point evaluations and all point evaluations are pure, but the sequence is not strongly positive, because $(1 + x + x^3)^m \cdot (1 - x + x^2)$ has a negative coefficient for all m.

On the other hand, if the P_i are unimodal, then sufficient for strong positivity is that all pure harmonic functions be point evaluations (no "$[0, \infty]$" is necessary).

In any event, the presence of pure states (of $R(\{P_i\})$) that are not point evaluations is an obstruction to strong positivity. One particularly interesting source of these obstructions arises from a possible convergence of ratios phenomenon, which we now investigate. We can obviously replace each P_i by a scalar multiple of itself, P_i/α_i (typically $\alpha_i = P_i(1)$), or multiply each P_i by a monomial in x, say $x^{k(i)}$. Any sequence of such modifications of $\{P_i\}$ is called a *regularizing process.* Abbreviate $P^{n,0} = P_1 \cdot P_2 \cdot \cdots \cdot P_n$ to $P^{(n)}$. Suppose that for all j, after a possible regularization,

$$\lim_{n\to\infty} \frac{(P^{(n)}, x^j)}{(P^{(n)}, x^0)} \quad \text{exists and equals } \alpha_j.$$

Define a formal (Laurent) power series

$$F(z) = \sum_{j\in\mathbf{Z}} \alpha_j z^j.$$

For each j such that α_j is not zero, we may define a state or states on $R(\{P_i\})$ by telescoping the Ps and then setting

$$\gamma^j\left(\frac{f}{P^{(n)}}\right) = \lim_{m\to\infty} \frac{(f \cdot P^{m,n}, x^j)}{(P^{(m+n)}, x^0)}.$$

Without a suitable telescoping, the limit need not exist. Now suppose that the Laurent series F converges on the annulus $r_0 < |z| < R_0$, where either inequality may be weakened to less than or equal and $0 \le r_0 < R_0 \le \infty$. Then γ^j has the simpler and explicit form (not requiring telescoping)

$$\gamma^j(h) = \frac{(hF, x^j)}{(F, x^j)} \qquad \text{for } h \in R(\{P_i\})$$

(The inner product notation is extended to power series.) By the standard Laurent series representation, for positive real r within the annulus of convergence, $hF(r) = \sum (F, x^j) r^j \gamma^j(h)$, and so if the state given by point evaluation at r is denoted γ_r, we deduce

$$\gamma_r = \sum \frac{(F, x^j) r^j}{F(r)} \gamma^j.$$

This expresses γ_r as a σ-convex combination of the states γ^j. It is routine to verify that if F actually converges, then none of the γ^j can be point evaluations (except when $j = 0$). Thus we have deduced:

LEMMA 2.2. *If the formal Laurent series F converges on an annulus containing r, then the state corresponding to point evaluation at r is a σ-convex combination of the states γ^j arising from F. In particular, point evaluation at r is not a pure state.*

There may be several or even arbitrarily many Laurent series associated to a single sequence $\{P_i\}$ (different regularizations may be used), with disjoint annuli of convergence. If each P_i is strongly unimodal, the convergence phenomenon is very well behaved; this remark is also valid if the sequence of total degrees of the P_i's is bounded; we define the *total degree* of a Laurent polynomial P or its corresponding random variable X as

$$\max\{j \in \mathbf{Z} \mid \Pr(X = j) \neq 0\} - \min\{j \in \mathbf{Z} \mid \Pr(X = j) \neq 0\}.$$

LEMMA 2.3A. *Let $\{P_i\}$ be a sequence of Laurent polynomials for which there is a bound on the total degrees. Suppose that after some regularization, $\prod P_i(z)$ converges (absolutely) in a disk (in the complex plane) containing a positive real number. Then convergence occurs on all of $\mathbf{C} \setminus \{0\}$. In particular, no point evaluations are pure.*

PROOF. Suppose that convergence occurs in a disk of radius δ containing the positive real number r_0. By reparameterizing, we may assume that $r_0 = 1$ (with δ dilated accordingly). Let d_i denote the minimal exponent occurring in P_i, and D_i the maximal one. So $\sup\{D_i - d_i\}$ is bounded. If convergence occurs, we claim that both $\{d_i\}$ and $\{D_i\}$ must be bounded. To see this, we may first suppose that $P_i(1) = 1$, so for real $z \geq 1$, $P_i(z) \geq z^{d_i}$ (if $d_i > 0$). Hence $|P_i(z) - 1| \geq z^{d_i} - 1 > 0$. We conclude $\{d_i\}$ must be bounded above.

Hence $D = \sup\{D_i\}$ and $d = \inf\{d_i\}$ exist. Set $e = D - d$, and observe that the norm $\| \ \|_\delta$ on $\mathbf{R}^e$ given by $(r_1, r_2, \ldots, r_e) \longmapsto \sup\{ | \sum r_{j-1+d} z^j | \mid |z - 1| < \delta/2\}$ is equivalent to the l^1 norm (because all norms are equivalent). Hence there exists a positive number K such that $\sum |r_j| < K \|(r_j)\|_\delta$. It follows immediately from the uniform convergence of $\sum |P_i(z) - 1|$ (on a small disk) that the sums of the individual coefficients converge as well—but this forces convergence of $\sum |P_i(z) - 1|$ on all of $\mathbf{C} \setminus \{0\}$. ∎

For P a Laurent polynomial with no negative coefficients and λ a positive real number, recall the definition of the *reparameterization of P at λ*, $P(\lambda \cdot)$, obtained by replacing x^j by $(\lambda x)^j$. We say that P is *unimodal at λ* if $P(\lambda \cdot)$ is unimodal. This is equivalent to

$$\lambda \notin \bigcup_k \left\{ \alpha \in \mathbf{R} \;\middle|\; \frac{(P, x^{k+1})}{(P, x^k)} > \frac{1}{\alpha} > \frac{(P, x^k)}{(P, x^{k-1})} \right\}.$$

By 1.5, P is unimodal at all λ if and only if P is strongly unimodal. If $\{P_i\}$ is a sequence of Laurent polynomials with no negative coefficients, we say that it has a *common interval of unimodality* if there exist positive real numbers $a < b$ such that for all λ in the interval $[a, b]$, each P_i is unimodal at λ. For each such λ, $r(P_i(\lambda \cdot))$ is defined.

LEMMA 2.3B. *Let $\{P_i\}$ be a sequence of Laurent polynomials admitting a common interval of unimodality $[a, b]$. Suppose that $\sum r(P_i(\lambda \cdot)) < \infty$ for one value of λ in the open interval (a, b). Then there exists a sequence of integers $\{n(i)\}$ so that if $Q_i = x^{n(i)} P_i$,*

$$\prod_i \frac{Q_i(\lambda z)}{(Q_i(\lambda \cdot), x^0)} \qquad \textit{converges uniformly on the disk } a < |z| < b.$$

PROOF. By reparameterizing, we may assume that $\lambda = 1$, so that $a < 1 < b$. On multiplying each P_i by a power of x depending on i, we may assume that $(P_i, x^0) \geq (P_i, x^k)$ for all k—this defines $n(i)$. Since $\sum r(P_i) < \infty$, it follows that for all but finitely many

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values of i, $r(P_i) < \min\{1/b, a\}$. Discard those P_j for which $r(P_j) > \min\{1/b, a\}$. We note that $(P_i, x^{-1})/b < (P_i, x^0) > b(P_i, x^1)$, so that a local maximum occurs at 0 for $P_i(b\cdot)$. Since the latter is unimodal, *the* maximum occurs at 0. Hence for $k > 0$, $(P_i, x^k)b^k \leq (P_i, x^{k-1})b^{k-1} \leq \cdots \leq (P_i, x^1)b$. Thus $(P_i, x^k) \leq b^{1-k}(P_i, x^1)$. Similarly for $k < 0$, $(P_i, x^k) \leq a^{-k-1}(P_i, x^{-1})$.

For z a complex number,

$$|P_i(z) - (P_i, x^0)| \leq b \sum_{k>0} \left(|z|/b\right)^k \left(P_i, x^1\right) + a^{-1} \sum_{k<0} \left(|z|/a\right)^k \left(P_i, x^{-1}\right).$$

Now divide by (P_i, x^0); note that $(P_i, x^{\pm 1})/(P_i, x^0) \leq r(P_i)$. If $|z|/b < 1$ and $|z|/a > 1$ (that is, $a < |z| < b$), we deduce

$$\left|\frac{P_i(z)}{(P_i, x^0)} - 1\right| < (K + K')r(P_i)$$

where K and K' are constants (and $|z|$ is fixed). As $\sum r(P_i) < \infty$, the infinite product converges. ∎

COROLLARY 2.3C. *Let $\{P_i\}$ be a sequence of Laurent polynomials admitting a common interval of unimodality $[a, b]$. Then $\{P_i\}$ is strongly positive if and only if for one (hence any) value of λ in the open interval (a, b), $\sum r(P_i(\lambda\cdot)) = \infty$.*

If there is only a single point λ at which all the P_i are unimodal, then we may assume that all the P_i are already unimodal, by the obvious reparameterization. Proposition 2.6 gives a result about the pure states in this case.

Now we give some examples. After the purity criterion is established (2.5), we shall discuss more fully the purity of point evaluation harmonic functions.

EXAMPLES 2.4.

(a) Set $P_i = i^2 + x + \cdots + x^i$. Then $\prod P_i/i^2$ converges on the open unit disk; nowhere else is there even a regularization under which the infinite product will converge. It follows that for $0 < r < 1$, point evaluation at r is not pure. Purity of the other point evaluations will be discussed prior to 2.6.

(b) Set $P_i = i^3 + x + \cdots + x^i$. This time, $\prod P_i/i^3$ converges on the closed unit disk, and so the point evaluations in the interval $(0, 1]$ are not pure.

(c) Let $P_i = 1 + \frac{1}{i^2}x + x^2 + x^3$; then $\alpha_1 = \lim(P^{(n)}, x^1)/(P^{(n)}, x^0)$ exists (and equals $\pi^2/6$), but $\alpha_2, \alpha_3, \ldots$ do not exist. Consequently, there is no Laurent series expansion. Nevertheless, α_1 does yield a state; it is pure and not a point evaluation. In this example, all the point evaluations are pure, as we shall show later.

(d) Suppose that $P_i = i + \frac{1}{i}(x + x^2 + \cdots + x^{i^2}) + x^{2^i}(1 + i^2x)$. We note that for $|z| < 1$, $\prod P_i(z)/i$ converges absolutely, whereas for $|z| > 1$, the same holds for $\prod(z^{-2^i-1})P_i(z)/i^2$. Hence all point evaluations except that at 1 are not pure (we are excluding the evaluations at 0 and ∞ from this discussion).

(e) Set $P_i = 1 + x/i^2 + i^ix^i(1 + ix)$. The formal Laurent (power) series $F(z) = \sum_{j\geq 0} \alpha_j z^j$ exists but does not converge anywhere. We see that for j a fixed positive integer,

$$(P^{(n)}, x^j)/(P^{(n)}, x^0)$$

converges (as n increases), because only finitely many of the large coefficients contribute to the numerator. Thus we can define γ^j as we did earlier (formally), and it is routine to verify that none of them can be a point evaluation (except γ^0, which is point evaluation at 0). It is easy to check directly that F converges nowhere; this will also follow once we prove all point evaluations are pure.

(f) Set $P_i = 1 + x^{2^{i-1}}$, so that $P^{(n)} = 1 + x + \cdots + x^{2^n - 1}$. As every monomial in $P^{(n)}$ is attained by a unique product of terms, it is straightforward to see that the pure harmonic functions are in natural bijection with the Boolean space $\{0,1\}^{\mathbf{N}}$. There is no convergence of infinite products here and none of the point evaluations can be pure.

We now obtain the relevant purity criterion for point evaluations. We shall show that point evaluation at 1 is pure if and only if the flatness condition from the conclusion of Mineka's theorem holds. A necessary and sufficient criterion for purity of the other point evaluations then follows from reparameterization. As for evaluation at 0 or ∞, it is easy to verify that they are always pure.

Recall the definition of $\mathcal{F}(P)$ from Section 1. It is immediate that $\mathcal{F}(P) = \|(1-x)P\| / P(1)$ (using the l^1 norm), and that $\mathcal{F}(P)$ can also be computed as twice the sum of the local maxima (among the coefficients of P) less twice the sum of the local minima. Let $P = \sum \lambda_j x^j$ and suppose that $P(1) = 1$; then define $I_1(P) = \sum \min\{\lambda_j, \lambda_{j+1}\}$. Clearly, $2I_1(P) + \mathcal{F}(P) = 2$. Similarly, define $I_k(P) = \sum \min\{\lambda_j, \lambda_{j+k}\}$, and observe that $2I_k(P) \geq 2 - k\mathcal{F}(P)$.

THEOREM 2.5. *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients. Suppose that*

$$\lim_{N\to\infty} \mathcal{F}(P^{N,k}) = 0 \qquad \textit{for all positive integers } k. \tag{2}$$

Then point evaluation at 1 is a pure state of $R(\{P_i\})$ and of $S(\{P_i\})$, and the corresponding harmonic function is pure, even when viewed as a $[0,\infty]$-valued harmonic function. Conversely, if the sequence $\{P_i\}$ is projectively faithful, then purity of point evaluation at 1 entails that (2) hold.

PROOF. Since $R(\{P_i\})$ is a dimension group admitting an order unit (the constant function $\mathbf{1}$), by [GH, 3.1(b)], a normalized state is pure if and only if

(‡) For all a and b in $R(\{P_i\})^+$ and for all $\varepsilon > 0$, there exists z in $R(\{P_i\})^+$ such that $z \leq a$, $z \leq b$, and $\gamma(z) > \min\{\gamma(a), \gamma(b)\} - \varepsilon$.

We proceed to verify the criterion when γ is point evaluation at 1. Obviously, we may assume that $P_i(1) = 1$ for all i. Given a and b as indicated, we may also assume that $\gamma(a) = \gamma(b)$: if $\lambda = \gamma(b)/\gamma(a) > 1$, find the z corresponding to the pair a and b/λ; the same z works for a and b.

Now we may assume $a(1) = b(1) = 1$ and $P^{(k)}(1) = 1$ for all k. There exists an integer k so that $a = f/P^{(k)}$ and $b = g/P^{(k)}$, where f and g belong to $\mathbf{R}[x, x^{-1}]^+$, and each of

$\operatorname{Log} f$ and $\operatorname{Log} g$ is contained in $\operatorname{Log} P^{(k)}$. Since $P^{(k)}(1) = 1$, we have that $f(1) = g(1) = 1$. Write

$$f = \sum \lambda_j x^j, \qquad g = \sum \mu_j x^j \qquad \text{where } \lambda_j \geq 0 \text{ and } \mu_j \geq 0.$$

Since $\sum \lambda_j = \sum \mu_j$, applying Riesz decomposition over the real numbers, we may find non-negative real numbers λ_{jm} (with j and m in $\operatorname{Log} P^{(k)}$) such that

$$\text{for all } j, \quad \lambda_j = \sum_m \lambda_{jm} \qquad \text{and} \qquad \text{for all } m, \quad \mu_m = \sum_j \lambda_{jm}.$$

Select $\varepsilon > 0$. For a pair (j, m), define elements of $R(\{P_i\})^+$, $a_j = x^j / P^{(k)}$ and $a_m = x^m / P^{(k)}$. Their values at 1 are 1. We shall find z_{jm} in $R(\{P_i\})^+$ so that $0 \leq z_{jm} \leq a_j,\ a_m$ and $\gamma(z_{jm}) > 1 - \varepsilon$.

Suppose that $j < m$. Choose M (depending on j and m) so large that for all $N \geq M$, $(m-j)\mathcal{F}(P^{N,k}) < 2\varepsilon$. Define a polynomial c_{jm}

$$c_{jm} = \sum_{t \in \mathbf{Z}} \min\left\{(x^j P^{N,k}, x^t), (x^m P^{N,k}, x^t)\right\} \cdot x^t$$

for some N exceeding M. Then c_{jm} has no negative coefficients. Obviously, within the ordered vector space $S(\{P_i\})$, $c_{jm} \leq x^j P^{N,k}$ and $c_{jm} \leq x^m P^{N,k}$. Let $z_{jm} = c_{jm} / P^{(N)}$. Since j and m belong to $\operatorname{Log} P^{(k)}$, z_{jm} belongs to $R(\{P_i\})^+$, and obviously $0 \leq z_{jm} \leq a_j,\ a_m$. It remains to show that $z_{jm}(1) > 1 - \varepsilon$, or equivalently, $c_{jm}(1) > 1 - \varepsilon$.

Thus we must show

$$\sum_{t \in \mathbf{Z}} \min\left\{(x^j P^{N,k}, x^t), (x^m P^{N,k}, x^t)\right\} > 1 - \varepsilon.$$

The sum is simply $I_{m-j}(P^{N,k})$; by the earlier comments,

$$\begin{aligned} I_{m-j}(P^{N,k}) &\geq 1 - \frac{1}{2}(m-j)\mathcal{F}(P^{N,k}) \\ &> 1 - \varepsilon. \end{aligned}$$

Thus $z_{jm}(1) > 1 - \varepsilon$. Now define

$$z = \sum_{\{(j,m) \in \operatorname{Log} P^{(k)} \times \operatorname{Log} P^{(k)} \mid j \neq m\}} z_{jm} \lambda_{jm}$$

Then we have:

$$\begin{aligned} z &= \sum_{j,m} z_{jm} \lambda_{jm} \\ &\leq \sum_m \sum_j a_m \lambda_{jm} \\ &= \sum_m a_m \Big(\sum_j \lambda_{jm} \Big) \\ &= \sum_m a_m \lambda_m \\ &= a. \end{aligned}$$

Similarly $z \leq b$. Obviously, z belongs to $R(\{P_i\})^+$, so it remains to show that $z(1) > 1-\varepsilon$. We observe:

$$\begin{aligned} z(1) &= \sum_m \sum_j z_{jm}(1)\lambda_{jm} \\ &> (1-\varepsilon)\sum_m \sum_j \lambda_{jm} \\ &= (1-\varepsilon)\sum_j \lambda_j = 1-\varepsilon. \end{aligned}$$

The criterion having been satisfied, evaluation at 1 is a pure state of $R(\{P_i\})$. Since $R(\{P_i\})$ is an order ideal in $S(\{P_i\})$ and point evaluation extends to a state on the latter, it routinely follows that the point evaluation state is also pure on $S(\{P_i\})$.

Now we prove the converse. Assume that evaluation at 1 is a pure state of $R(\{P_i\})$, and that the sequence $\{P_i\}$ is projectively faithful; by telescoping and multiplying the P_i by monomials, we may assume that for all i, (P_i, x^0) and (P_i, x^1) are greater than zero. Then for any fixed k, the rational functions $a_0 = 1/P^{(k)}$ and $a_1 = x/P^{(k)}$ belong to $R(\{P_i\})^+$. From (‡), for some $N > k$, there exists $z = f/P^{(N)}$ in $R(\{P_i\})^+$ such that

$$0 \leq z \leq a_0, a_1 \quad \text{and} \quad z(1) > 1-\varepsilon.$$

As usual, we may assume that f has no negative coefficients and $\operatorname{Log} f \subseteq \operatorname{Log} P^{(N)}$. From $f/P^{(N)} \leq 1/P^{(k)}$, we deduce that for some $M \geq N$, $f \cdot P^{(M)}/P^{(N)} \leq P^{(M)}/P^{(k)}$ as computed with respect to $\mathbf{R}[x,x^{-1}]^+$. Similarly, $f \cdot P^{(M)}/P^{(N)} \leq x \cdot P^{(M)}/P^{(k)}$. As $\mathbf{R}[x,x^{-1}]$ is a lattice with respect to the usual ordering, the infimum of the two Laurent polynomials $P^{(M)}/P^{(k)}$ and $x \cdot P^{(M)}/P^{(k)}$ exists and is given by

$$d_{01} = \sum_{t\in\mathbf{Z}} \min\left\{(P^{(M)}/P^{(k)}, x^t), (xP^{(M)}/P^{(k)}, x^t)\right\} x^t.$$

By the lattice property, $f \cdot P^{(M)}/P^{(N)} \leq P^{(M)}/P^{(k)}, xP^{(M)}/P^{(k)}$ computed with respect to the ordering in $\mathbf{R}[x,x^{-1}]$. On dividing by $P^{(M)}$, we obtain (with respect to the ordering on $S(\{P_i\})$), $z \leq d_{01}/P^{(M)} \leq a_0$, a_1. In particular, it follows that $d_{01}/P^{(M)}$ belongs to $R(\{P_i\})^+$. Evaluating this inequality at $x=1$, we deduce

$$1-\varepsilon < \sum_{t\in\mathbf{Z}} \min\left\{(P^{(M)}/P^{(k)}, x^t), (xP^{(M)}/P^{(k)}, x^t)\right\} \leq 1.$$

The middle term is $I_1(P^{(M)}/P^{(k)})$, so that $\mathcal{F}(P^{(M)}/P^{(k)}) < 2\varepsilon$. ∎

This criterion for purity of evaluation at 1 can be modified to obtain a criterion for purity of any point evaluation. To test for purity of evaluation at r on the sequence $\{P_i\}$, define the reparameterized polynomials $P_i(r\cdot)$ by $P_i(r\cdot x) = P_i(rx)$. Then point evaluation at 1 is pure for $\{P_i(r\cdot)\}$ if and only if point evaluation at r is pure for the original sequence. Mineka's criterion is the standard test for sufficiency. While in general it is not necessary that $\sum 2 - \mathcal{F}(P_i)$ diverge, this is the case when all the P_i are unimodal (as we shall see after completing our discussion of the examples).

EXAMPLES (REVISITED). (See 2.4)

(a) It remains to consider point evaluations at $r \geq 1$. The normalized reparameterized polynomials are $\left(i^2 + \sum_{j \leq i} x^j r^j\right) / \left(i^2 + \sum_{j \leq i} r^j\right)$. With r fixed, for almost all i, the maximal coefficients occur at $j = 0, i$ and the only minimal coefficient occurs at $j = 1$. A simple computation reveals that at $r = 1$, we have that $2 - \mathcal{F}(P_i)$ behaves as $1/i$, so the sum diverges, while for $r > 1$, we find that $2 - \mathcal{F}\left(P_i(r\cdot)\right)$ behaves as $1/r$ and again divergence occurs. By Mineka's criterion, the point evaluation states are pure for $r \geq 1$. We have already seen that they are impure for $0 < r < 1$.

(b) Essentially the same computations as in (a) yield that for $r > 1$, the point evaluations are pure. We already knew that the point evaluations were not pure for $0 < r \leq 1$. In both cases (a) and (b), it turned out that if $\sum 2 - \mathcal{F}(P_i(r\cdot)) < \infty$, then point evaluation at r is not pure; that is, the converse to Mineka's criterion holds. This is no coincidence—in both cases, the P_i are all unimodal (although of course, their reparameterizations are not).

(c) Here $\mathcal{F}(P_i(r\cdot))$ behaves as $2/(3r)$, so that all point evaluations are pure.

(d) We saw earlier that all point evaluations except possibly that at 1 are not pure. To test the remaining point evaluation, we compute $\mathcal{F}(P_i) = 2 - 2/(i+1)$, so Mineka's criterion yields purity.

(e) For all sufficiently large i, $\mathcal{F}(P_i(r\cdot))$ behaves as $2 - 2/(ri)$, so that for all r, the point evaluation at r is pure.

As promised, the strongest possible converse to Mineka's theorem holds when $\{P_i\}$ consists of unimodal polynomials; however, the converse does not hold in general—consider the sequence defined by means of $P_{2i} = 1 + x^2$ and $P_{2i+1} = 1 + x^3$).

PROPOSITION 2.6. *(A converse to Mineka's Theorem, without blocking.) Let $\{P_i\}$ consist of unimodal polynomials. For r a positive real number, the state given by point evaluation at r is pure if and only if*

$$\textit{either } \sum 2 - \mathcal{F}\left(P_i(r\cdot)\right) \textit{ diverges} \quad \textit{or} \quad \sum r(P_i) \textit{ diverges.}$$

PROOF. If the sum of the ratios diverges, then $\{P_i\}$ is strongly positive, and so all point evaluations are pure. If the other sum diverges, purity of the point evaluation at r follows from Mineka's theorem and 2.5. So assume that both sums converge for some r. We shall show that evaluation at r is not pure. In particular, $r(P_i)$ tends to zero, and we may assume that the maximum coefficient occurs at 0, that is, (P_i, x^0) is the largest coefficient. It follows that for almost all i, the constant coefficient is a local maximum among the coefficients of $P_i(r\cdot)$. We now argue in the case that $r \geq 1$ (if $r < 1$, repeat the argument below with $P_i(x^{-1})$ replacing $P_i(x)$).

Let $P = \sum_{-e \leq j \leq d} \beta_j x^j$ be unimodal with the constant term being its maximal coefficient. The local maxima among the coefficients of $P(r\cdot)$ occur at those j such that $\beta_j/\beta_{j-1} \geq 1/r \geq \beta_{j+1}/\beta_j$, and local minima occur when the inequalities are reversed.

Denote by $\mathcal{M} = \mathcal{M}_r(P)$ the set of local maxima, and by $m = m_r(P)$, the set of local minima. If j belongs to $\mathcal{M}_r(P)$, it contributes $2\beta_j r^j / P(r)$ to $\mathcal{F}(P(r\cdot))$; if it belongs to $m_r(P)$, its contribution is $-2\beta_j r^j / P(r)$. For $j > 0$, $\beta_{j-1} \geq \beta_j$, so that $\beta_{j-1} r^{j-1} \geq \frac{1}{r}\beta_j r^j$. When $j < 0$, replace $j-1$ by $j+1$. In any event,

$$\begin{aligned} 2 - \mathcal{F}(P(r\cdot)) &= \frac{2\sum_{j\notin\mathcal{M}} \beta_j r^j + 2\sum_{j\in m} \beta_j r^j}{P(r)} \\ &\geq \frac{1}{r}\frac{\sum_{j\neq 0}\beta_j r^j}{P(r)} \\ &= \frac{1}{r}\frac{P(r)-\beta_0}{P(r)}. \end{aligned}$$

Thus convergence of $\sum 2 - \mathcal{F}(P_i(r\cdot))$ entails that $\sum \frac{P_i(r)-(P_i,x^0)}{P_i(r)}$ converges. This permits us to conclude that the infinite product, $\prod \frac{(P_i,x^0)}{P_i(r)}$ converges—meaning that it converges to a nonzero number. However, the coefficient of x^0 in

$$\frac{P_1(r\cdot)\cdot P_2(r\cdot)\cdot \cdots \cdot P_N(r\cdot)}{P_1(r)\cdot P_2(r)\cdot \cdots \cdot P_N(r)}$$

is always greater than this number. This obviously contradicts $\mathcal{F}(P_1(r\cdot)\cdot P_2(r\cdot)\cdot \cdots \cdot P_N(r\cdot)) \longrightarrow 0$. ∎

If we specialize further, we can obtain sharper results. Let us additionally assume that each P_i is *increasing* at r; say each P_i is increasing on the interval $I = (r_1, r)$. (Note that P_i is unimodal and its maximum coefficient is the constant term, so this condition is not automatically satisfied.) For example, if P is symmetric about the origin (that is, $(P, x^j) = (P, x^{-j})$ for all j), then P is monotone increasing to the right of 1. Suppose that $\sum 2 - \mathcal{F}(P_i(r\cdot))$ converges. From the proof, this forces $\sum \frac{P_i(r)-(P_i,x^0)}{P_i(r)}$ to converge. Then for any r' in I, we have that $\sum \frac{P_i(r')-(P_i,x^0)}{P_i(r)}$ also converges. If the complex number z belongs to the open annulus centred at the origin with radii r_1 and r, it easily follows that

$$\sum \frac{P_i(z)-(P_i,x^0)}{P_i(r)}$$

converges. From this, we deduce that the infinite product $\prod P_i(z)/P_i(r)$ converges on a disk containing I. In particular, the point evaluations in I are all impure. If P_i are all decreasing, then the corresponding interval of convergence is to the right of r.

When each P_i is symmetric unimodal, the polynomials are monotone decreasing going to the left from 1 and monotone increasing to the right. Thus if $r \neq 1$ and evaluation at r is not pure, then (i) all the point evaluations in the open interval with endpoints $1/r$ and r are not pure, and (ii) for every point in the interior, the infinite product converges. So there is an annulus of convergence for the infinite product, F; the annulus is centred at the origin and has $1/r$ and r as its inner and outer radii (from the invariance of the polynomials under the involution $x \mapsto x^{-1}$). We summarize this discussion:

PROPOSITION 2.7. *Let $\{P_i\}$ be a sequence of symmetric unimodal polynomials. If the set*

$$D(\{P_i\}) = \{ r \in \mathbf{R}^+ \mid \gamma_r \text{ is not a pure state of } R(\{P_i\})\}$$

is neither empty nor all of $\mathbf{R}^+$, then it must be of the form $(1/r, r)$ or $[1/r, r]$ for some real number $r \geq 1$. Moreover, the infinite product $\prod P_i(z)/P_i(1)$ converges on the centred annulus containing this interval, and on no bigger annulus.

All of the possibilities discussed in 2.7 can be obtained. For this purpose (and also for Section 3), we define a *T-function* to be a symmetric unimodal polynomial of the form P where

$$(P, x^j) = \begin{cases} h & \text{if } j = 0 \\ 1 & \text{if } 1 \leq |j| \leq d \\ 0 & \text{if } |j| > d \end{cases}$$

for some integer d and positive real number h. The distribution of the coefficients has the shape of an inverted letter T. The number d is called the *semi-width* of P, and $h = 1/r(P)$ is called the *height*. Any positive scalar multiple will also be called a T-function.

Let $\{P_i\}$ be a sequence of T-functions, with corresponding semi-widths $\{d(i)\}$ and heights $\{h(i)\}$. We observe that

$$\sum 2 - \mathcal{F}(P_i) \quad \text{converges iff } \sum \frac{d(i)}{h(i)} \text{ converges}$$

$$\sum 2 - \mathcal{F}(P_i(r\cdot\,)) \text{ converges iff } \sum \frac{r^{d(i)}}{h(i)} \text{ converges (for } r > 1).$$

(In the latter case, note there are generically just two local maxima in the coefficients of $P_i(r\cdot\,)$—at 0 and the right endpoint of the support, $d(i)$.) By 2.7, the formulas above permit us to decide whether a given point evaluation is pure.

EXAMPLE 2.8. We fix $h(i) = i^2$ and choose $d(i)$ so that the corresponding sequence of T-functions $\{P_i\}$ will have the indicated intervals of impure point evaluation states. Brackets ([ ]) will denote the greatest integer function.

(a) $D(\{P_i\}) = \mathbf{R}$. Select $d(i)$ to satisfy $d(i) = \mathbf{o}(\log i)$ (e.g., $d(i) = 1$). Then both sums converge, the latter for all $r > 1$, so the infinite product converges on the whole punctured plane.

(b) $D(\{P_i\}) = [1/r, r]$ for some $r > 1$. Set $d(i) = [\log_r i - 2\log_r \log_r i]$; one easily computes that $\sum \frac{r'^{d(i)}}{i^2}$ converges at $r' > 1$ if and only if $r' \leq r$.

(c) $D(\{P_i\}) = (1/r, r)$ for some $r > 1$. Set $d(i) = [\log_r i]$; the computation is straightforward.

(d) $D(\{P_i\}) = \{1\}$. Define $d(i)$ so that for some $\varepsilon > 0$, $(\ln i)^{1+\varepsilon} \leq d(i) \leq i/(\ln i)^{1+\varepsilon}$. Then the second sum diverges for all $r > 1$, and all point evaluations at $r \neq 1$ are pure. However, the first sum is bounded by $\sum 1/i(\ln i)^{1+\varepsilon}$ and so converges.

(e) $D(\{P_i\})$ is empty. One might conjecture that in the (very special) case of symmetric unimodal polynomials, if all point evaluations are pure, then the sequence is strongly positive. If we consider two closely related choices for $d(i)$, namely $d(i) = i$ and $d(i) = [i/\ln i]$, we see immediately that all point evaluations are pure. However, in the first case, the sequence is strongly positive (3.9(c)), while in the second, it is not (3.9(b))! These results come from the strong positivity criteria (applicable more generally to symmetric unimodal polynomials) developed in the next section. In fact, there is an extra family of pure states in the latter example, arising from limiting ratios not equal to 1.

3. **Strong positivity criteria for symmetric unimodal polynomials.** In this section, we give sufficient (as well as some necessary) criteria for a sequence of symmetric unimodal Laurent polynomials, $\{P_i\}$, to be strongly positive. In the first section, we saw that for general unimodal polynomials, divergence of $\sum r(P_i)$ is sufficient, but not necessary. Here we assume that $\sum r(P_i)$ converges, and derive some numerical criteria which will guarantee strong positivity. In the case of T-functions (defined at the end of Section 2), some sharp criteria can be given.

We present a reasonably effective criterion (3.4) for strong positivity of sequences of symmetric unimodal polynomials. This criterion is considerably improved in 3.6. In 3.8, we obtain necessary conditions for a sequence of T-functions to be strongly positive—these involve functions of the heights and semi-widths. In this case, we modify the distribution (beyond the origin), and show that after multiplying the modified polynomials, the resulting ratio of the product is dominated by the ratio of the original product.

Let P be a symmetric unimodal polynomial (with no negative coefficients). For a (small) positive real number δ, define $\delta(P)$ to be the polynomial obtained from P by multiplying everything but the constant coefficient by $1-\delta$, and leaving the constant term untouched; in other words,

$$(\delta(P), x^i) = \begin{cases} (1-\delta)(P, x^i) & \text{if } i \neq 0 \\ (P, x^0) & \text{if } i = 0. \end{cases}$$

Define the following property for a sequence $\{P_i\}$ (of symmetric unimodal polynomials):

$$(\ddagger) \qquad \lim_{N\to\infty} \frac{\prod_{i=1}^{N}(P_i, x^0)}{(P^{(N)}, x^0)} = 0.$$

It is true in special cases that if $\{P_i\}$ satisfies $(\ddagger)$, then it is strongly positive, and we believe it is true in general. Although we cannot prove this, the "δ" version of it is true (see Proposition 3.2), and in view of our subsequent computations, this is enough to obtain large classes of strongly positive sequences.

For a symmetric unimodal polynomial P (not necessarily a T-function), let $d(P)$ denote the semi-width of P, that is, $\max\{\, i \in \mathbf{N} \mid (P, x^i) \neq 0\}$, and $h(P)$ will be the height, that is, $1/r(P)$.

LEMMA 3.1. *Suppose that the sequence of symmetric unimodal polynomials* $\{P_i\}_{i=1}^N$ *satisfies the following two conditions, where* $d(i) = d(P_i)$*:*

(a) Viewed as a map on a finite set of integers, d is strictly monotone.

(b) There exists $\alpha > 0$ *such that for all i and for* $1 \leq j < d(i)$, $(P_i, x^{j+1}) \geq \alpha(P_i, x^j)$.

If $\frac{\prod_{i=1}^N (P_i, x^0)}{(P^{(N)}, x^0)} < \varepsilon$, *then* $\frac{(P^{(N)}, x^1)}{(P^{(N)}, x^0)} \geq \frac{1-\varepsilon}{1+1/\alpha}$.

PROOF. Form the finite measure space $K(i) = \{-d(i), \dots, -1, 0, 1, \dots, d(i)\}$, with measure obtained from the polynomial P_i—that is, the measure of $\{j\}$ is (P_i, x^j). Let K be the cartesian product of the first N of the $K(i)$'s. The product measure on K is determined by $\mu(\{k\}) = \prod (P_i, x^{k(i)})$, for $k = \big(k(1), k(2), \dots, k(N)\big)$. Let $S: K \longrightarrow \mathbf{Z}$ denote the addition map, $S(k) = \sum k(i)$. Then

$$(P^{(N)}, x^j) = \sum_{k \in S^{-1}\{j\}} \mu(\{k\}) = \mu(S^{-1}\{j\}).$$

Set $k_0 = (0, 0, \dots, 0)$, and let $S' = S^{-1}\{0\} \setminus \{k_0\}$, and for an integer t, let $S_t = S^{-1}(\{t\})$. Define a map ϕ_{-1} with domain in S_{-1} and range in S'. Choose k in S_{-1} such that if i is the smallest integer for which $k(i) = \max\{k(j) \mid 1 \leq j \leq N\}$, then $0 \leq k(i) < d(i)$. Define $\phi_{-1}(k)$ in S' by replacing $k(i)$ by $k(i) + 1$. Since the P_i are unimodal with maximal coefficient at the origin, $\mu(\{\phi_{-1}(k)\}) \leq \mu(\{k\})$; it follows that the measure of the domain of ϕ_{-1} is at least as large as the measure of its range.

The elements of S' not in the range of ϕ_{-1} are precisely those k in S' for which the cardinality of $\{i \mid k(i) = \max_{j \leq N} k(j)\}$ exceeds one. Define ϕ_1 on a subset of S_1 as follows. Select k in S_1 such that $\{i \mid k(i) = \max k(j)\}$ contains just one element, and in addition the maximal value of $k(j)$ exceeds one. If i is the unique integer for which $k(i)$ is maximal, replace it by $k(i) - 1$. This yields $\phi_1(k)$ in S', and by (b), $\alpha\mu(\{\phi_1(k)\}) \leq \mu(\{k\})$. Hence $\mu(\operatorname{range}\phi_1) \leq \alpha^{-1}\mu(\operatorname{dom}\phi_1)$.

We claim that $S' = \operatorname{range}\phi_1 \cup \operatorname{range}\phi_{-1}$. If k is an element of S' and $k(i_1) = k(i_2) = \max k(j)$, then $k(i_1) > 0$ and by (a), either $k(i_1) < d(i_1)$ or $k(i_2) < d(i_2)$. Select i such that $k(i) = k(i_1)$ and $k(i) < d(i)$ and replace $k(i)$ by $k(i) + 1$. This yields k_1 in S_1 and in the domain of ϕ_1 such that $\phi_1(k_1) = k$.

Thus $\mu(S') \leq \mu(\operatorname{range}\phi_1) + \alpha^{-1}\mu(\operatorname{range}\phi_{-1}) \leq (1+\alpha^{-1})\mu(S_1)$. Since $(1-\varepsilon)\mu(S_0) \leq \mu(S')$ by hypothesis, we deduce $(P^{(N)}, x^1)/(P^{(N)}, x^0) \geq (1-\varepsilon)/(1+1/\alpha)$, as desired. ∎

That the following was likely to be true with $\delta = 0$ was suggested by an anonymous probabilist. We do not know if this is the case, although it obviously follows when the hypotheses of 3.1 hold on setting $N = \infty$. Even when it is true, we cannot eliminate the ε term that appears in 3.6 and 3.7.

PROPOSITION 3.2. *Suppose that* $\{P_i\}$ *is a sequence of symmetric unimodal polynomials such that for some* $\delta > 0$, $\{\delta(P_i)\}$ *satisfies* $(\ddagger)$ *(see page 29). Then* $\{P_i\}$ *is strongly positive.*

PROOF. To begin with, we show that $(\ddagger)$ persists when finitely many terms are deleted from the sequence. We define for integers $n_0 < n$,

$$f(n_0, n) = \frac{\prod_{i=n_0}^n \big(P_i, x^0\big)}{\big(P_{n_0+1} \cdot P_{n_0+2} \cdot \dots \cdot P_n, x^0\big)}.$$

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We claim that if $f(0,n) \longrightarrow 0$ as $n \longrightarrow \infty$, then for all $n_0, f(n_0,n) \longrightarrow 0$ as $n \longrightarrow \infty$.

To see this, we observe that

$$f(n_0,n) = \frac{1}{f(0,n_0)} \frac{\left(P^{(n)},x^0\right)}{\left(P^{(n)}/P^{(n_0)},x^0\right)\left(P^{(n_0)},x^0\right)} f(0,n).$$

For symmetric unimodal polynomials P and Q, it is immediate that $\left(PQ,x^0\right) \leq Q(1)\left(P,x^0\right)$. Setting $P = P^{(n)}/P^{(n_0)}$ and $Q = P^{(n_0)}$, we have that $f(n_0,n) \leq Kf(0,n)$ where K depends only on n_0.

This permits us to delete any finite number of terms from the sequence.

Now let P be any symmetric unimodal polynomial, and write $P = \delta(P) + \delta \sum_{j>0}(P,x^j)(x^j + x^{-j})$. We alter P to a symmetric unimodal polynomial P' satisfying various properties; the idea is that the original sequence $\{P_i\}$ will be transformed into another one $\{P_i'\}$ which satisfies the hypotheses of the previous lemma. The polynomial will satisfy:

(i) Given an integer $M \geq d(P)$, $d(P') \in \{M, M+1\}$;

(ii) P satisfies condition (b) of the preceding lemma, with $\alpha = \delta/2$;

(iii) for all i, $(P',x^i) \geq (\delta(P),x^i)$; moreover, if $d(P) < |i| < d(P')$, then $(P',x^i) \leq \delta \cdot (P,x^1)/2^{i+k-d(P)}$ for some $k \leq d(P)$.

Set $P^0 = \delta(P)$ and find the smallest integer i with $1 < i \leq d(P)$ for which $(P^0,x^i) < (P^0,x^{i-1})/2$. Alter P^0 by adding $\min\left\{\frac{\delta}{2}(P,x^1), \frac{1}{2}(P^0,x^{i-1})\right\}$ to the coefficients of x^i and x^{-i}; then add exactly half this amount to the coefficients of $x^{\pm(i+1)}$. Label the resulting polynomial P^0 and observe that it is still unimodal and the ratio of consecutive coefficients $(P^0,x^j)/(P^0,x^{j-1})$ is at least $\delta/2$ for $1 < j \leq i+1$. In the new P^0 find the smallest integer i exceeding the previous i such that $(P^0,x^i)/(P^0,x^{i-1}) < 1/2$. Add $\min\left\{\frac{\delta}{2\cdot 2^2}(P,x^1), \frac{1}{2}(P^0,x^{i-1})\right\}$ to the coefficients of x^i and x^{-i} and add exactly half this amount to the coefficients of $x^{\pm(i+1)}$. Iterate this procedure until the resulting polynomial has degree M or $M+1$, and call it P'.

It is an easy verification that all of (i), (ii), and (iii) hold. What we have done is to redistribute some of the extra mass accruing from $\delta \cdot (P,x^1)$ of P over $\delta(P)$.

Now suppose that $\{\delta(P_i)\}$ satisfies (‡). Let $M_1, M_2, \ldots$ be a sequence of positive integers with the property that $M_i > M_{i+1}, d(P_i)$ for all i. Using these as the integers "M", construct P_i' with the process above. We note that the degrees of the P_i' are strictly increasing, so in particular are distinct; moreover, each P_i' satisfies condition (b) of the previous lemma.

Now we show (in order): $\{P_i'\}$ satisfies (‡); $\{P_i'\}$ is strongly positive; strong positivity of $\{P_i'\}$ implies that of $\{P_i\}$.

We check that $\{P_i'\}$ satisfies (‡) as a consequence of $\{\delta(P_i)\}$ satisfying it. We note that $P_i' = \delta(P_i) + Z_i$ where Z_i has no negative coefficients, is symmetric, and $(Z_i,x^0) = 0$. A simple computation yields that for all symmetric unimodal polynomials Q, we have $(P_i'Q,x^0) \geq (\delta(P_i)Q,x^0)$. It follows that $(\prod P_i',x^0) \geq (\prod \delta(P_i),x^0)$. Since $(P_i',x^0) = (\delta(P_i),x^0)$, (‡) follows.

By the previous lemma, there exists an integer N so that $\prod_{i=1}^{N} P'_i$ has ratio at least $\delta/4$. By the first paragraph of this proof, we may delete the first N terms from the sequence and repeat the process described above; hence there is a telescoping of $\{P'_i\}$ such that each term has ratio $\delta/4$. By Corollary 1.9, $\{P'_i\}$ is strongly positive.

Now we show strong positivity of $\{P'_i\}$ entails that of $\{P_i\}$. Suppose that T and U are symmetric unimodal polynomials with no negative coefficients such that

$$T = U - \varepsilon(U,x^1)(x^1+x^{-1}) - \varepsilon'(U,x^k)(x^k+x^{-k}) + \varepsilon''(U,x^l)(x^l+x^{-l}),$$

where $\varepsilon'' < \varepsilon/2$. (Note that P' can be obtained from P by a sequence of such operations, with suitable choices of parameters.) We show that $r(TQ) \leq r(UQ)$ if the latter is less than $1/4$. We calculate

$$\begin{aligned}
(TQ,x^0) &= (UQ,x^0) - 2\varepsilon(U,x^1)(Q,x^1) - 2\varepsilon'(U,x^k)(Q,x^k) + 2\varepsilon''(U,x^l)(Q,x^l)\\
(TQ,x^1) &= (UQ,x^1) - \varepsilon(U,x^1)((Q,x^0)+(Q,x^2)) - \varepsilon'(U,x^k)\big((Q,x^{k-1})+(Q,x^{k-1})\big)\\
&\qquad + \varepsilon''(U,x^l)\big((Q,x^{l+1})+(Q,x^{l-1})\big)\\
&\leq (UQ,x^1) - \frac{\varepsilon}{2}(U,x^1)\big((Q,x^0)+(Q,x^2)\big) - \varepsilon'(U,x^k)(Q,x^{k-1}).
\end{aligned}$$

Thus $r(TQ) \leq \frac{(UQ,x^1)-a}{(UQ,x^0)-b}$ and $b < 4a$. Thus the former is bounded above by the expression $(UQ,x^1)/(UQ,x^0) = r(UQ)$.

Iterating, we find that if Q is such that $r(PQ) \leq 1/4$, then $r(P'Q) < r(PQ)$. If $\{P_i\}$ were not strongly positive, after removing finitely many terms from the sequence, we could assume that for all N, $r(P_1P_2\cdots P_N) < 1/4$. Thus

$$\begin{aligned}
\frac{1}{4} \geq r(P_1P_2\cdots P_N) &\geq r\big((P_1P_2\cdots P_{N-1})P'_N\big)\\
&\geq r((P_1P_2\cdots P_{N-2}P'_N)P'_{N-1}) \geq \cdots \geq r(P'_1\cdots P'_N).
\end{aligned}$$

As $\{P'_i\}$ is strongly positive, there exists N so that $r(P'_1\cdots P'_N)$ exceeds $1/4$; this is a contradiction. ∎

We now show that the converse of 3.2 also holds.

PROPOSITION 3.3. *Suppose that $\{P_i\}$ is a strongly positive sequence of symmetric unimodal polynomials. Then it satisfies* (‡).

PROOF. We show that if

$$\limsup_{n\to\infty} \frac{\prod_{i=1}^{n}(P_i,x^0)}{(P^{(n)},x^0)} > 0$$

then $\{P_i\}$ is not strongly positive. We first note that $\big(P^{(n+1)},x^0\big) \geq \big(P_{n+1},x^0\big)\cdot\big(P^{(n)},x^0\big)$; it follows that $f(0,n) \geq f(0,n+1)$, where f is defined in Proposition 3.2. Thus the lim sup condition can be replaced by $f(0,n) > \delta$ for all n.

If $\{P_i\}$ were a strongly positive sequence, there would be a telescoping

$$Q_1 = P_1 \cdot P_2 \cdot \cdots \cdot P_{m(1)},$$
$$Q_2 = P_{m(1)+1} \cdot \cdots \cdot P_{m(2)},$$
$$\ldots$$

so that $r(Q_k) > \theta$ for all k, for some $\theta > 0$ (we can make θ as close to 1 as we wish, but this is not important). To see this, note that any telescoping consists of symmetric unimodal polynomials. As remarked in the introduction, ε-convexity is a consequence of the eventual nonnegativity of the product with a polynomial of the form $x+x^{-1}-(2-\varepsilon)$. This ensures that we can make the telescoped polynomials ε-convex for arbitrarily small ε. This translates to a ratio arbitarily close to 1 (look at the coefficients of $x^{\pm 1}$ and the constant term). We may thus write $Q_k = \left(Q_k, x^0\right)\left(1 + \theta \cdot (x + x^{-1})\right) + R_k$ where R_k is a symmetric polynomial having no negative coefficients. It is immediate that

$$\left(\prod_{k=1}^{M} Q_k, x^0\right) \geq \prod_{k=1}^{M} \left(Q_k, x^0\right) \cdot \left(1 + \theta^2 \binom{M}{2}\right).$$

However, $\left(\prod_{k=1}^{M} Q_k, x^0\right) = \left(P^{(m(M))}, x^0\right)$, and

$$\prod_{k=1}^{M} \left(Q_k, x^0\right) = \prod_{k=1}^{M} \left(P^{(m(k))}/\, P^{(m(k-1))}, x^0\right) \geq \prod_{i=1}^{m(M)} \left(P_i, x^0\right).$$

Therefore $\left(\prod_{k=1}^{M} Q_k, x^0\right) \leq \left(\prod_{k=1}^{M} \left(Q_k, x^0\right)\right) / \delta$. We arrive at a contradiction if we choose M large enough that $1/\delta < 1 + \theta^2 \binom{M}{2}$. ∎

For a sequence of symmetric unimodal polynomials $\{P_i\}$, let $D(N) = \sum_{i=1}^{N} d(i)$ (recall that $d(i) = d(P_i)$ is the semi-width of P_i). We obtain a crude but useful criterion.

PROPOSITION 3.4. *Let $\{P_i\}$ be a sequence of symmetric unimodal polynomials normalized so that $P_i(1) = 1$ for which there exists $\delta > 0$ such that*

$$\frac{\prod_{i=1}^{N} \left(1 + (1-\delta)\frac{1-(P_i, x^0)}{(P_i, x^0)}\right)}{D(N)} \longrightarrow \infty \quad \text{as } N \longrightarrow \infty. \tag{1}$$

Then $\{P_i\}$ is strongly positive.

PROOF. We verify the criterion of Proposition 3.2. Note that $(\prod \delta(P_i), x^0) \geq \prod \delta(P_i)(1)/D(N)$ (the crudest possible estimate!), and $\frac{(\delta(P_i), x^0)}{\delta(P_i)(1)} = \frac{(\delta(P_i), x^0)}{1-\delta(1-(\delta(P_i), x^0))}$. Substitute these into the reciprocal of the expression in (‡), and we are done. ∎

In order to use the criterion of 3.4, it is sometimes necessary to truncate the polynomials, and this is permissible as a result of the superposition principal. This idea is refined in the proof of Theorem 3.6. If we restrict ourselves to sequences of T-functions (with semi-widths $d(i)$ and heights $h(i)$) for the moment, the criterion amounts to deciding if for some $\varepsilon > 0$,

$$\frac{\prod_{i=1}^{N} \left(1 + (2-\varepsilon)\frac{d(i)}{h(i)}\right)}{D(N)} \longrightarrow \infty \quad \text{as } N \longrightarrow \infty. \tag{†}$$

Since point evaluation at 1 must be pure, a necessary condition for strong positivity is that $\sum d(i)/h(i)$ diverge. If $d(i)/h(i) \geq \mathbf{O}(1)$ (see Example 3.9(e), where $d(i)$ grows at least exponentially), then the criterion in 3.4 is sharp. If $d(i)/h(i) = \mathbf{O}(1/i)$ (see Example 3.9(c) where $d(i)$ grows polynomially), the criterion is useful but not sharp. In order to refine Proposition 3.4, we obtain estimates on the mass over certain large intervals.

Let $\{P_i\}$ be a sequence of polynomials with semi-widths $d(i)$. For an integer k, define $D_k:\mathbf{N} \longrightarrow \mathbf{N}$ by $D_k(N) = \sum_{i=0}^{k-1} d(N-i)$ for $N \geq k$.

PROPOSITION 3.5. *Let $\{P_i\}$ be a sequence of symmetric unimodal polynomials with semi-widths $d(i)$, arranged so that $d\colon \mathbf{N} \longrightarrow \mathbf{N}$ is monotone nondecreasing. Then*

$$\Pr\Big(\Big|\sum_{j=1}^{q} X_j\Big| \leq D_k(q)\Big) \geq \prod_{j=1}^{q}\left(1-(1-(P_j,x^0))\cdot\frac{d(j)}{4D_k(j-1)}\right).$$

PROOF. Fix q and renormalize all the P_i so that $P_i(1) = 1$. Let Q (depending on q, but this will be suppressed in the notation) be the polynomial defined by $(Q,x^i) = (P^{(q)},x^i)$ if $|i| \leq D_k(q)$ and $(Q,x^i) = 0$ otherwise. Thus $Q(1)$ is the expression on the left. We estimate the total mass of QP_{q+1} in the interval $[-D_k(q+1), D_k(q+1)]$, by getting an upper bound for the mass outside this interval.

Since all the polynomials are symmetric unimodal and since $D_k(q) \leq D_k(q+1)$, the constant coefficient of P_{q+1} does not contribute to the mass beyond $D_k(q{+}1)$ in the product QP_{q+1}. Similarly, since $d(q+1) < D_k(q+1)$, neither does the constant coefficient of Q. Hence the amount outside the interval is bounded above by

$$2Q(1)((1-(P_{q+1},x^0))/(2d(q+1)2D_k(q))$$

times the mass obtained by convolving two indicator functions of intervals of length $d(q+1) - D_k(q+1) + D_k(q) \leq d(q-k+1)+1$. (The “2” in the numerator comes from having to also keep track of the negative exponents, and the rest of the terms are obtained by assuming the worst possible cases for the distributions of coefficients, absolutely flat, except for the constant term of P.) Let $M^{(q)}$ denote $\Pr\Big(\Big|\sum_{j=1}^{j=q} X_j\Big| \leq D_k(q)\Big)$.

Then

$$\begin{aligned} M^{(q+1)} &\geq Q(1) - \frac{2Q(1)(1-(P_{q+1},x^0))}{4d(q+1)D_k(q)}\,\frac{(d(q-k+1)+1)^2}{2} \\ &\geq Q(1)\left(1-(1-(P_{q+1},x^0))\cdot\frac{d(q+1)}{4D_k(q)}\right). \end{aligned}$$

Since $M^{(q)} = Q(1)$, we telescope this inequality, and obtain the desired lower bound. ■

THEOREM 3.6. *Let $\{P_i\}$ be a sequence of symmetric unimodal Laurent polynomials having semi-widths $\{d(i)\}$, and with $P_i(1) = 1$ for all i. Suppose $\{d(i)\}$ is monotone non-decreasing and $\lim_{j\to\infty} d(j)/d(j-1) = 1$. Then $\{P_i\}$ is strongly positive if there exists a positive real number ε such that*

$$\frac{\prod_{i=1}^{N}\left(1+(1-\varepsilon)\frac{1-(P_i,x^0)}{(P_i,x^0)}\right)}{d(N)} \longrightarrow \infty.$$

PROOF. *We use Proposition 3.5 to apply Proposition 3.2. Let N be a large integer. For any δ (to be chosen later), the mass in the interval $[-D_k(N), D_k(N)]$ of the product polynomial $\prod_{j=1}^{N} \delta(P_j)/\delta(P_j)(1)$ is at least as large as*

$$\prod_{j=1}^{N}\left(1-(1-\frac{(P_j,x^0)}{1-\delta+\delta(P_j,x^0)}\cdot\frac{d(j)}{4D_k(j-1)}\right)=\prod_{j=1}^{N}(1-a_j\cdot b_j).$$

This serves to define a_j and b_j. (The expression involving δ arose from renormalizing $\delta(P_j)$ so that its mass is 1.) Thus, the mass at the origin is at least as large as $(\prod(1-a_j\cdot b_j))/D_k(N)$. To simplify computations, we note that if $(1+c)(1-a)=1$, then $(1+c)(1-ab)=1+c(1-b)$ (this defines $c_j=a_j/(1-a_j)$). Hence

$$\frac{\prod_{j=1}^{N}(\frac{\delta(P_j)}{\delta(P_j)(1)},x^0)}{(\prod_{j=1}^{N}\frac{\delta(P_j)}{\delta(P_j)(1)},x^0)}\leq\frac{D_k(N)\prod_{j=1}^{N}(1-a_j)}{\prod_{j=1}^{N}(1-a_j\cdot b_j)}$$
$$=\frac{D_k(N)}{\prod_{j=1}^{N}(1+c_j\cdot(1-b_j))}.$$

However, $b_j=d(j)/4D_k(j-1)$. For all sufficiently large j, $d(j)<2d(j-k)$. Thus $1-b_j\geq 1-1/(2k)$. Also, c_j works out to $(1-\delta)\big(1-(P_j,x^0)\big)/(P_j,x^0)$. Choose k large enough and δ small enough so that $(1-\delta)(1-1/(2k))>1-\varepsilon$. Since $D_k(N)\leq kd(N)$, 3.2 applies. ∎

For T-functions, 3.6 immediately yields:

PROPOSITION 3.7. *Let $\{P_i\}$ be a sequence of T-functions with semi-widths $\{d(i)\}$ and heights $\{h(i)=1/r(P_i)\}$ such that $\lim_{j\to\infty} d(j)/d(j-1)=1$ and $\{d(i)\}$ is monotone non-decreasing. Then $\{P_i\}$ is strongly positive if there exists a positive real number ε such that*

$$\frac{\prod_{i=1}^{N}\left(1+(2-\varepsilon)\frac{d(i)}{h(i)}\right)}{d(N)}\longrightarrow\infty.$$

Before giving examples, we establish necessary conditions for strong positivity of a sequence of T-functions. The formula in 3.8 is unwieldy in appearance but is not difficult to compute with.

PROPOSITION 3.8. *Let $\{P_i\}$ be a sequence of T-functions having semi-widths $\{d(i)\}$ and heights $\{h(i)\}$. If $\{P_i\}$ is strongly positive, then*

$$\sum_{j=2}^{\infty}\frac{\prod_{i=1}^{j-1}\left(1+\frac{2d(i)}{h(i)}\right)}{h(j)}=\infty.$$

PROOF. We estimate the ratio, $r(P^{(j)})$ in terms of its predecessors, and use an additive (rather than multiplicative) telescoping. Write $P^{(n-1)}=a_0+\sum a_j(x^j+x^{-j})$, where the sum of the coefficients is 1. Writing $P_n = h + \sum_{j=1}^{d}(x^j + x^{-j})$ (so $d(n)$ and $h(n)$ are abbreviated d and h respectively), direct computation yields (with $a_j = 0$ if $|j| > \sum_1^{n-1} d(i)$):

$$
\begin{aligned}
r(P^{(n)}) = \frac{(P^{(n)}, x^1)}{(P^{(n)}, x^0)} &= \frac{a_1 h + 2\sum_1^d a_j + (a_{d+1} + a_0 - a_1 - a_d)}{a_0 h + 2\sum_1^d a_j} \\
&= \frac{a_1 h + 2\frac{a_1}{a_0}\sum_1^d a_j}{a_0 h + 2\sum_1^d a_j} + \frac{(2 - 2\frac{a_1}{a_0})\sum_1^d a_j + (a_{d+1} - a_d) + a_0 - a_1}{a_0 h + 2\sum_1^d a_j} \\
&\leq \frac{a_1}{a_0} + \frac{a_0 + 2\sum_1^d a_j}{a_0 h + 2\sum_1^d a_j} \leq r(P^{(n-1)}) + \frac{1}{a_0 h}
\end{aligned}
$$

Thus

$$
r(P^{(n)}) - r(P^{(n-1)}) \leq \frac{1}{a_0 h}. \tag{3}
$$

This inequality telescopes. Moreover, a_0 is simply $(P^{(n-1)}, x^0)$, and since $P^{(n-1)}$ has already been normalized, $a_0 \geq \prod_1^{n-1} h(i)/(h(i) + 2d(i))$. Employing the telescoped version of (3), we obtain (after replacing n by $N - k$)

$$
r(P^{N,k}) \leq \frac{1}{h(k)} + \sum_{j=k}^{N+k} \frac{\prod_{i=1}^{j-1}\left(1 + \frac{2d(i)}{h(i)}\right)}{h(j)}.
$$

If $\{P_i\}$ is strongly positive, for all choices of k, there exists N such that $r(P^{N,k})$ is at least $\frac{1}{2}$. It follows that

$$
\sum_{j=2}^{\infty} \frac{\prod_{i=1}^{j-1}\left(1 + \frac{2d(i)}{h(i)}\right)}{h(j)}
$$

must diverge. ∎

EXAMPLES 3.9. In this set of examples, the polynomials P_i will be T-functions, with semi-widths $d(i) \geq 1$ and heights $h(i) \geq 1$, so the sequences $\{P_i\}$ are specified simply by the sequences of semi-widths and heights. The letters C and β will denote positive real numbers, which may be viewed as parameters of the sequences, and [ ] will be the greatest integer function.

(a) If $\sum 1/h(i)$ diverges, the sequence is strongly positive, regardless of the choice of semi-widths. Obviously, $r(P_i) = 1/h(i)$, and strong positivity follows from 1.6.

(b) (Very slow growth.) If $d(i)/h(i) = C/(i \log i)$, we note that

$$
\prod_{i=1}^{N}\left(1 + (2 - \varepsilon)C/(i \log i)\right)
$$

is asymptotic to $(\log N)^{(2-\varepsilon)C}$. If we apply the criterion in 3.7, we see that strong positivity will result if $h(i) = \mathbf{O}(i(\log i)^{(2-\varepsilon)C-1})$. Explicitly, if $d(i) = [\log i]$ and $h(i) \leq (2 - \varepsilon)i(\log i)^2$ for some $\varepsilon > 0$, then strong positivity results. However, if instead, $h(i) = (2+\varepsilon)i(\log i)^2$ and we apply the criterion of 3.8, we see that strong positivity fails. If $d(i) = [Ci/\log i]$ and $h(i) = i^2$, then strong positivity fails regardless of the choice of C, again by 3.8. Note that the test in 3.4 is useless in these examples.

(c) (Slow growth.) Suppose that $d(i) = [Ci^\beta]$ and $h(i) = i^{\beta+1}$. (These sequences are referred to in the Introduction.) Then $d(i)/h(i) \sim C/i$, and 3.7 and 3.8 together yield that $\{P_i\}$ is strongly positive if $\beta < 2C$ and is not strongly positive if $\beta > 2C$. This provides an example of the phenomenon noted after 1.11. Note that the criterion of 3.4 only yields strong positivity if $\beta \leq 2C - 1$, so that test is useful but not optimal in this case. The status of the sequence for $\beta = 2C$ is unknown.

(d) (Subexponential growth.) If $d(i)/h(i)$ is asymptotic to $1/\sqrt{i}$, then the numerator of the expression in 3.7 (the product) is approximately $\sqrt{N}\exp((2-\varepsilon)\sqrt{N})$. Thus the sequence will be strongly positive if the growth of d is at most $\exp((2-\varepsilon)\sqrt{N})$—in particular, if the growth of d is at most polynomial. On the other hand, the sequence will fail to be strongly positive if the growth is exponential, by 3.8.

(e) (Exponential growth.) Suppose $d(i)/h(i)$ tends to the constant C. Here the criterion of 3.4 can be used. The product expression is approximately $(2C+1)^N$, so that if $d(i) \leq (2C+1-\varepsilon)^i$, the sequence will be strongly positive by 3.4. Notice that the exponential growth rate precludes the applicability of 3.7. On the other hand, if $d(i) \geq (2C+1+\varepsilon)^i$, strong positivity fails by 3.8. In particular, if $d(i) = h(i) = [K^i]$ for some constant K, strong positivity occurs if $K < 3$, it fails if $K > 3$, and its status is unknown if $K = 3$.

(f) (Superexponential growth.) Set $h(i) = i!$ and $d(i) = i!i^C$. Then the numerator in the expression (†) is asymptotic to $2^N(N!)^C$, so that when $C \geq 1$, the sequence is strongly positive. If $C < 1$, we apply the criterion of 3.8; it follows that the sequence is not strongly positive.

Cases (c), (e), and (f) above exhibited a phenomenon analoguous to phase transition, in the parameter C. In the super-exponential situation, even the status at the critical point could be determined (although if $h(i) = i!$ and $d(i) = [Ci!i]$, we do not know what happens at the critical point, $C = 1/2$). This is interesting, but what is more interesting is that our methods could detect these transitions. For example, the argument in 3.8 uses extremely crude estimates, and both 3.4 and 3.6 employ a lower bound for the mass at the origin obtained by taking the average mass over a very large interval.

In the cases discussed above wherein strong positivity fails, what is the obstruction? All the point evaluation states are pure (this requires only that $\sum d(i)/h(i)$ diverge). One observation is that the ratios of $P^{N,k}$ cannot go to 1 for all k. The states defined in Section 2 from the ratios α_i cannot be point evaluations (because the α_i are not all one). We find that the mass at the origin (and every other point) goes to zero as N increases but the ratios around the origin do not tend to 1.

4. **Harmonic functions and eventual positivity for strongly unimodal sequences.** In this section, we consider sequences of Laurent polynomials $\{P_i\}$ where either each P_i is strongly unimodal or each is unimodal and there is a bound on the total degrees. We have seen (1.9) that $\sum r(P_i) = \infty$ is sufficient for strong positivity, and otherwise no point evaluation is pure. We show that in this non-strongly positive case, the pure harmonic functions are precisely those obtained from the Laurent series expansion of the infinite product. This is achieved by means of a positivity theorem that decides when a polynomial f is eventually positive with respect to the sequence $\{P_i\}$. The following is an immediate consequence of 2.3A or 2.3C:

THEOREM 4.1. *Suppose that $\{P_i\}$ is a sequence of Laurent polynomials and either every P_i is strongly unimodal, or the P_i are unimodal with a bound on the total degree. Then the following are equivalent:*

(a) $\{P_i\}$ is not strongly positive;
(b) $\sum r(P_i) < \infty$;
(c) After a suitable regularization, $\prod P_i(z)$ converges in $\mathbf{C}\setminus\{0\}$;
(d) For all positive real numbers r, evaluation at r is not a pure state;
(e) There exists a positive real number r whose corresponding point evaluation is not pure;
(f) Evaluation at 1 is not a pure state.

LEMMA 4.2. *Let $\{P_i\}$ satisfy the hypotheses and the equivalent properties of Theorem 4.1. Suppose F denotes the infinite product, $\prod P_i(z)$, and j is a fixed integer. On forming the Laurent series expansion of $F/P^{(j)}(z) = \sum \alpha_k z^k$, we have*

$$\lim_{k\to\infty} \frac{\alpha_{k+1}}{\alpha_k} = 0 \qquad \text{and } \lim_{k\to\infty} \frac{\alpha_{-k-1}}{\alpha_{-k}} = 0$$

(adopting the convention that $0/0 = 0$), and moreover, there are no gaps in the coefficients of F (that is, $(F, x^i), (F, x^k) > 0$ for integers $i < j < k$ implies $(F, x^j) > 0$).

PROOF. This is routine once we observe that it suffices to prove the result for F itself. ∎

If I is a closed bounded subset of $\mathbf{R}$, then we define $\partial_e I$ to be the two point set consisting of its infimum and supremum.

PROPOSITION 4.3. *Let $\{P_i\}$ be a sequence of Laurent polynomials with no negative coefficients such that $P_i(1) = 1$ for all i, and $\prod P_i(z)$ converges absolutely on $\mathbf{C}\setminus\{0\}$ to F which has no gaps. Suppose additionally that for each $j = 0, 1, 2, \ldots$, the Laurent expansion $F/P^{(j)}(z) = \sum \alpha_k z^k$ satisfies*

$$\lim_{k\to\infty} \frac{\alpha_{k+1}}{\alpha_k} = 0 \text{ and } \lim_{k\to\infty} \frac{\alpha_{-k-1}}{\alpha_{-k}} = 0$$

Suppose that f in $\mathbf{R}[x, x^{-1}]$ satisfies

(i) For some j, $\operatorname{Log} P^{(j)} \supseteq \operatorname{Log} f \supseteq \partial_e \operatorname{Log} P^{(j)}$;
(ii) $\lim_{t\to 0}(f/P^{(j)})(t) > 0$, and $\lim_{t\to\infty}(f/P^{(j)})(t) > 0$;

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(iii) There exists $\delta > 0$ *so that for all* k, $\big((F\cdot f)/P^{(j)}, x^k\big) \geq \delta(F, x^k)$.
Then there exists an integer N *so that* $P_{j+1}\cdot P_{j+2}\cdot\cdots\cdot P_{j+N}\cdot f$ *has no negative coefficients.*

PROOF. By fixing j and multiplying by an appropriate power of x (once only), we may assume that $f = \sum_0^d r_i x^i$ where r_0 and r_d are nonzero. From (ii), both endpoint coefficients must be positive. Write $P^{(n+j)}/P^{(j)} = \sum p_{n,t}x^t$. From the uniform convergence (on compact subsets of $\mathbf{C}\setminus\{0\}$) of the products, we have that $\{p_{n,t}\} \longrightarrow (F, x^t)$ uniformly in t, and the no gaps condition of 4.2 ensures that

$$\lim_n \frac{p_{n,t}}{p_{n,t+1}} = 0 \qquad \text{if } t < 0$$
$$\lim_n \frac{p_{n,t+1}}{p_{n,t}} = 0 \qquad \text{if } t \geq 0$$

$(0/0$ is interpreted as $0)$. For $k \geq d$ and $p_{n,k-d} > 0$,

$$\begin{aligned} f\cdot(P^{(n+j)}/P^{(j)}, x^k) &= \sum_{i=0}^{d} r_i p_{n,k-i} \\ &= p_{n,k-d}\left(r_d + r_{d-1}\frac{p_{n,k-d+1}}{p_{n,k-d}} + \cdots + r_0\frac{p_{n,k}}{p_{n,k-d}}\right). \end{aligned}$$

Since $r_d > 0$ and the fractions in the big parentheses all go to zero as $n \longrightarrow \infty$, there exists N_0 such that $n \geq N_0$ entails $(P^{(n+j)}/P^{(j)}\cdot f, x^k) \geq 0$ for all $k \geq d_0 = \max\{d, \min\{i \mid p_{n,i} > 0\}\}$. (If the latter set is empty, the inequality comes for free.) Similarly, there exists $d' < 0$ and N' such that $n \geq N'$ entails $(P^{(n+j)}/P^{(j)}\cdot f, x^k) \geq 0$ for all $k \leq d'$. By (iii), for each integer $d' \leq s \leq d_0$, there exists an integer N_s such that $n \geq N_s$ implies $(P^{(n+j)}/P^{(j)}\cdot f, x^s) \geq \frac{1}{2}\delta\big((F/P^{(j)}), x^s\big) \geq 0$. Set $N = \max\{N_0, N', N_s\}$. For all k, $\big((P^{(N+j)}/P^{(j)})\cdot f, x^k\big) \geq 0$; in other words, $\big(P^{(N+j)}/P^{(j)}\big)\cdot f$ has no negative coefficients. ∎

In the following, if $\alpha_i = 0$ for $i = -1, -2, \ldots$, then γ^0 is seen to be evaluation at 0.

COROLLARY 4.4. *If* $\{P_i\}$ *are as hypothesized in Theorem 4.1, then the pure states of* $R(\{P_i\})$ *are precisely:*

$$\{\gamma^i \mid i \in \mathbf{Z},\ \alpha_i \neq 0\} \cup \{\text{ evaluation at } 0\} \cup \{\text{ evaluation at } \infty\}$$

where $\gamma^i(h) = \frac{1}{2\pi\sqrt{-1}}\oint h\cdot F/z^{i+1}\,dz$ *for* h *in* $R(\{P_i\})$ *(vide Section 2).*

PROOF. Let X denote the set of states listed in the statement of this result. It is not hard to show that if $\alpha_i \neq 0$ for infinitely many $i < 0$, then $\lim_{i\to\infty}\gamma^i$ is evaluation at 0, and similarly, the limit in the opposite direction is evaluation at ∞, and moreover, no other limit points exist. Thus X is compact.

Now we observe that if $f/P^{(k)} = h$ where h in $R(\{P_i\})$ satisfies $\gamma(h) > \delta$ for all γ in X, then h satisfies the hypotheses of 4.3; hence it is positive. It follows immediately from this and the compactness of X that the pure state space is included in X.

To show that every point in X is pure is now straightforward (since a non-pure γ^i would have to be a σ-linear combination of the others). ∎

After this paper had been accepted, Professor S. V. Kerov of the University of Leningrad pointed out to us that he had obtained similar results in the case that $P_n = 1 + a_n x$ where $\sum a_n < \infty$ [K]. He also discussed some of the states arising in the situation that $P_n = 1 + nx$.

PROPOSITION 4.5. *Let $\{P_i\}$ satisfy the hypotheses of 4.2, and let f be an element of $\mathbf{R}[x,x^{-1}]$. Then there exists N so that $P_1 \cdot P_2 \cdot \cdots \cdot P_N \cdot f$ has no negative coefficients if and only if there exists $\delta > 0$, so that for all i,*

$$(f \cdot F, x^i) \geq \delta (P_0 \cdot F, x^i)$$

where $P_0 = \sum_i x^j$, the sum taken over cvx $\operatorname{Log} f \cap \mathbf{Z}$.

PROOF. Form $R(\{P_0, P_1, P_2, \dots\})$; then f/P_0 satisfies (i),(ii),(iii) of 4.3, as an element of that ordered group; by 4.3, it lies in the positive cone. However, this means that there exists N so that $P_1 \cdot P_2 \cdot \cdots \cdot P_N \cdot f$ has no negative coefficients (note that P_0 is strongly unimodal, so its presence will not affect the hypotheses of 4.3). The converse follows from f/P being an order unit in $R(\{P_i\})$. ∎

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