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A Classification of Tsirelson Type Spaces

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Abstract. We give a complete classification of mixed Tsirelson spaces $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ for finitely many pairs of given compact and hereditary families $\mathcal{F}_i$ of finite sets of integers and $0<\theta_i<1$ in terms of the Cantor–Bendixson indices of the families $\mathcal{F}_i$, and θ_i $(1\le i\le r)$. We prove that there are unique countable ordinal α and $0<\theta<1$ such that every block sequence of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ has a subsequence equivalent to a subsequence of the natural basis of the $T(\mathcal{S}_{\omega^\alpha},\theta)$. Finally, we give a complete criterion of comparison in between two of these mixed Tsirelson spaces.

Introduction

The line of research we continue in this paper has been initiated by an old problem of S. Banach asking if every Banach space contains a subspace isomorphic to c_0 or some ℓ_p. This problem was solved negatively by B. S. Tsirelson [19] who provided the first example of a Banach space that does not contain any of the spaces c_0, ℓ_p, $1\le p<\infty$. The idea of Tsirelson's construction became particularly apparent after T. Figiel and W. B. Johnson [11] showed that the norm of the dual of a Tsirelson space satisfies the following implicit equation

$$(*)\qquad \Big\|\sum_n a_n e_n\Big\| = \max\Big\{\sup_n |a_n|, \frac{1}{2}\sup\sum_{i=1}^d \Big\|E_i\Big(\sum_n a_n e_n\Big)\Big\|\Big\},$$

where the sequences $(E_i)_{i=1}^d$ considered above consist of successive subsets of integers with the property that $d\le \min E_1$, $d\in\mathbb{N}$, and $E_i(\sum_n a_n e_n)=\sum_{n\in E_i} a_n e_n$ is the restriction of $\sum_n a_n e_n$ on the set E_i. We refer to [10] for an extended study of Tsirelson space T. A first systematic abstract study on Tsirelson's construction was given by S. Bellenot [7] and S. A. Argyros and I. Deliyanni [4]. Given a real number $0<\theta<1$ and an arbitrary *compact* and *hereditary* family $\mathcal{F}$ of finite sets of integers, one defines the Tsirelson type Banach space $T(\mathcal{F},\theta)$ as the completion of c_{00} with the implicitly given norm $(*)$ replacing $1/2$ by θ and using sequences $(E_i)_i$ of finite sets of integers which are $\mathcal{F}$-admissible, *i.e.,* there is some $\{m_i\}_{i=1}^d\in\mathcal{F}$ such that

$$m_1\le \min E_1\le \max E_1 < m_2\le \min E_2\le \max E_2<\cdots<m_d\le \min E_d\le \max E_d.$$

In this notation, Tsirelson's example is the space $T(\mathcal{S},1/2)$, where $\mathcal{S}=\{s\subseteq\mathbb{N} : \#s\le \min s\}$ is the so-called *Schreier family*. It was proved in [4] that if the *Cantor–Bendixson index* $\iota(\mathcal{F})$ and θ satisfy the inequality $\theta\cdot\iota(\mathcal{F})>1$, then the space $T(\mathcal{F},\theta)$

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is reflexive. Moreover, in the case of $\iota(\mathcal{F}) \geq \omega$, they proved that the space $T(\mathcal{F}, \theta)$ does not contain any of the classical spaces c_0 or ℓ_p, $1 \leq p < \infty$. In the case that $\mathcal{F}$ is chosen to be the family of the finite subsets of $\mathbb{N}$ with cardinality at most $n \geq 2$, denoted by $[\mathbb{N}]^{\leq n}$, it was shown [4, 7] that the corresponding space $T([\mathbb{N}]^{\leq n}, \theta)$ is isomorphic to c_0 if $n\theta \leq 1$ and is isomorphic to ℓ_p $(1 < p < \infty)$ if $\theta = n^{-1/q}$, where q is the conjugate of p, *i.e.,* $1/p + 1/q = 1$.

Further examples of Tsirelson type spaces with interesting properties are the spaces $T(\mathcal{S}_\alpha, \theta)$ considered in [1, 4], where the compact and hereditary families $\mathcal{S}_\alpha$ are the α*-Schreier families*, the natural generalizations of the Schreier family of index ω^α ($\mathcal{S}_1 = \mathcal{S}$). These spaces share many properties with the original Tsirelson space, and their natural Schauder bases are examples of w-null sequences with large oscillation indices. A basic property of any $\mathcal{S}_\alpha$ is that it is spreading (see definition below). This is used to show that every normalized block sequence with respect to their natural bases (e_n) is equivalent to a subsequence of (e_n), a property that c_0 and ℓ_p also have. From this, and the fact that the Cantor–Bendixson indices of the families $\mathcal{S}_\alpha$ and $[\mathbb{N}]^{\leq n}$ are very much different, it can be explained why $T(\mathcal{S}_\alpha, 1/2)$ does not contain isomorphic copies of $\ell_p \cong T([\mathbb{N}]^{\leq n}, n^{-1/q})$ or $c_0 \cong T([\mathbb{N}]^{\leq n}, 1/n)$.

The aim of this paper is to understand in these terms the so-called *mixed Tsirelson spaces* $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$, whose norms are defined implicitly by

$$\|x\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} = \max\Big\{ \|x\|_\infty, \sup\big\{ \theta_i \sum_{j=1}^n \|E_j x\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} : (E_j)_{j=1}^n \text{ is } \mathcal{F}_i\text{-admissible}, 1 \leq i \leq r\big\} \Big\},$$

for arbitrary compact and hereditary families $\mathcal{F}_i$ and establish a criterion of comparability in between them. The first step in this direction was done by J. Bernues and I. Deliyanni [8] and J. Bernues and J. Pascual [9] who proved the following two results:

- If the Cantor–Bendixson indices of the families are finite, then $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$ is saturated by either c_0 or some ℓ_p, $1 < p < \infty$.
- If the Cantor–Bendixson index of $\mathcal{F}$ is equal to $\omega + 1$, then $T(\mathcal{F}, \theta)$ contains a subspace isomorphic to a subspace of $T(\mathcal{S}, \theta)$.

The only case left is when one of the families has infinite index. Recall that every ordinal $\alpha > 0$ has a unique decomposition as $\alpha = \omega^\beta k + \delta$, where $\delta < \omega^\beta$ and $k \in \mathbb{N}$. Using it twice it follows that every infinite ordinal α has the unique decomposition $\alpha = \omega^{\omega^\gamma n + \xi} m + \eta$ (see [18]). Now given a compact family $\mathcal{F}$, let $\gamma(\mathcal{F})$ and $n(\mathcal{F})$ be ω^{ω^γ} and n in the previous decomposition for α equal to the Cantor–Bendixson index of $\mathcal{F}$. Following this notation, our main result is the following.

Theorem *Fix $(\mathcal{F}_i, \theta_i)_{i=1}^r$ such that at least one of the families has infinite index. Then there is some $1 \leq i_0 \leq r$ such that for every compact and hereditary family $\mathcal{G}$ the following are equivalent.*

(i) $\gamma(\mathcal{G}) = \gamma(\mathcal{F}_{i_0})$.

(ii) *Every infinite-dimensional closed subspace of $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$ contains a subspace isomorphic to a subspace of $T(\mathcal{G}, \theta_{i_0}^{n(\mathcal{G})/n(\mathcal{F}_{i_0})})$.*

(iii) *Every normalized block sequence* (x_n) *of* $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ *has a subsequence* $(x_n)_{n\in M}$ *equivalent to the subsequence* $(e_{\min\operatorname{supp} x_n})_{n\in M}$ *of the basis of* $T(\mathcal{G},\theta_{i_0}^{n(\mathcal{G})/n(\mathcal{F}_{i_0})})$.

It readily follows that:

- Every normalized block sequence (x_n) of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ has a subsequence $(x_n)_{n\in M}$ equivalent to the subsequence $(e_{\min\operatorname{supp} x_n})_{n\in M}$ of the basis of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$.
- There are unique countable ordinal α and $0<\theta<1$ such that every normalized block sequence with respect to the basis (e_n) of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ has a subsequence equivalent to a subsequence of the basis (e_n) of $T(\mathcal{S}_{\omega^\alpha},\theta)$.

So, for example $T(\mathcal{S}_{\omega^3 4+\omega 5},1/2^4)$ and $T(\mathcal{S}_{\omega^3},1/2)$ are mutually saturated, while $T(\mathcal{S}_{\omega^3},1/2)$ and $T(\mathcal{S}_{\omega^4},1/2)$ are totally incomparable.

Another consequence is that every subspace of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ contains an $\mathcal{S}_{\omega^\alpha}-\ell_1$ spreading model, that is, there exists a constant $K>1$ such that for every sequence of coefficients $(a_n)_n$

$$\Big\|\sum_{n\in s}a_nx_n\Big\|\geq\frac{1}{K}\sum_{n\in s}|a_n|\quad(s\in\mathcal{S}_{\omega^\alpha}).$$

In particular, every subspace of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ contains an asymptotic ℓ_1-subspace. Asymptotic ℓ_1-spaces, the structure of these spaces as well as the structure of the spreading models of a Banach space is a current research topic, which provides interesting examples and structural results in Banach space theory (see [2, 16]).

The proofs given in this paper use four main ingredients: we work with the equivalent reformulation of the implicit norm of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ given by the norming set $K((\mathcal{F}_i,\theta_i)_{i=1}^r)$, and the so-called tree analysis of a functional of $K(\mathcal{F},\theta)$ (see Section 4). In particular, given a normalized block sequence (x_n) of the basis (e_n) we provide an algorithm to estimate the norm of a linear combination $\sum_n a_nx_n$ in terms of a corresponding linear combination of a subsequence of the basis (e_n) of an auxiliary space $T[(\mathcal{G}_i,\theta_i)_{i=1}^r]$, much in the spirit of well-known works in this field. Secondly, we use the well-known fact (see [6, 12]) that given two compact and hereditary families $\mathcal{F}$ and $\mathcal{G}$, there is an infinite set M such that either $\mathcal{F}\restriction N=\{s\in\mathcal{F}:s\subseteq N\}\subseteq\mathcal{G}\restriction N=\{s\in\mathcal{G}:s\subseteq N\}$ or vice versa. This is indeed a consequence of the fact that for every compact and hereditary family $\mathcal{F}$, there is an infinite set M such that $\mathcal{F}\restriction M$ is, what we call here, *homogeneous* on M. It turns out that the $\subseteq$-maximal elements of such families have the *Ramsey property*, which we will use here to avoid some combinatorial computations.

Finally, we reduce the study of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ for compact and hereditary families $\mathcal{F}_i$'s to the case of $T(\mathcal{G},\theta)$ for some *regular* family $\mathcal{G}$, *i.e.,* a compact hereditary family $\mathcal{G}$ that is in addition *spreading* (see below). This additional regularity property of families $\mathcal{G}$ has two main advantages: the first is that the associated norming set $K(\mathcal{G},\theta)$ has a simpler form; the second one is that their Cantor–Bendixson index is preserved if we restrict them to an infinite set.

The paper is organized as follows: in the first section we introduce notation, basic combinatorial definitions, and mixed Tsirelson spaces. In the second section we study the behavior of subsequences of the natural basis of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ in the case of regular families. An important outcome of this section is the reduction we make from finitely many families to one.

The third section is devoted to an abstract study of compact and hereditary families of finite sets of integers. In particular, we introduce homogeneous and uniform families and we prove two combinatorial results, basic tools for this work. This section provide the link between mixed Tsirelson spaces built by compact and hereditary families with Tsirelson type spaces constructed using a regular family.

In the last section we show that every block sequence of a mixed Tsirelson space $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ has a further subsequence equivalent to a subsequence of its basis. As a consequence of this and of the results of the previous sections we provide several saturation results. Using special convex combinations, we also give two criteria to obtain incomparability for Tsirelson type spaces. Finally, we expose the classification of mixed Tsirelson spaces $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$.

1 Basic Facts

Throughout this paper we are going to deal with families of finite sets of integers. The family of all finite sets of integers is denoted here by FIN. Given $s,t\in$ FIN we write $s<t$ (resp. $s\le t$) to denote that $\max s<\min t$ (resp. $\max s\le \min t$), and for an integer n, we write $n<s$ ($n\le s$) whenever $\{n\}<s$ (resp. $\{n\}\le s$). These orders can be easily extended to vectors $x,y\in c_{00}(\mathbb{N})$: $x<y$ ($x\le y$) if and only if $\operatorname{supp} x<\operatorname{supp} y$ (resp. $\operatorname{supp} x\le \operatorname{supp} y$), where for $x\in c_{00}$, $\operatorname{supp} x=\{n\in\mathbb{N}: x(n)\neq 0\}$. We say that a sequence (s_n) of non-empty finite sets of integers is a *block sequence* if $s_n<s_{n+1}$ for every n. In a similar manner one defines the corresponding notion of *block sequence of vectors* of c_{00}.

Given an infinite set M and a finite set s we denote $M/s=\{n\in M: n>s\}$, and for a given integer n, let $M/n=M/\{n\}$. The *shift* of a non-empty set A of integers is ${}_*A=A\setminus\{\min A\}$. Given two sets A and B we set $A\setminus B=\{n\in A: n\notin B\}$, and $M\setminus m=\{n\in M: n\ge m\}$. For a given family $\mathcal{F}\subseteq$ FIN, an infinite set $M\subseteq\mathbb{N}$ and a finite set s, let $\mathcal{F}\restriction M=\{t\in\mathcal{F}: t\subseteq M\}$ be the *restriction* of $\mathcal{F}$ in M, and let $\mathcal{F}_s=\{t\in \text{FIN}: s<t,\ s\cup t\in\mathcal{F}\}$. Given a finite set s we use $\#s$ to denote its cardinality. Finally, every time we write an enumeration $A=\{m_i\}$ of a set A we mean a strictly increasing enumeration.

Concerning topological aspects, observe that the family of all finite sets of integers has the natural topology induced by the product topology on the Cantor space $\{0,1\}^{\mathbb{N}}$, simply by identifying every finite set s with its characteristic function $\xi_s\colon\mathbb{N}\to\{0,1\}$. We say then that a family $\mathcal{F}\subseteq$ FIN is *compact* if $\mathcal{F}$ is closed with respect to the previous topology. In particular, there is no infinite sequence $(s_n)\subseteq\mathcal{F}$ such that $s_n\subsetneq s_{n+1}$ (this is characterization if $\mathcal{F}$ is a family closed under inclusion). Given a compact family $\mathcal{F}$, recall that $\partial\mathcal{F}$ is the set of all proper accumulation points of $\mathcal{F}$ and that $\partial^{(\alpha)}(\mathcal{F})=\bigcap_{\beta<\alpha}\partial(\partial^{(\beta)}(\mathcal{F}))$. The rank is well defined since $\mathcal{F}$ is countable and therefore a scattered compactum, so the sequence $(\partial^{(\alpha)}(\mathcal{F}))_\alpha$ of iterated derivatives must vanish. We define, as in [8], the Cantor–Bendixson index $\iota(\mathcal{F})$ of a compact family $\mathcal{F}$ as the minimal ordinal α such that $\partial^{(\alpha)}\mathcal{F}\subseteq\{\varnothing\}$. Observe that this definition is a slight variation of the standard one, where one considers the first ordinal α such that $\partial^{(\alpha)}\mathcal{F}$ vanishes. Let us point out the reason to take this definition of the index of a family $\mathcal{F}$: while for families with infinite index the results we present in this paper have exactly the same form using the standard notion of Cantor–Bendixson index, for families with finite index the standard Cantor–Bendixson index cannot be used to characterize the corresponding mixed Tsirelson spaces (see [8]).

A family $\mathcal{F}$ is called *hereditary* if and only if $s \subseteq t \in \mathcal{F}$ implies that $s \in \mathcal{F}$. Another relevant order of FIN is $\preceq$: given two finite sets s and t we write $s \preceq t$ if and only if $|t| = |s|$ and the only strictly increasing map $\sigma\colon t \to s$ satisfies that $n \geq \sigma(n)$ for all $n \in t$, or equivalently, if $s = \{p_1, \dots, p_d\}$, then $t = \{q_1, \dots, q_d\}$ and $p_i \leq q_i$ for every $i \leq d$. We say that a family $\mathcal{F}$ of finite subsets of an infinite set M is *spreading* on M if $s \preceq t \subseteq M$ and $s \in \mathcal{F}$ implies $t \in \mathcal{F}$. We say that $\mathcal{F}$ is spreading if it is spreading on $\mathbb{N}$. We say that $\mathcal{F}$ is *regular* on M if and only if it is compact hereditary and spreading on M, and that $\mathcal{F}$ is regular if it is regular on $\mathbb{N}$.

Examples of regular families are the families of subsets of M with cardinality $\leq n$, denoted by $[M]^{\leq n}$, and with index n. Indeed, we will see that every regular family with finite index is, when restricted to some tail $\mathbb{N}/n$, of this form (see Proposition 3.4). A regular family of index ω is the well-known *Schreier family*

$$\mathcal{S} = \{s \in \text{FIN} : \#s \leq \min s\}.$$

In general, for a countable ordinal α we can define inductively on α an *α-Schreier family* by $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_{\alpha+1} = \{s_1 \cup \dots \cup s_n : (s_i) \subseteq \mathcal{S}_\alpha \text{ is } \mathcal{S}\text{-admissible}\}$, and if α is a limit ordinal, $\mathcal{S}_\alpha = \bigcup_n \mathcal{S}_{\alpha_n} \restriction (\mathbb{N} \setminus n)$, where (α_n) is a fixed increasing sequence of ordinals with limit α. It can be shown that $\mathcal{S}_\alpha$ is a regular family with index ω^α [1]. We introduce now two well known operations between families of finite sets.

Definition 1.1 Fix two families $\mathcal{F}$ and $\mathcal{G}$ of finite sets. Recall the following from [3]:

$$\mathcal{F} \oplus \mathcal{G} = \{s \cup t : s < t,\ s \in \mathcal{G},\ t \in \mathcal{F}\},$$

$$\mathcal{F} \otimes \mathcal{G} = \{s_1 \cup \dots \cup s_n : (s_i) \text{ is a block sequence}, \{s_i\} \subseteq \mathcal{F} \text{ and } \{\min s_i\} \in \mathcal{G}\}.$$

The operation $\mathcal{F} \oplus \mathcal{G}$ is called a *block sum* while the operation $\mathcal{F} \otimes \mathcal{G}$ is called a *convolution*. Observe that $\alpha + 1$-Schreier families are defined inductively by the formula $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \otimes \mathcal{S}$. Also, it is well known that the index of the families $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$ are equal to $\iota(\mathcal{F}) + \iota(\mathcal{G})$ and $\iota(\mathcal{F})\iota(\mathcal{G})$, respectively, assuming that $\mathcal{F}, \mathcal{G}$ are regular (see Proposition 3.4). So, if α has *Cantor normal form* $\alpha = \omega^{\alpha_0} n_0 + \dots + \omega^{\alpha_k} n_k$ (see [18] for standard properties of ordinal arithmetic), the regular family $(\mathcal{S}_{\alpha_0} \otimes [\mathbb{N}]^{\leq n_0}) \oplus \dots \oplus (\mathcal{S}_{\alpha_k} \otimes [\mathbb{N}]^{\leq n_k})$ is of index α.

It is not difficult to prove that $\oplus$ and $\otimes$ share many properties with the addition and multiplication of ordinals. For example, $\oplus$ and $\otimes$ are associative, and they have the distributive law $\mathcal{F} \otimes (\mathcal{G} \oplus \mathcal{H}) = (\mathcal{F} \otimes \mathcal{G}) \oplus (\mathcal{F} \otimes \mathcal{H})$, while in general the two operations are not commutative or $(\mathcal{F} \oplus \mathcal{G}) \otimes \mathcal{H} \neq (\mathcal{F} \otimes \mathcal{H}) \oplus (\mathcal{G} \otimes \mathcal{H})$ (as for the addition and multiplication of ordinals).

We introduce the following simplifying notation.

Notation By $(\mathcal{F}_i, \theta_i)_{i \in I}$ we shall always mean a sequence of pairs of compact and hereditary families $\mathcal{F}_i$ and real numbers $0 < \theta_i < 1$ $(i \in I)$. We call a sequence

$(\mathcal{F}_i,\theta_i)_{i\in I}$ *regular* if in addition every $\mathcal{F}_i$ is regular. Given two sequences $(\mathcal{F}_i,\theta_i)_{i\in I}$ and $(\mathcal{F}_i,\theta_i)_{i\in J}$ we use $(\mathcal{F}_i,\theta_i)_{i\in I}{}^\frown(\mathcal{F}_i,\theta_i)_{i\in J}$ to denote the disjoint concatenation $(\mathcal{F}_i,\theta_i)_{i\in I\sqcup J}$ of the two sequences. Given $\mathcal{F}\subseteq \mathrm{FIN}$ and $m\in\mathbb{N}$ let

$$\mathcal{F}^{\otimes(m)}=\mathcal{F}\otimes\overset{(m)}{\cdots}\otimes\mathcal{F}.$$

We are now ready to give the definition of mixed Tsirelson spaces.

Definition 1.2 Given a sequence $(\mathcal{F}_i,\theta_i)_{i\in I}$ the norm $\|\cdot\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}$ on c_{00} is defined as follows. For $x\in c_{00}$ let

$$(1.1)\quad \|x\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}=$$
$$\max\Big\{\|x\|_\infty,\sup\Big\{\theta_i\sum_{j=1}^n\|E_jx\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}:(E_j)_{j=1}^n \text{ is } \mathcal{F}_i\text{-admissible}, i\in I\Big\}\Big\}.$$

Next, $T[(\mathcal{F}_i,\theta_i)_{i\in I}]$ denotes the completion of $(c_{00},\|\cdot\|_{(\mathcal{F}_i,\theta_i)_{i\in I}})$. Observe that a Tsirelson type space $T(\mathcal{F},\theta)$ is nothing else but the mixed Tsirelson space $T[(\mathcal{F},\theta)]$.

Remark 1.3. (i) The property of being hereditary of every family $\mathcal{F}_i$ is not needed for the previous definition to make sense. Moreover, if the $\mathcal{F}_i'$s are only compact, then the standard Hamel basis (e_n) of c_{00} is a 1-sign-unconditional normalized Schauder basis of $T[(\mathcal{F}_i,\theta_i)_{i\in I}]$ (in order to prove the unconditionality it is more convenient to work with the equivalent presentation of the norm $\|\cdot\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}$ given just before Remark 1.4). In the sequel, whenever we consider block sequences they will be with respect the basis $(e_n)_n$.

(ii) If I is finite, or if $\lim_{i\in I}\theta_i=0$, then (e_n) is shrinking, while if there exists $i\in I$ with $\theta_i>1/\iota(\mathcal{F}_i)$ (with the convention $1/\iota(\mathcal{F}_i)=0$ for $\iota(\mathcal{F}_i)$ is infinite), then the basis (e_n) is boundedly complete (see [6] for more details).

(iii) Observe that if in the previous definition of the norm $\|\cdot\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}$ we do not impose that $\mathcal{F}_i$ are necessarily hereditary but only $\sqsubseteq$-hereditary ($s\sqsubseteq t$ if $s\subseteq t$ and $s<t\setminus s$), then in the corresponding completion $T[(\mathcal{F}_i,\theta_i)_{i\in I}]$ the sequence (e_n) is still a bimonotone Schauder basis 1-sign unconditional.

(iv) It can be shown that the implicit formula (1.1) remains true for every $x\in T[(\mathcal{F}_i,\theta_i)_{i\in I}]$ (see [14] or Remark 1.4 below).

(v) If we allow some of the families $\mathcal{F}_i$ to be non-compact, *i.e.,* if some of their closures contain an infinite set, then it follows easily that $T[(\mathcal{F}_i,\theta_i)_{i\in I}]$ is ℓ_1-saturated. Indeed, every seminormalized block sequence contains a further subsequence, for which every finite initial subsequence is $\mathcal{F}_i$-admissible for a non-compact family $\mathcal{F}_i$, and hence equivalent to the natural basis of ℓ_1.

Now we present a standard alternative description of the norm of the space $T[(\mathcal{F}_i,\theta_i)_{i\in I}]$, closer to the spirit of Tsirelson's original definition. Let us denote by $K((\mathcal{F}_i,\theta_i)_{i\in I})$ the minimal subset of c_{00}

- containing $\pm e_n^*$ ($n\in\mathbb{N}$)
- it is closed under the $(\mathcal{F}_i,\theta_i)$-operation ($i\in I$): $\theta_i(f_1+\cdots+f_n)\in K((\mathcal{F}_i,\theta_i)_{i\in I})$ for every $\mathcal{F}_i$-admissible sequence $(f_i)_{i=1}^n\subseteq K((\mathcal{F}_i,\theta_i)_{i\in I})$.

The norm induced by $K((\mathcal{F}_i,\theta_i)_{i\in I})$, *i.e.*,

$$\|x\|_{K((\mathcal{F}_i,\theta_i)_{i\in I})} = \sup\{f(x) : f \in K((\mathcal{F}_i,\theta_i)_{i\in I})\}, \text{ for } x \in c_{00},$$

is exactly the norm $\|x\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}$ defined above. Given an infinite set M of integers we set $K^M((\mathcal{F}_i,\theta_i)_{i\in I}) = \{\phi \in K((\mathcal{F}_i,\theta_i)_{i\in I}) : \operatorname{supp}\phi \subseteq M\}\}$.

Remark 1.4. (i) It is easy to see that the closure under the pointwise convergence topology of $\operatorname{conv} K((\mathcal{F}_i,\theta_i)_{i\in I})$ is the unit dual ball $B_{T[(\mathcal{F}_i,\theta_i)_{i\in I}]^*}$. It follows that $B_{T[(\mathcal{F}_i,\theta_i)_{i\in I}]^*}$ is closed under the $(\mathcal{F}_i,\theta_i)$-operation ($i \in I$). It is also easy to see that even if the families $\mathcal{F}_i$'s are not necessarily hereditary, the corresponding norming set $K((\mathcal{F}_i,\theta_i)_{i\in I})$ is closed sign modification. If $(a_n)_{n\in\mathbb{N}} \in K((\mathcal{F}_i,\theta_i)_{i\in I})$, and $(\varepsilon_n)_n \subseteq \{-1,1\}$, then $(\varepsilon_n a_n)_{n\in\mathbb{N}} \in K((\mathcal{F}_i,\theta_i)_{i\in I})$. This proves that (e_n) is always 1-unconditional.

(ii) For every infinite set M of integers and every sequence $(a_n)_{n\in M}$ of scalars we have

$$\Big\|\sum_{n\in M} a_n e_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i\in I}} = \Big\|\sum_{n\in M} a_n e_n\Big\|_{K^M((\mathcal{F}_i,\theta_i)_{i\in I})}.$$

Observe that $K^M((\mathcal{F}_i,\theta_i)_{i\in I}) = K^M((\mathcal{F}_i\restriction M,\theta_i)_{i\in I})$ if $\mathcal{F}_i$ is regular for every $i \in I$, but that in general the previous equality is not true.

Notice that, by minimality of $K((\mathcal{F}_i,\theta_i)_{i\in I})$, every functional from $K((\mathcal{F}_i,\theta_i)_{i\in I})$ either has the form $\pm e_n^*$ ($n \in \mathbb{N}$), or it is the result of a $(\mathcal{F}_i,\theta_i)$-operation to some sequence in $K((\mathcal{F}_i,\theta_i)_{i\in I})$ and $i \in I$. This suggests that somehow every element of $K((\mathcal{F}_i,\theta_i)_{i\in I})$ has a complexity that increases in every use of the $(\mathcal{F}_i,\theta_i)$-operations. This is captured by the following definition.

Definition 1.5 [3] A family $(f_t)_{t\in\mathcal{T}} \subseteq K((\mathcal{F}_i,\theta_i)_{i\in I})$ is called a *tree analysis* of a functional $f \in K((\mathcal{F}_i,\theta_i)_{i\in I})$ if the following are satisfied:

(i) $\mathcal{T} = (\mathcal{T}, \preceq_{\mathcal{T}})$ is a finite tree with a unique root denoted by $\varnothing$, and $f_\varnothing = f$.

(ii) For every $t \in \mathcal{T}$ maximal node, $f_t = \varepsilon_t e_{k_t}^*$ where $\varepsilon_t = \pm 1$.

(iii) For every $t \in \mathcal{T}$ which is not maximal, there exists $i \in I$ such that $(f_s)_{s\in S_t}$ is $\mathcal{F}_i$-admissible and $f_t = \theta_i \sum_{s\in S_t} f_s$, where S_t denotes the set of immediate successor nodes of t.

Note that S_t is well-ordered by $s_0 < s_1$ if and only if $\operatorname{supp} f_{s_0} < \operatorname{supp} f_{s_1}$. Whenever there is no possible confusion we will write $\preceq$ in order to denote $\preceq_{\mathcal{T}}$.

It is not difficult to see, by the minimality of the set $K((\mathcal{F}_i,\theta_i)_{i\in I})$, that every functional of $K((\mathcal{F}_i,\theta_i)_{i\in I})$ admits a tree analysis.

As we mentioned before in Remark 1.4, in general it is not true that

$$K^M((\mathcal{F}_i,\theta_i)_{i\in I}) = K^M((\mathcal{F}_i\restriction M,\theta_i)_{i\in I})$$

for a given infinite set M of integers, so, *a priori*, it does not suffice to control the restrictions $\mathcal{F}_i\restriction M$ ($i \in I$) for the understanding of norms $\|\sum_{n\in M} a_n e_n\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}$. We will see soon that the following is a key definition for this purpose.

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Definition 1.6 Given a family $\mathcal{F}$ we define the family of all $\mathcal{F}$-admissible sets as follows. We say that a finite set $t = \{m_i\}_{i=0}^{k-1}$ *interpolates* the block sequence $(s_i)_{i=0}^{k-1}$ of finite sets if and only if $m_0 \le s_0 < m_1 \le s_1 < \cdots < m_{n-1} \le s_{n-1}$. We say that $t = \{n_i\}$ *interpolates* $s = \{m_i\}$ if and only if t interpolates the block sequence $(\{m_i\})$.

Given a family $\mathcal{F}$ of finite sets, a block sequence $(s_i)_{i=0}^{n-1}$ of finite sets is $\mathcal{F}$-*admissible* if there is some $t \in \mathcal{F}$ which interpolates $(s_i)_{i=0}^{n-1}$. We define

$$\mathrm{Ad}(\mathcal{F}) = \{\{m_i\}_{i=0}^{n} \in \mathrm{FIN} : (\{m_i\})_{i=0}^{n} \text{ is } \mathcal{F}\text{-admissible}\},$$

the family of all $\mathcal{F}$-admissible finite sets.

Notice that if $M \subseteq \mathbb{N}$ and (s_i) is an $\mathcal{F}$-admissible sequence of subsets of M, then $\{\min s_i\} \in \mathrm{Ad}(\mathcal{F}) \restriction M$. The converse is not true in general.

We list some properties of the $\mathcal{F}$-admissible sets. Particularly interesting is the characterization of spread of a family in terms of its $\mathcal{F}$-admissible sets.

Proposition 1.7 (i) $\mathcal{F} \subseteq \mathrm{Ad}(\mathcal{F})$.

(ii) *If $\mathcal{F}$ is compact or hereditary, then so is* $\mathrm{Ad}(\mathcal{F})$.

(iii) *$\mathcal{F}$ is spreading on M if and only if* $\mathrm{Ad}(\mathcal{F}) \restriction M = \mathcal{F}$.

(iv) *Set* $\mathrm{Ad}^{(n+1)}(\mathcal{F}) = \mathrm{Ad}(\mathrm{Ad}^{(n)}\,\mathcal{F})$, $\mathrm{Ad}^{(0)}(\mathcal{F}) = \mathcal{F}$. *Then*

$$\mathrm{spread}(\mathcal{F}) = \{s : \exists t \in \mathcal{F}\,(t \preceq s)\} = \bigcup_n \mathrm{Ad}^{(n)}(\mathcal{F})$$

is the minimal spreading family on $\mathbb{N}$ containing $\mathcal{F}$. If $\mathcal{F}$ is compact or hereditary, then so is $\mathrm{spread}(\mathcal{F})$*, and if $\mathcal{F}$ is regular on some set M,* $\mathrm{spread}(\mathcal{F}) \restriction M = \mathcal{F}$.

Proof (i) and (ii) are easily proved.

(iii) If $\mathcal{F}$ is spreading on M, and $t \in \mathcal{F}$ interpolates some $s \subseteq M$, then, in particular, $t \preceq s$ and hence $s \in \mathcal{F}$. Suppose that $\mathrm{Ad}(\mathcal{F}) \restriction M = \mathcal{F}$ and suppose that $s \preceq t$ with $s \in \mathcal{F}$ and $t \subseteq M$. Set $s = \{m_i\}_{i=1}^{k}$ and $t = \{n_i\}_{i=1}^{k}$. For each $0 \le j \le k$ let $t_j = \{m_i : 1 \le i \le k - j\} \cup \{n_i : k - j + 1 \le i \le k\}$. Observe that $t_0 = s \in \mathcal{F}$, t_j interpolates t_{j+1} and that $t_k = t$, so an easy inductive argument finishes the proof of (iii). (iv) follows by arguments similar to (iii). ■

Finally, let us recall the following from [12].

Theorem 1.8 *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two compact and hereditary families. Then there is some infinite set M such that either $\mathcal{F} \restriction M \subseteq \mathcal{G} \restriction M$ or $\mathcal{G} \restriction M \subseteq \mathcal{F} \restriction M$.*

As for regular families $\mathcal{F}$, we have that $\iota(\mathcal{F} \restriction M) = \iota(\mathcal{F})$ for every M (see Proposition 3.4), it follows that if $\mathcal{F}$ and $\mathcal{G}$ are two regular families with $\iota(\mathcal{F}) < \iota(\mathcal{G})$, then for every M there is some $N \subseteq M$ such that $\mathcal{F} \restriction N \subseteq \mathcal{G} \restriction N$. In other words, strict inequalities between indices of regular families imply, modulo restrictions, strict inclusion between those families.

2 Subsequences of the Basis for Regular Families

The purpose of this section is to understand, for regular families, the relationship between the operations $\oplus$ and $\otimes$ on regular families and corresponding norming sets. For example, what is the relation between $K(\mathcal{F}\oplus\mathcal{F},\theta)$ and $K(\mathcal{F},\theta)$? It is well known that if the family $\mathcal{F}$ has finite index, then these two norming sets are, in general, different as the corresponding Tsirelson type spaces are isomorphic to different ℓ_ps. However if $\mathcal{F}$ is, for example, the Schreier family $\mathcal{S}$, then it can be easily shown that $[\mathbb{N}]^{\le 3}\otimes\mathcal{S}\subseteq\mathcal{S}\otimes[\mathbb{N}]^{\le 2}$, and hence

$$\begin{aligned}[\mathbb{N}]^{\le 8}\otimes(\mathcal{S}\otimes[\mathbb{N}]^{\le 2}) &\subseteq ([\mathbb{N}]^{\le 3}\otimes([\mathbb{N}]^{\le 3}\otimes\mathcal{S}))\otimes[\mathbb{N}]^{\le 2}\\ &\subseteq (\mathcal{S}\otimes[\mathbb{N}]^{\le 4})\otimes[\mathbb{N}]^{\le 2}=\mathcal{S}\otimes[\mathbb{N}]^{\le 8}.\end{aligned}$$

It follows by induction on the complexity of $\phi\in K(\mathcal{S}\otimes[\mathbb{N}]^{\le 2},\theta)$ that $\phi=\phi_1+\cdots+\phi_8$ for some block sequence $(\phi_i)_{i=1}^8\subseteq K(\mathcal{S},\theta)$. This clearly implies that

$$\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{S},\theta)}\le\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{S}\otimes[\mathbb{N}]^{\le 2},\theta)}\le 8\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{S},\theta)}$$

for every $0<\theta<1$ and every sequence (a_n) of scalars. As one can guess, this reasoning cannot be applied to an arbitrary regular family $\mathcal{F}$ with infinite index since we do not have an explicit presentation of $\mathcal{F}$ as for the Schreier family. However, we do have the index of the family, and by the properties of the ordinals we have that

$$\iota([\mathbb{N}]^{\le 3}\otimes(\mathcal{F}\otimes[\mathbb{N}]^{\le 2}))=3(\iota(\mathcal{F})2)<\iota(\mathcal{F})2+\omega\le\iota(\mathcal{F})3,$$

and, since $\mathcal{F}$ is regular, there is some infinite set M of integers such that $[M]^{\le 3}\otimes(\mathcal{F}\restriction M\otimes[M]^{\le 2})\subseteq\mathcal{F}\otimes[\mathbb{N}]^{\le 3}$, hence

$$\Big\|\sum_{n\in M} a_ne_n\Big\|_{(\mathcal{F},\theta)}\le\Big\|\sum_{n\in M} a_ne_n\Big\|_{(\mathcal{F}\otimes[\mathbb{N}]^{\le 2},\theta)}\le 3\Big\|\sum_{n\in M} a_ne_n\Big\|_{(\mathcal{F},\theta)},$$

so the two subsequences $(e_n)_{n\in M}\subseteq T(\mathcal{F},\theta)$ and $(e_n)_{n\in M}\subseteq T(\mathcal{F}\otimes[\mathbb{N}]^{\le 2},\theta)$ of the corresponding natural bases are 3-equivalent.

We start with the following simple fact that readily follows from the definitions of the norms.

Fact Suppose that $(\mathcal{F}_i,\theta_i)_{i\in I}$, $(\mathcal{G}_i,\theta_i)_{i\in I}$ and $M\subseteq\mathbb{N}$ have the property that every $\mathcal{G}_i$-admissible sequence of subsets of M is $\mathcal{F}_i$-admissible ($i\in I$). Then for every sequence $(a_n)_{n\in M}$ of scalars

$$\Big\|\sum_{n\in M} a_ne_n\Big\|_{(\mathcal{G}_i,\theta_i)_{i\in I}}\le\Big\|\sum_{n\in M} a_ne_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}.$$

The next result is a simple generalization of the above fact which will be used repeatedly.

Proposition 2.1 *Suppose that $(\mathcal{F}_i,\theta_i)_{i\in I}$, $(\mathcal{G}_i,\theta_i)_{i\in I}$, $M\subseteq\mathbb{N}$ and $k\in\mathbb{N}$ have the property that*

$$[M]^{\leq k}\otimes \mathrm{Ad}(\mathcal{F}_i)\restriction M\subseteq \mathcal{G}_i\restriction M\otimes[M]^{\leq k}\quad (i\in I). \tag{2.1}$$

Then for every sequence $(a_n)_{n\in M}$ of scalars

$$\Big\|\sum_{n\in M}a_ne_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i\in I}}\leq k\Big\|\sum_{n\in M}a_ne_n\Big\|_{(\mathcal{G}_i,\theta_i)_{i\in I}}.$$

Proof We are going to show, using (2.1), that for every $\phi\in K^M((\mathcal{F}_i,\theta_i)_{i\in I})$ there are $\psi_0<\cdots<\psi_{l-1}$ in $K^M((\mathcal{G}_i,\theta_i)_{i\in I})$, $l\leq k$, such that $\phi=\psi_0+\cdots+\psi_{l-1}$. The proof is by induction on the *complexity of* ϕ. If $\phi=e_n^*$, there is nothing to prove. Suppose that $\phi=\theta_i(\phi_0+\cdots+\phi_n)$, where $(\phi_i)_{i=0}^n\subseteq K^M((\mathcal{F}_i,\theta_i)_{i\in I})$ is $\mathcal{F}_i$-admissible. By inductive hypothesis find for every j a set u_j of cardinality at most k and a block sequence $(\psi_s)_{s\in u_j}\subseteq K^M((\mathcal{G}_i,\theta_i)_{i\in I})$ such that $\phi_j=\sum_{s\in u_j}\psi_s$ $(j=0,\dots,n)$. Observe that since $(\phi_j)_{j=0}^n$ is $\mathcal{F}_i$-admissible, $\{\min\phi_j\}_{j=0}^n\in\mathrm{Ad}(\mathcal{F}_i)$. Hence by our hypothesis (2.1),

$$t=\bigcup_{j=0}^n\{\min\psi_s:s\in u_j\}\in[M]^{\leq k}\otimes(\mathrm{Ad}(\mathcal{F}_i))\restriction M\subseteq\mathcal{G}_i\restriction M\otimes[M]^{\leq k}.$$

So there are $t_0<\cdots<t_{l-1}$ in $\mathcal{G}_i\restriction M$ $(l\leq k)$ such that $t=t_0\cup\cdots\cup t_{l-1}$. For $0\leq m\leq l-1$ set

$$\psi^{(m)}=\theta_i\Big(\sum_{\min\psi_s\in t_m}\psi_s\Big)\in K^M((\mathcal{G}_i,\theta_i)_{i\in I}).$$

Then $\phi=\psi^{(0)}+\cdots+\psi^{(l-1)}$, as desired. ∎

As a consequence we obtain the following two results. The first one is the general version of the examples considered in the introduction to this section.

Corollary 2.2 *Let $(\mathcal{B}_i,\theta_i)_{i=1}^r$ and $(\mathcal{C}_i,\theta_i)_{i=1}^r$ be regular sequences such that*

$$\omega\leq\iota(\mathcal{C}_i)\leq\iota(\mathcal{B}_i)\leq\iota(\mathcal{C}_i)k\quad 1\leq i\leq r$$

for some integer $k\geq 1$. Then for every M there is some $N\subseteq M$ such that the subsequences $(e_n)_{n\in N}$ of the basis of $T[(\mathcal{B}_i,\theta_i)_{i=1}^r]$ and $T[(\mathcal{C}_i,\theta_i)_{i=1}^r]$ are $2(k+1)$-equivalent.

Proof By our assumption on the indices of the families we obtain that

$$\iota([\mathbb{N}]^{\leq k+1}\otimes\mathcal{B}_i)=(k+1)\iota(\mathcal{B}_i)<\iota(\mathcal{B}_i)+\omega\leq\iota(\mathcal{C}_i\otimes[\mathbb{N}]^{\leq k+1})$$

for every $1\leq i\leq r$. Hence there is some $N_0\subseteq M$ such that $[N_0]^{\leq k+1}\otimes\mathcal{B}_i\restriction N_0\subseteq\mathcal{C}_i\otimes[N_0]^{\leq k+1}$ for every $1\leq i\leq r$. Proposition 2.1 yields

$$\Big\|\sum_{n\in N_0}a_ne_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r}\leq(k+1)\Big\|\sum_{n\in N_0}a_ne_n\Big\|_{(\mathcal{C}_i,\theta_i)_{i=1}^r}.$$

By Theorem 1.8 there exists $N \subseteq N_0$ such that

$$[N]^{\leq 2} \otimes \mathcal{C}_i \restriction N \subseteq \mathcal{B}_i \otimes [N]^{\leq 2} \text{ for every } i \leq r.$$

Proposition 2.1 yields

$$\| \sum_{n \in N} a_n e_n \|_{(\mathcal{C}_i, \theta_i)_{i=1}^r} \leq 2 \| \sum_{n \in N} a_n e_n \|_{(\mathcal{B}_i, \theta_i)_{i=1}^r}$$

which completes the proof. ∎

The next result says the *shift* operator, when restricted to some subsequence of the basis, is always bounded. For a given set N and $n \in N$, let $n^+ \in N$ be the immediate successor of n in N, *i.e.*, $n^+ = \min N/n$.

Corollary 2.3 *Let $(\mathcal{B}_i, \theta_i)_{i=1}^r$ be a regular sequence. Then for every M there is some $N \subseteq M$ such that for every sequence of scalars $(a_n)_{n \in N}$,*

$$\Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r} \leq \Big\| \sum_{n \in N} a_n e_{n^+} \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r} \leq 2 \Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r}.$$

Proof We set $I = \{1 \leq i \leq r : \iota(\mathcal{B}_i) \text{ is finite}\}$ and J for the complement of I. By Theorem 1.8 we can find $N \subseteq M$ such that and

$$[N]^{\leq 2} \otimes \big((\mathcal{B}_i \restriction N) \oplus [N]^{\leq 1} \big) \subseteq (\mathcal{B}_i \restriction N) \otimes [N]^{\leq 2} \quad (i \in J).$$

Moreover, we may assume that $\mathcal{B}_i \restriction N = [N]^{\leq \iota(\mathcal{B}_i)}$ for every $i \in I$ (see Proposition 3.4). By Proposition 2.1 we get

$$\Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i \in I} {}^\frown (\mathcal{B}_i \oplus [\mathbb{N}]^{\leq 1}, \theta_i)_{i \in J}} \leq 2 \Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r}. \tag{2.2}$$

For a given finite set $s \subseteq N$, we set $s^+ = \{n^+ : n \in s\}$. Suppose that $s^+ \in \mathcal{B}_i$. Then if $i \in I$ we have that $s \in \mathcal{B}_i$; if $i \in J$, then $(s^+) \setminus \{\max s^+\} \preceq {}_*s$, so ${}_*s \in \mathcal{B}_i$ because $\mathcal{B}_i$ is spreading. This implies that in this case $s \in \mathcal{B}_i \oplus [\mathbb{N}]^{\leq 1}$. This fact proves that

$$\Big\| \sum_{n \in N} a_n e_{n^+} \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r} \leq \Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i \in I} {}^\frown (\mathcal{B}_i \oplus [\mathbb{N}]^{\leq 1}, \theta_i)_{i \in J}}. \tag{2.3}$$

Now, using that $\mathcal{B}_i$ are spreading, by (2.2) and (2.3) we get,

$$\Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r} \leq \Big\| \sum_{n \in N} a_n e_{n^+} \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r} \leq 2 \Big\| \sum_{n \in N} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r}. \qquad ∎$$

We examine the effect of the power operation $\mathcal{B}^{\otimes(m)}$ for regular families $\mathcal{B}$ on the corresponding norming set. We follow some of the ideas used in the proof of the corresponding result for Schreier families (see [13, 15]).

Lemma 2.4 *Fix an infinite set M of integers, $m \in \mathbb{N}$, and a regular sequence $(\mathcal{B}_i, \theta_i)_{i=1}^r$. Then for every sequence $(a_n)_{n\in M}$ of scalars*

$$\Big\|\sum_{n\in M} a_n e_n\Big\|_{(\mathcal{B}_1^{\otimes(m)},\theta_1^m)^\frown(\mathcal{B}_i,\theta_i)_{i=2}^r} \le \Big\|\sum_{n\in M} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r}.$$

Proof For simplicity, using that the families considered here are regular, we may assume that $M = \mathbb{N}$. Suppose that $\phi \in K((\mathcal{B}_1^{\otimes(m)}, \theta_1^m)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$. We will show that

$$\phi\Big(\sum_n a_n e_n\Big) \le \Big\|\sum_n a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r}. \tag{2.4}$$

It can easily be shown by induction on m that if $(s_i)_{i=1}^k$ is $\mathcal{B}_1^{\otimes(m)}$-admissible, then

$$\theta_1^m \sum_{i=1}^k \Big\|\sum_{n\in s_i} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r} \le \Big\|\sum_{n\in\bigcup_{i=1}^k s_i} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r}.$$

It is not difficult to show by induction on the complexity of ϕ that the last inequality gives (2.4). ■

Lemma 2.5 *Suppose that M is an infinite set and that $(\mathcal{B}_i, \theta_i)_{i=1}^r$ is a regular sequence such that*

$$\mathcal{B}_1\restriction M \otimes \mathcal{B}_i \subseteq \mathcal{B}_i \otimes \mathcal{B}_1 \tag{2.5}$$

for every $1 \le i \le r$. Then for every integer m,

$$\begin{aligned}\theta_1^{m-1}\Big\|\sum_{n\in M} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r} &\le \Big\|\sum_{n\in M} a_n e_n\Big\|_{(\mathcal{B}_1^{\otimes(m)},\theta_1^m)^\frown(\mathcal{B}_i,\theta_i)_{i=2}^r}\\ &\le \Big\|\sum_{n\in M} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r}.\end{aligned} \tag{2.6}$$

Proof The second inequality is given by Lemma 2.4. We assume that $M = \mathbb{N}$. In order to prove the first inequality of (2.6) we are going to show that

$$\phi\Big(\sum_n a_n e_n\Big) \le \frac{1}{\theta_1^{m-1}}\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{B}_1^{\otimes(m)},\theta_1^m)^\frown(\mathcal{B}_i,\theta_i)_{i=2}^r)}$$

for every if $\phi \in K((\mathcal{B}_i, \theta_i)_{i=1}^r)$. For suppose that $(\phi_t)_{t\in\mathcal{T}}$ is a tree analysis of ϕ. For every $s \preceq t$ and $1 \le i \le r$ let

$$l_i(s,t) = \#\Big(\Big\{u : s \preceq u \precneqq t \text{ and } \phi_u = \theta_i \sum_{v\in S_u} \phi_v\Big\}\Big).$$

So we have the decomposition

$$\phi = \sum_{t\in\mathcal{A}}\Big(\prod_{i=1}^r \theta_i^{n_i(t)}\Big)(-1)^{\varepsilon_t} e_{m_t},$$

where $\mathcal{A}$ is the set of terminal nodes of $\mathcal{T}$, $n_i(t) = l_i(\varnothing, t)$, $\varepsilon_t \in \{0,1\}$, and m_t is an integer.

Claim *Suppose that there is some $0 \le d < m$ such that $n_1(t) \equiv d \pmod{m}$ for every $t \in \mathcal{A}$. Then there are $(\psi_i)_{i=1}^{l} \subseteq K\big((\mathcal{B}_1^{\otimes(m)}, \theta_1^m) ^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r\big)$ such that*

(i) *$\phi = \theta_1^d(\psi_1 + \cdots + \psi_l)$;*
(ii) *$(\psi_i)_{i=1}^l$ is $\mathcal{B}_1^{\otimes(d)}$-admissible.*

Assuming the claim, for every $t \in \mathcal{A}$, let $0 \le d_t < m$ be such that $n_1(t) + d_t \equiv 0 \pmod{m}$, and let

$$\psi = \sum_{t\in\mathcal{A}} \Big(\theta_1^{d_t} \prod_{i=1}^{r} \theta_i^{n_i(t)}\Big)(-1)^{\varepsilon_t} e_{m_t}.$$

By the claim we have that $\psi \in K((\mathcal{B}_1^{\otimes(m)}, \theta_1^m) ^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r)$. Finally,

$$\Big|\phi\Big(\sum_n a_n e_n\Big)\Big| \le \frac{1}{\theta_1^{m-1}}\Big|\psi\Big(\sum_n a_n e_n\Big)\Big| \le \frac{1}{\theta_1^{m-1}}\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{B}_1^{\otimes(m)}, \theta_1^m) ^\frown (\mathcal{B}_i,\theta_i)_{i=2}^r}.$$

which completes the proof of the lemma. ∎

Proof of Claim The proof is by induction on the complexity of ϕ. Suppose first that $\phi = \pm e_s$. Then $d = 0$ and the desired result is clearly true. Now suppose that $\phi = \theta_j(\phi_1 + \cdots + \phi_k)$. There are two cases to consider. If $j = 1$, then by inductive hypothesis applied to each ϕ_i $(1 \le i \le k)$, we have that for every $1 \le i \le k$,

$$\phi_i = \theta_1^{\bar d}(\psi_1^{(i)} + \cdots + \psi_{s_i}^{(i)}),$$

where $0 \le \bar d < m$ is such that $\bar d \equiv d - 1 \pmod{m}$ and

$$(\psi_l^{(i)})_{l=1}^{s_i} \subseteq K((\mathcal{B}_1^{\otimes(m)}, \theta_1^m) ^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r)$$

is $\mathcal{B}^{\otimes(\bar d)}$-admissible. It follows that

$$\phi = \theta_1(\phi_1 + \cdots + \phi_k) = \begin{cases} \theta_1^m(\sum_{i=1}^k(\sum_{l=1}^{s_i}\psi_l^{(i)})) & \text{if } d = 0,\\ \theta_1^d(\sum_{i=1}^k(\sum_{l=1}^{s_i}\psi_l^{(i)})) & \text{if } d > 0.\end{cases}$$

Using that $(\phi_i)_{i=1}^k$ is $\mathcal{B}_1$-admissible we obtain that

$$\bigcup_{i=1}^{k} \{\min \psi_j^{(i)}\}_{j=1}^{s_i} \in \begin{cases} \mathcal{B}_1^{\otimes(m)} & \text{if } d = 0,\\ \mathcal{B}_1^{\otimes(d)} & \text{if } d > 0.\end{cases}$$

So if $d = 0$, we obtain that $\phi \in K((\mathcal{B}_1^{\otimes(m)}, \theta_1^m) ^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r)$, as desired; otherwise, (i) and (ii) in the claim are clearly true for ϕ.

Now suppose that $j > 1$. By inductive hypothesis applied to each ϕ_i ($1 \le i \le k$), we have that $\phi_i = \theta_1^d(\psi_1^{(i)} + \cdots + \psi_{s_i}^{(i)})$, where

$$(\psi_l^{(i)})_{l=1}^{k_i} \subseteq K((\mathcal{B}_1^{\otimes(m)}, \theta_1^m)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$$

is $\mathcal{B}_1^{\otimes(d)}$-admissible. It follows that the sequence $(\psi_1^{(1)}, \dots, \psi_{s_1}^{(1)}, \dots, \psi_1^{(k)}, \dots, \psi_{s_k}^{(k)})$ is $(\mathcal{B}_1^{\otimes(d)}) \otimes \mathcal{B}_j$-admissible. Observe that (2.5) and the associative property of $\otimes$ give that

$$(\mathcal{B}_1^{\otimes(d)}) \otimes \mathcal{B}_j = (\mathcal{B}_1 \otimes \overset{(d)}{\cdots} \otimes \mathcal{B}_1) \otimes \mathcal{B}_j \subseteq \mathcal{B}_j \otimes (\mathcal{B}_1^{\otimes(d)}),$$

so it follows that $(\psi_1^{(1)}, \dots, \psi_{s_1}^{(1)}, \dots, \psi_1^{(k)}, \dots, \psi_{s_k}^{(k)})$ is also $\mathcal{B}_j \otimes (\mathcal{B}_1^{\otimes(d)})$-admissible. Let $(t_i)_{i=1}^h$ be a block sequence of finite sets such that

$$\{\min \psi_p^{(i)} : 1 \le i \le k,\ 1 \le p \le s_i\} = \bigcup_{i=1}^h t_i$$

with $\{t_i\}_{i=1}^h \subseteq \mathcal{B}_j$ and $\{\min t_i\}_{i=1}^h \in \mathcal{B}_1^{\otimes(d)}$. For every $1 \le l \le h$ let

$$\xi_l = \theta_j \sum_{\min \psi_p^{(i)} \in t_l} \psi_p^{(i)} \in K((\mathcal{B}_1^{\otimes(m)}, \theta_1^m)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r),$$

whence we obtain the decomposition $\phi = \theta_1^d \sum_{l=1}^h \xi_l$, giving the desired result. ■

As a consequence of the previous lemma, we get the next proposition, which is the natural generalization of a well-known fact for the Schreier families $\mathcal{S}_n$ ($n \in \mathbb{N}$).

Proposition 2.6 *Let $\mathcal{B}$ be a regular family. Then for every $0 < \theta < 1$, every m, and every sequence of scalars (a_n)*

$$\Big\| \sum_n a_n e_n \Big\|_{(\mathcal{B}^{\otimes(m)}, \theta^m)} \le \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{B}, \theta)} \le \frac{1}{\theta^{m-1}} \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{B}^{\otimes(m)}, \theta^m)}.$$

The next lemma analyzes the case of indices $\iota(\mathcal{B}) = \omega^{\alpha+\beta}$ and $\iota(\mathcal{C}) = \omega^\alpha$ with $\beta + \alpha = \alpha$, as for example $\mathcal{B} = \mathcal{S}_{\omega^2+\omega}$ and $\mathcal{C} = \mathcal{S}_{\omega^2}$.

Lemma 2.7 *Let M be an infinite set of integers, $\mathcal{C}$, $\mathcal{B}_i$ be regular families ($1 \le i \le r$) such that $[M]^{\le 2} \subseteq \mathcal{C}$ and*

$$[M]^{\le 2} \otimes \mathcal{C}\restriction M \otimes \mathcal{B}_i\restriction M \subseteq \mathcal{B}_i \otimes [\mathbb{N}]^{\le 2} \quad (1 \le i \le r). \tag{2.7}$$

Then for every sequence $(\theta_i)_{i=1}^r \subset (0,1)$ and every sequence of scalars $(a_n)_{n\in M}$,

$$\Big\| \sum_{n\in M} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r} \le \Big\| \sum_{n\in M} a_n e_n \Big\|_{(\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r} \le \frac{2}{\theta_1} \Big\| \sum_{n\in M} a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r}.$$

Proof The first inequality is clear. Let us show the second one. In order to keep the notation simpler, we may assume, since all families here are regular, that $M = \mathbb{N}$.

Claim *Every $\phi \in K((\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$ has a decomposition $\phi = \phi_1 + \cdots + \phi_n$ where $(\phi_i)_{i=1}^n \subseteq K((\mathcal{C} \otimes \mathcal{B}_1, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$ is $\mathcal{C}$-admissible.*

Proof of Claim Fix $\phi \in K((\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$. If $\phi = \pm e_n^*$, the claim is clear. Now there are two cases to consider.

Case 1. $\phi = \theta_1(\phi_1 + \cdots + \phi_n)$, where $(\phi_i)_{i=1}^n \subseteq K((\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$ is $\mathcal{B}_1 \otimes \mathcal{C}$-admissible. By inductive hypothesis, for each $i = 1, \ldots, n$,

$$\phi_i = \sum_{j=1}^{n_i} \psi_j^{(i)}$$

where

$$(\psi_j^{(i)})_{j=1}^{n_i} \subseteq K((\mathcal{C} \otimes \mathcal{B}_1, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$$

is $\mathcal{C}$-admissible, *i.e.*, $s_i = \{\min \psi_j^{(i)}\}_{j=1}^{n_i} \in \mathcal{C}$. Since for every $i = 1, \ldots, n$, $\min s_i = \min \operatorname{supp} \phi_i$ we obtain that

$$s_1 \cup \cdots \cup s_n \in \mathcal{C} \otimes (\mathcal{B}_1 \otimes \mathcal{C}) = (\mathcal{C} \otimes \mathcal{B}_1) \otimes \mathcal{C}.$$

Hence we can find a block sequence $(t_i)_{i=1}^m$ such that $s_1 \cup \cdots \cup s_n = t_1 \cup \cdots \cup t_m$ and such that $(t_i)_{i=1}^m \subseteq \mathcal{C} \otimes \mathcal{B}_1$ is $\mathcal{C}$-admissible. For every $k \in t_1 \cup \cdots \cup t_m$, let $i(k)$, $j(k)$ be such that $\min \psi_{j(k)}^{(i(k))} = k$. For every $i = 1, \ldots, m$, let

$$\psi_i = \theta_1 \Big(\sum_{k \in t_i} \psi_{j(k)}^{i(k)} \Big).$$

Since $(\psi_{j(k)}^{(i(k))})_{k \in t_i}$ is a block sequence, and since $\{\min \psi_{j(k)}^{(i(k))}\}_{k \in t_i} = t_i \in \mathcal{C} \otimes \mathcal{B}_1$ we obtain that $\psi_i \in K((\mathcal{C} \otimes \mathcal{B}_1, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$. It is clear that

$$\phi = \theta_1(\phi_1 + \cdots + \phi_n) = \theta_1 \Big(\sum_{i=1}^n \sum_{j=1}^{n_i} \psi_j^{(i)} \Big) = \theta_1 \sum_{i=1}^m \sum_{k \in t_i} \psi_{j(k)}^{(i(k))}$$

$$= \sum_{i=1}^m \theta_1 \sum_{k \in t_i} \psi_{j(k)}^{(i(k))} = \psi_1 + \cdots + \psi_m.$$

Note that $\min \psi_i = \min t_i$ $(1 \leq i \leq m)$, hence $\{\min \psi_i\}_{i=1}^m = \{\min t_i\}_{i=1}^m \in \mathcal{C}$, so we are done.

Case 2. $\phi = \theta_j(\phi_1 + \cdots + \phi_n)$, where $(\phi_i)_{i=1}^n \subseteq K((\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown(\mathcal{B}_i, \theta_i)_{i=2}^r)$ is $\mathcal{B}_j$-admissible for some $2 \leq j \leq r$. By inductive hypothesis, for each $i = 1, \ldots, n$,

$$\phi_i = \sum_{j=1}^{k_i} \psi_j^{(i)}$$

where $(\psi_j^{(i)})_{j=1}^{k_i} \subseteq K\big((\mathcal{C} \otimes \mathcal{B}_1, \theta_1)^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r)\big)$ is $\mathcal{C}$-admissible, *i.e.*,

$$s_i = \{\min \psi_j^{(i)}\}_{j=1}^{k_i} \in \mathcal{C}.$$

It follows by (2.7) and the fact that $[\mathbb{N}]^{\leq 2} \subseteq \mathcal{C}$ that

$$s_1 \cup \dots \cup s_n \in \mathcal{C} \otimes \mathcal{B}_j \subseteq \mathcal{B}_j \otimes [\mathbb{N}]^{\leq 2} \subseteq \mathcal{B}_j \otimes \mathcal{C}.$$

Following ideas similar to those in the proof of Case 1, one can easily find the desired decomposition of ϕ. ■

From the claim we obtain that $\theta_1 \phi \in K((\mathcal{C} \otimes \mathcal{B}_1)^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r)$ for every $\phi \in K((\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r)$. Now this fact implies that for every sequence $(a_n)_n$ of scalars

$$\Big\| \sum_n a_n e_n \Big\|_{(\mathcal{B}_1 \otimes \mathcal{C}, \theta_1)^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r} \leq \frac{1}{\theta_1} \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{C} \otimes \mathcal{B}_1, \theta_1)^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r}. \tag{2.8}$$

Since (2.7) holds, we can apply Proposition 2.1 to get that

$$\Big\| \sum_n a_n e_n \Big\|_{(\mathcal{C} \otimes \mathcal{B}_1, \theta_1)^\frown (\mathcal{B}_i, \theta_i)_{i=2}^r} \leq 2 \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{B}_i, \theta_i)_{i=1}^r}. \tag{2.9}$$

Finally we obtain the desired inequality by joining (2.8) and (2.9). ■

The previous combinatorial lemma gives the following.

Theorem 2.8 *Suppose that $\mathcal{B}_0$ and $\mathcal{B}_1$ are two regular families such that $\iota(\mathcal{B}_0) = \omega^{\alpha+\beta}$, $\iota(\mathcal{B}_1) = \omega^\alpha$, with $\beta + \alpha = \alpha$. Then for every infinite set M of integers there is an infinite $N \subseteq M$ such that $(e_n)_{n \in N} \subseteq T(\mathcal{B}_0, \theta)$ and $(e_n)_{n \in N} \subseteq T(\mathcal{B}_1, \theta)$ are equivalent.*

Proof Let $\mathcal{C}$ be a regular family with $\iota(\mathcal{C}) = \omega^\beta$. Since $\iota(\mathcal{B}_1 \otimes \mathcal{C}) = \omega^{\alpha+\beta} = \iota(\mathcal{B}_0)$ passing to a subset N of M if needed, we may assume, by Corollary 2.2, that the subsequence $(e_n)_{n \in N}$ is equivalent in the spaces $T(\mathcal{B}_0, \theta)$ and $T(\mathcal{B}_1 \otimes \mathcal{C}, \theta)$, and hence we may assume that $\mathcal{B}_0 = \mathcal{B}_1 \otimes \mathcal{C}$. Then

$$\iota([M]^{\leq 2} \otimes \mathcal{C}{\restriction}M \otimes \mathcal{B}_1{\restriction}M) = 2\omega^\beta \omega^\alpha = \omega^\alpha < \omega^\alpha 2.$$

So we may find $N \subseteq M$ such that $[N]^{\leq 2} \otimes \mathcal{C}{\restriction}N \otimes \mathcal{B}_1{\restriction}M \subseteq \mathcal{B}_1 \otimes [\mathbb{N}]^{\leq 2}$, (see the comment after Theorem 1.8). Hence the result follows from the previous lemma. ■

2.1 Reduction from Finite to One

The aim of this subsection is to reduce finite regular sequences to one, more precisely, we show in Theorem 2.13 that for every finite regular sequence $(\mathcal{F}_i, \theta_i)_{i=1}^r$ there is some $1 \leq i_0 \leq r$ and some infinite set M of integers such that $(e_n)_{n \in M} \subseteq T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$ and $(e_n)_{n \in M} \subseteq T(\mathcal{F}_{i_0}, \theta_{i_0})$ are equivalent, where i_0 will come from a certain ordering of the pairs $(\mathcal{F}_i, \theta_i)$.

Definition 2.9 Recall that every ordinal $\alpha > 0$ has the unique decomposition

$$\alpha = \omega^{\lambda(\alpha)} l(\alpha) + \xi(\alpha)$$

with $l(\alpha) > 0$ an integer and $\xi(\alpha) < \omega^{\lambda(\alpha)}$. Define

$$\gamma(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ is finite,} \\ \omega^{\omega^{\lambda(\lambda(\alpha))}} & \text{if } \alpha \text{ is infinite,} \end{cases}$$

$$n(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is finite,} \\ l(\lambda(\alpha)) & \text{if } \alpha \text{ is infinite,} \end{cases}$$

$$k(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is finite,} \\ l(\alpha) & \text{if } \alpha \text{ is infinite.} \end{cases}$$

For example, $\gamma(\omega^{\omega^2 3+\omega} 4 + \omega^5) = \omega^{\omega^2}$, $n(\omega^{\omega^2 3+\omega} 4 + \omega^5) = 3$ and $\gamma(m) = m$ for every integer $m \neq 0$. In general for an arbitrary ordinal $\alpha > 0$ we have the decomposition $\alpha = \gamma(\alpha)^{n(\alpha)} \omega^{\xi(\lambda(\alpha))} k(\alpha) + \xi(\alpha)$, with the convention of $\xi(0) = 0$.

We want to compare two Tsirelson type spaces $T(\mathcal{F}_0, \theta_0)$ and $T(\mathcal{F}_1, \theta_1)$. There is the following natural relation of domination: we write $(\mathcal{F}_0, \theta_0) \leq' (\mathcal{F}_1, \theta_1)$ if and only if there is some $C \geq 1$ such that every subsequence $(e_n)_{n\in M}$ of the basis of $T(\mathcal{F}_0, \theta_0)$ has a further subsequence $(e_n)_{n\in N}$ such that

$$\Big\| \sum_{n\in N} a_n e_n \Big\|_{(\mathcal{F}_0,\theta_0)} \leq C \Big\| \sum_{n\in N} a_n e_n \Big\|_{(\mathcal{F}_1,\theta_1)}.$$

It is clear that if $\mathcal{F}_0 \subseteq \mathcal{F}_1$ and $\theta_0 \leq \theta_1$, then $(\mathcal{F}_0, \theta_0) \leq' (\mathcal{F}_1, \theta_1)$. As we have already seen in Proposition 2.6, for every integer n the pair $(\mathcal{F}, \theta) \leq'$-dominates $(\mathcal{F}^{\otimes(n)}, \theta^n)$ and vice versa. This suggests the following more appropriate relation: $(\mathcal{F}_0, \theta_0) \leq'' (\mathcal{F}_1, \theta_1)$ if and only if there are $n_0, n_1 \in \mathbb{N}$ such that for every M there is $N \subseteq M$ such that $\mathcal{F}_0^{\otimes(n_0)} \restriction N \subseteq \mathcal{F}_1^{\otimes(n_1)}$ and $\theta_0^{n_0} \leq \theta_1^{n_1}$.

As we have also shown that

$$(\mathcal{S}_{\omega^\alpha}, \theta) \leq'' (\mathcal{S}_{\omega^{\alpha}+\beta}, \theta) \leq'' (\mathcal{S}_{\omega^{\alpha}+\beta} \otimes [N]^{\leq k}, \theta) \leq'' (\mathcal{S}_{\omega^\alpha}, \theta),$$

we end up with the following definition.

Definition 2.10 For pairs (α, θ) of ordinals > 0 and real numbers we write $(\alpha_0, \theta_0) \leq_{\mathrm{T}} (\alpha_1, \theta_1)$ if and only if: $\alpha_0 = 1$, or $1 < \alpha_0, \alpha_1 < \omega$ and $\log_{\gamma(\alpha_0)} \theta_0 \leq \log_{\gamma(\alpha_1)} \theta_1$, or $\alpha_0\alpha_1 \geq \omega$ and there are integers m_0, m_1 such that $\gamma(\alpha_0)^{n(\alpha_0)m_0} \leq \gamma(\alpha_1)^{n(\alpha_1)m_1}$ and $\theta_0^{m_0} \leq \theta_1^{m_1}$.

We write $(\mathcal{F}_0, \theta_0) \leq_{\mathrm{T}} (\mathcal{F}_1, \theta_1)$ if and only if $(\iota(\mathcal{F}_0), \theta_0) \leq_{\mathrm{T}} (\iota(\mathcal{F}_1), \theta_1)$.

To simplify the notation we will write $\gamma(\mathcal{F})$ for $\gamma(\iota(\mathcal{F}))$ and $n(\mathcal{F})$ for $n(\iota(\mathcal{F}))$.

Proposition 2.11 (i) *Suppose that* $\max\{\alpha_0, \alpha_1\} \geq \omega$. *Then* $(\alpha_0, \theta_0) \leq_{\mathrm{T}} (\alpha_1, \theta_1)$ *if and only if* $\gamma(\alpha_0) < \gamma(\alpha_1)$, *or* $\gamma(\alpha_0) = \gamma(\alpha_1)$ *and* $\theta_0^{n(\alpha_1)} \leq \theta_1^{n(\alpha_0)}$.

(ii) $\leq_{\mathrm{T}}$ *is a total pre-ordering,* i.e., *reflexive and transitive.*

Proof (ii) We show that $<_{\mathrm{T}}$ is total. So, fix two pairs $(\alpha_i, \theta_i), i = 0, 1$. Suppose first that $\alpha_i\omega \leq \alpha_j$ for $i \neq j$. Then let n be such that $\theta_i^n < \theta_j$. Then clearly $\alpha_i n < \alpha_j$, and $\theta_i^n < \theta_j$, so $(\alpha_i, \theta_i) <_{\mathrm{T}} (\alpha_j, \theta_j)$. Suppose now that $\gamma(\alpha_0) = \gamma(\alpha_1)$. Then if $\theta_0^{n(\alpha_1)} \leq \theta_1^{n(\alpha_0)}$, we obtain that $(\alpha_0, \theta_0) \leq_{\mathrm{T}} (\alpha_1, \theta_1)$, and $(\alpha_1, \theta_1) \leq_{\mathrm{T}} (\alpha_0, \theta_0)$ otherwise. ■

Lemma 2.12 *Suppose that* Γ *is a finite set of countable ordinals and* $n \in \mathbb{N}$. *There is a sequence* $(\mathcal{B}_\gamma)_{\gamma\in\Gamma}$ *of regular families such that:*

(i) $\iota(\mathcal{B}_\gamma) = \gamma$ *for every* $\gamma \in \Gamma$*;*

(ii) $\mathcal{B}_\gamma = [\mathbb{N}]^{\leq\gamma}$ *if* $\gamma \in \Gamma$ *is finite;*

(iii) *for every* $m_1, m_2 \leq n$ *and every* $f_i : \{1, \ldots, m_i\} \to \Gamma$ *(*$i = 1, 2$*),*

$$\textit{if} \prod_{i\leq m_1} f_1(i) < \prod_{i\leq m_2} f_2(i), \textit{ then } \mathcal{B}_{f_1(1)} \otimes \cdots \otimes \mathcal{B}_{f_1(m_1)} \subseteq \mathcal{B}_{f_2(1)} \otimes \cdots \otimes \mathcal{B}_{f_2(m_2)}.$$

Proof For every $\gamma \in \Gamma$, fix a regular family $\mathcal{C}_\gamma$ of index γ with the extra requirement that if γ is finite, then $\mathcal{C}_\gamma = [\mathbb{N}]^{\leq\gamma}$. Since

$$\{\mathcal{C}_{f(1)} \otimes \cdots \otimes \mathcal{C}_{f(m)} : m \leq n \text{ and } f\colon \{1, \ldots, m\} \to \Gamma, \}$$

is a finite set of regular families, we can find an infinite set M such that for every $m_1, m_2 \leq n$ and every $f_i\colon \{1, \ldots, m_i\} \to \Gamma$, where $i = 1, 2$. If $\prod_{i\leq m_1} f_1(i) < \prod_{i\leq m_2} f_2(i)$, then

$$\mathcal{C}_{f_1(1)}\restriction M \otimes \cdots \otimes \mathcal{C}_{f_1(m_1)}\restriction M \subseteq \mathcal{C}_{f_2(1)}\restriction M \otimes \cdots \otimes \mathcal{C}_{f_2(m_2)}\restriction M.$$

Let $\Theta\colon M \to \mathbb{N}$ be the unique order-preserving onto mapping between M and $\mathbb{N}$. Then $(\Theta''(\mathcal{C}_\gamma\restriction M))_{\gamma\in\Gamma}$ is the desired sequence. ■

Theorem 2.13 *Suppose that* $(\mathcal{B}_i, \theta_i)_{i=1}^r$ *is a regular sequence with at least one of the families with infinite index. Let* i_0 *be such that*

$$(\iota(\mathcal{B}_{i_0}), \theta_{i_0}) = \max_{<_{\mathrm{T}}}\{(\iota(\mathcal{B}_i), \theta_i) : 1 \leq i \leq r\}.$$

Then every subsequence $(e_n)_{n\in M}$ *of the natural basis of* $T[(\mathcal{B}_i, \theta_i)_{i=1}^r]$ *has a further subsequence* $(e_n)_{n\in N}$ *equivalent to the corresponding subsequence* $(e_n)_{n\in N}$ *of the natural basis of* $T(\mathcal{B}_{i_0}, \theta_{i_0})$.

Proof To simplify the notation, we assume that $M = \mathbb{N}$. We re-order $(\mathcal{B}_i, \theta_i)_{i=1}^r$ in such a way that $(\mathcal{B}_i, \theta_i) \leq_{\mathrm{T}} (\mathcal{B}_j, \theta_j)$ for every $1 \leq i \leq j \leq r$.

Recall the decomposition (see Definition 2.9) $\iota(\mathcal{B}_i) = \gamma_i^{n_i}\delta_i k_i + \xi_i$, where $\gamma_i = \gamma(\iota(\mathcal{B}_i))$, and if $\iota(\mathcal{B}_i)$ is finite, then $\delta_i = n_i = k_i = 1, \xi_i = 0$, while if $\iota(\mathcal{B}_i)$ is infinite,

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then $n_i = n(\iota(\mathcal{B}_i))$, $\delta_i = \omega^{\xi(\lambda(\iota(\mathcal{B}_i)))}$, $k_i = k(\iota(\mathcal{B}_i))$ and $\xi_i = \xi(\iota(\mathcal{B}_i))$. Observe that $\gamma_r = \max\{\gamma_i : 1 \le i \le r\}$ is infinite. Define $m_i \in \mathbb{N}$ $(1 \le i \le r-1)$ as

$$m_i = \begin{cases} [\log_{\theta_i} \theta_r] + 1 & \text{if } \gamma_i < \gamma_r, \\ n_r & \text{if } \gamma_i = \gamma_r, \end{cases}$$

where $[a]$ stands for the integer part of a. Use the Lemma 2.12 for $\Gamma = \{\gamma_i, \delta_i : 1 \le i \le r\} \cup \{2\}$ and n large enough (for example $n = 2\max\{n_i m_i : 1 \le i \le r\} + 2$) to find the corresponding sequence $(\mathcal{H}_\gamma)_{\gamma \in \Gamma}$ of regular families.

For $1 \le i \le r$, let $\mathcal{C}_i = (\mathcal{H}_{\gamma_i})^{\otimes(n_i)} \otimes \mathcal{H}_{\delta_i}$. Observe that $\iota(\mathcal{C}_i) = \gamma_i^{n_i}\omega^{\delta_i}$ for every $1 \le i \le r$. It readily follows that there is $N \subseteq M$ such that for every $1 \le i \le r$, if $\iota(\mathcal{B}_i)$ is infinite, then

$$\begin{aligned} &[N]^{\le 2} \otimes \mathcal{C}_i \restriction N \subseteq \mathcal{B}_i \otimes [N]^{\le 2}, \\ &[N]^{\le k_i+1} \otimes \mathcal{B}_i \restriction N \subseteq \mathcal{C}_i \restriction N \otimes [N]^{\le k_i+1}, \end{aligned} \tag{2.10}$$

while $\mathcal{B}_i \restriction N = \mathcal{C}_i \restriction N$ if $\iota(\mathcal{B}_i)$ is finite. Since the families $\mathcal{B}_i$ and $\mathcal{C}_i$ are regular $(1 \le i \le r)$, Proposition 2.1 gives that for every sequence of scalars $(a_n)_{n\in N}$ we have that

$$\begin{aligned} \frac{1}{2}\Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{C}_i,\theta_i)_{i=1}^r} &\le \Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r} \\ &\le \Big(1 + \max_{\substack{1\le i\le r \\ \iota(\mathcal{B}_i) \text{ infinite}}} k_i\Big)\Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{C}_i,\theta_i)_{i=1}^r}. \end{aligned} \tag{2.11}$$

Let $\{\varrho_i\}_{i=1}^s$ be the strictly increasing enumeration of the set

$$\{\gamma_i : 1 \le i \le r,\ \gamma_i \text{ infinite}\}.$$

Define

$$\begin{aligned} I_0 &= \{1 \le i \le r : \gamma_i \text{ is finite}\} \\ I_i &= \{1 \le j \le r-1 : \gamma_j = \varrho_i\} \quad (1 \le i \le s), \end{aligned}$$

and $I_{s+1} = \{r\}$.

Finally, set $J_i = I_i \cup \cdots \cup I_{s+1}$ $(0 \le i \le s+1)$. The next result is the reduction from $(\mathcal{C}_i, \theta_i)_{i=1}^r$ to $(\mathcal{C}_r, \theta_r)$.

Claim *For every $0 \le j \le s$ and every sequence of scalars (a_n) we have that*

$$\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{C}_i,\theta_i)_{i\in J_j}} \le \prod_{i\in I_j} \frac{1}{\theta_i^{m_i-1}} \prod_{i\in I_j,\,\delta_i>1} \frac{2}{\theta_i}\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{C}_i,\theta_i)_{i\in J_{j+1}}}. \tag{2.12}$$

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Proof of Claim Fix $0 \le j \le s$. Let $K_j = \{i \in I_j : \delta_i > 1\}$, and suppose it is non-empty. This implies, in particular, that $j > 0$. Notice that $\varrho_j = \min\{\gamma_k : k \in J_j\}$. So it follows that $\delta_k < \gamma_k = \varrho_j \le \gamma_i$ for $k \in K_j$ and $i \in J_j$. So,

$$2\delta_k \gamma_i^{n_i} \delta_i = \gamma_i^{n_i} \delta_i < \gamma_i^{n_i} \delta_i 2 \quad (k \in K_j,\ i \in J_j),$$
$$2\delta_k \gamma_i^{n_i} = \gamma_i^{n_i} < \gamma_i^{n_i} 2 \quad (i, k \in K_j).$$

Hence,

$$[\mathbb{N}]^{\le 2} \otimes \mathcal{H}_{\delta_k} \otimes \mathcal{C}_i \subseteq \mathcal{C}_i \otimes [\mathbb{N}]^{\le 2} \quad (k \in K_j,\ i \in J_j),$$
$$[\mathbb{N}]^{\le 2} \otimes \mathcal{H}_{\delta_k} \otimes \mathcal{H}_{\gamma_i}^{\otimes(n_i)} \subseteq \mathcal{H}_{\gamma_i}^{\otimes(n_i)} \otimes [\mathbb{N}]^{\le 2} \quad (i, k \in K_j).$$

Repeated application of Lemma 2.7 gives that

$$\Big\| \sum_n a_n e_n \Big\|_{(\mathcal{C}_i, \theta_i)_{i \in I_j}} \le \prod_{i \in K_j} \frac{2}{\theta_i} \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{H}_{\gamma_i}^{\otimes(n_i)}, \theta_i)_{i \in K_j} \frown (\mathcal{C}_i, \theta_i)_{i \in I_j \setminus K_j}}. \tag{2.13}$$

Using that $\gamma_k^{n_k} \gamma_i^{n_i} \delta_i = \gamma_i^{n_i} \delta_i < \gamma_i^{n_i} \delta_i \gamma_k^{n_k}$ $(k \in I_j, i \in J_{j+1})$, it follows that

$$\mathcal{H}_{\gamma_k}^{\otimes(n_k)} \otimes \mathcal{C}_i \subseteq \mathcal{C}_i \otimes \mathcal{H}_{\gamma_k}^{\otimes(n_k)} \quad (k \in I_j,\ i \in J_{j+1}).$$

Since it is trivial that $\mathcal{H}_{\varrho_j}^{\otimes(n_k)} \otimes \mathcal{H}_{\varrho_j}^{\otimes(n_i)} = \mathcal{H}_{\varrho_j}^{\otimes(n_k+n_i)} = \mathcal{H}_{\varrho_j}^{\otimes(n_i)} \otimes \mathcal{H}_{\varrho_j}^{\otimes(n_k)}$ $(i, k \in I_j)$, the assumptions of Lemma 2.5 are fulfilled, therefore

$$\begin{aligned} \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{H}_{\gamma_i}^{\otimes(n_i)}, \theta_i)_{i \in K_j} \frown (\mathcal{C}_i, \theta_i)_{i \in I_j \setminus K_j}} &= \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{H}_{\varrho_j}^{\otimes(n_i)}, \theta_i)_{i \in I_j} \frown (\mathcal{C}_i, \theta_i)_{i \in J_{j+1}}} \\ &\le \prod_{i \in I_j} \frac{1}{\theta_i^{m_i - 1}} \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{H}_{\varrho_j}^{\otimes(n_i m_i)}, \theta_i^{m_i})_{i \in I_j} \frown (\mathcal{C}_i, \theta_i)_{i \in J_{j+1}}}. \end{aligned} \tag{2.14}$$

It is not difficult to see by the choice of m_i's, that the relations

$$\begin{cases} \mathcal{H}_{\varrho_j}^{\otimes(n_i m_i)} \subseteq \mathcal{C}_r \text{ while } \theta_i^{m_i} \le \theta_r (i \in I_j) & \text{if } j < s, \text{ or} \\ \mathcal{H}_{\varrho_s}^{\otimes(n_i m_i)} = \mathcal{H}_{\varrho_s}^{\otimes(n_i n_r)} \subseteq \mathcal{C}_r^{\otimes(n_i)} \text{ and } \theta_i^{m_i} = \theta_i^{n_r} \le \theta_r^{n_i} (i \in I_j) & \text{if } j = s. \end{cases}$$

are true. Hence, by Lemma 2.4 in the case of $j = s$, we obtain that

$$\Big\| \sum_n a_n e_n \Big\|_{(\mathcal{H}_{\varrho_j}^{\otimes(n_i m_i)}, \theta_i^{m_i})_{i \in I_j} \frown (\mathcal{C}_i, \theta_i)_{i \in J_{j+1}}} \le \Big\| \sum_n a_n e_n \Big\|_{(\mathcal{C}_i, \theta_i)_{i \in J_{j+1}}}. \tag{2.15}$$

It is clear now that (2.12) follows from equations (2.13), (2.14) and (2.15). ∎

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Repeated application of the previous claim gives that

$$\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{C}_i,\theta_i)_{i=1}^r} \le \prod_{i=1}^{r-1}\frac{1}{\theta_i^{m_i-1}}\prod_{i=1,\,\delta_i>1}^{r-1}\frac{2}{\theta_i}\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{C}_r,\theta_r)}. \tag{2.16}$$

It follows from (2.10), (2.11) and (2.16) that

$$\Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r} \le 2\Big(1+\max_{\substack{1\le i\le r\\ \iota(\mathcal{B}_i)\text{ infinite}}} k_i\Big)\prod_{i=1}^{r-1}\frac{1}{\theta_i^{m_i-1}}\prod_{i=1,\,\delta_i>1}^{r-1}\frac{2}{\theta_i}\Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{B}_r,\theta_r)}.$$

∎

In Theorem 2.13 we made the assumption that at least one family $\mathcal{B}_i$ has infinite index ($1 \le i \le r$). The conclusion of this theorem is also true for families, all of them with finite indices, but its proof uses different methods (see [8]).

3 Topological and Combinatorial Aspects of Families of Finite Sets of Integers

The main result of this section is that for every compact and hereditary family $\mathcal{F}$ there is a regular family $\mathcal{B}$ with the same index as $\mathcal{F}$ and an infinite set M of integers such that every $\mathcal{B}$-admissible sequence of subsets of M is also $\mathcal{F}$-admissible. The main tool we use is the notion of *homogeneous* family.

We start with the following list of useful properties. We leave their proofs to the reader.

Proposition 3.1 *Fix a compact family $\mathcal{F}$, and a countable ordinal α. Then*

(i) *For every $m \in \mathbb{N}$, $(\partial^{(\alpha)}\mathcal{F})\restriction\mathbb{N}/m = \partial^{(\alpha)}(\mathcal{F}\restriction(\mathbb{N}/m))$.*

(ii) *$\varnothing \ne s \in \partial^{(\alpha)}\mathcal{F}$ if and only if ${}_*s \in \partial^{(\alpha)}(\mathcal{F}_{\{\min s\}})$.*

(iii) *For every $n \in \mathbb{N}$, $\partial^{(\alpha)}(\mathcal{F}_{\{n\}}) = (\partial^{(\alpha)}\mathcal{F})_{\{n\}}$.*

(iv) *$\varnothing \in \partial^\alpha\mathcal{F}$ if and only if $\varnothing \in \partial^\alpha(\mathcal{F}\restriction\mathbb{N}/n)$ for every $n \in \mathbb{N}$.*

(v) *$\iota(\mathcal{F}) = \alpha + 1$ if and only if*
   - (a) *$\{\varnothing\} \subsetneq \partial^\alpha\mathcal{F}$ is finite, or*
   - (b) *there is an infinite set M such that for every $m \in M$, $\partial^\alpha(\mathcal{F}_{\{m\}})$ is non-empty and finite.*

(vi) *Suppose that $\alpha > 0$ is a limit ordinal. Then the following are equivalent:*
   - (a) *$\iota(\mathcal{F}) = \alpha$.*
   - (b) *$\iota(\mathcal{F}\restriction(\mathbb{N}/n)) = \alpha$ for every $n \in \mathbb{N}$.*
   - (c) *$\partial^\alpha\mathcal{F} = \{\varnothing\}$.*
   - (d) *$\partial^\alpha(\mathcal{F}_{\{n\}}) = \varnothing$ for every $n \in \mathbb{N}$, and for every $\beta < \alpha$ there is n with $\beta \le \iota(\mathcal{F}_{\{n\}}) < \alpha$.*

   In the following, $\mathcal{F}$ is, in addition, hereditary.

(vii) *If α is limit, then $\iota(\mathcal{F}) = \alpha$ if and only if $\partial^\alpha\mathcal{F}_{\{n\}} = \varnothing$ for every $n \in \mathbb{N}$, and there is an infinite set $M \subseteq \mathbb{N}$ and $(\alpha_n)_{n\in M} \uparrow \alpha$ such that $\alpha_m \le \iota(\mathcal{F}_{\{m\}}\restriction(\mathbb{N}/n)) < \alpha$ for every $m < n$ in M.*

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(viii) $\iota(\mathcal{F}) = \alpha + 1$ *if and only if* $\partial^{\alpha+1}\mathcal{F}_{\{n\}} = \varnothing$ *for every* $n \in \mathbb{N}$, *and there is an* $m < n$ *such that* $\partial^{\alpha}(\mathcal{F}_{\{m\}} \restriction (\mathbb{N}/k)) = \{\varnothing\}$ *for every* $k \geq n$.

3.1 Homogeneous Families and Admissible Sets

For our study it would be very useful to have a characterization of *every* compact hereditary family in terms of a class of families with good structural properties allowing inductive arguments, as for example the Schreier families. This is indeed the case for the class of homogeneous families. The following definition is modeled on the notion of α-uniform family introduced by Pudlak and Rödl [6].

Definition 3.2 We say that a family $\mathcal{F}$ is *α-homogeneous on M* (α a countable ordinal) if and only if $\varnothing \in \mathcal{F}$ and the following hold:

- if $\alpha = 0$, then $\mathcal{F} = \{\varnothing\}$;
- if $\alpha = \beta + 1$, then $\mathcal{F}_{\{n\}}$ is β-homogeneous on M/n for every $n \in M$;
- if $\alpha > 0$ is a limit ordinal, then there is an increasing sequence $\{\alpha_n\}_{n \in M}$ of ordinals converging to α such that $\mathcal{F}_{\{n\}}$ is α_n-homogeneous on M/n for all $n \in M$.

The family $\mathcal{F}$ is called *homogeneous on M* if it is α-homogeneous on M for some countable ordinal α.

Recall the following well known combinatorial notion [6]. A family $\mathcal{F}$ is *α-uniform on M* (α a countable ordinal) if and only if $\mathcal{F} = \{\varnothing\}$ for $\alpha = 0$ or $\varnothing \notin \mathcal{F}$ and $\mathcal{F}$ satisfies (b) or (c) in the other cases, where homogeneous is replaced by uniform. Some of the similarities of uniform and homogeneous families will be exposed in Proposition 3.6 below.

Notation If $s, t \in \mathrm{FIN}$, we write $s \sqsubseteq t$ if and only if s is an initial segment of t.

Remark 3.3. (i) It is easy to see that the only n-homogeneous families on M are the families of subsets of M with cardinality $\leq n$, denoted by $[M]^{\leq n}$. A well-known ω-homogeneous family on $\mathbb{N}$ is the Schreier family, and, in general, ω-homogeneous families on M are of the form $\{s \subseteq M : \#s \leq f(\min s)\}$, with $f\colon M \to \mathbb{N}$ an unbounded and increasing mapping. Observe that all those examples are regular families.

(ii) In the same way, the only n-uniform families on M, $n \in \mathbb{N}$ are the families $[M]^n$ of subsets of M of size *exactly* equal to n. While the ω-uniform families on some M are the ones of the form $\{s \subseteq M : \#s = f(\min s)\}$, with $f\colon M \to \mathbb{N}$ an unbounded and increasing mapping. Observe that in these two cases the maximal nodes under the relation $\sqsubseteq$ of the α-homogeneous family considered coincide with the corresponding α-uniform family. And conversely, the closure under $\sqsubseteq$ of the α-uniform family is the corresponding α-homogeneous family. This is a general phenomenon that we will expose in Proposition 3.6.

(iii) In general, an arbitrary homogeneous family does not need to be regular. However, we will show that homogeneous families are always $\sqsubseteq$-closed, hence compact. Also, it can be shown that if $\mathcal{F}$ is a homogeneous family on M, there is $N \subseteq M$ such that $\mathcal{F} \restriction N$ is hereditary (see [6]).

Homogeneous and regular families have many properties in common. One of the most remarkable is the fact that the index of these families never decreases when taking restrictions. We expose this analogy and some others in the next proposition.

Proposition 3.4 *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are homogeneous (regular) families on M.*

(i) *If $\iota(\mathcal{F})$ is finite, then $\mathcal{F} = [M]^{\le \iota(\mathcal{F})}$ if $\mathcal{F}$ is homogeneous on M, while $\mathcal{F}\restriction(M/n) = [M/n]^{\le\iota(\mathcal{F})}$ for some $n \in M$ if $\mathcal{F}$ is regular on M.*
(ii) *If $\iota(\mathcal{F}) \neq 0$, then $\mathcal{F}_{\{n\}}$ is homogeneous (regular) on M/n for every $n \in M$.*
(iii) *If $\mathcal{F}$ is α-homogeneous, then $\partial^{(\alpha)}\mathcal{F} = \{\varnothing\}$. Hence $\iota(\mathcal{F}) = \alpha$.*
(iv) *if $\mathcal{F}$ is α-homogeneous (regular) on M and $N \subseteq M$, then $\mathcal{F}\restriction N$ is α-homogeneous (regular) and $\iota(\mathcal{F}\restriction N) = \iota(\mathcal{F})$ for every $N \subseteq M$.*
(v) *$\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$ are homogeneous (regular), $\iota(\mathcal{F} \oplus \mathcal{G}) = \iota(\mathcal{F}) + \iota(\mathcal{G})$ and $\iota(\mathcal{F} \otimes \mathcal{G}) = \iota(\mathcal{F})\iota(\mathcal{G})$.*
(vi) *If $\iota(\mathcal{F}) < \iota(\mathcal{G})$, then for every M there is $N \subseteq M$ such that $\mathcal{F}\restriction N \subsetneq \mathcal{G}\restriction N$.*

Proof Suppose first that $\mathcal{F}$ is homogeneous. (i) and (ii) can be shown by an easy inductive argument.

(iii) Suppose first that $\alpha = \beta + 1$. By the inductive hypothesis, for every $n \in M$ we have that $(\partial^{(\beta)}(\mathcal{F}))_{\{n\}} = \partial^{(\beta)}(\mathcal{F}_{\{n\}}) = \{\varnothing\}$. So, $[M]^{\le 1} = \partial^{(\beta)}(\mathcal{F})$ (since $\partial^{(\beta)}(\mathcal{F})$ is closed and it contains all singletons $\{n\}$ ($n \in M$)). Hence $\partial^{(\beta+1)}(\mathcal{F}) = \{\varnothing\}$. Suppose now that α is a limit ordinal. Now by the inductive hypothesis we can conclude that for every $n \in M$,

$$\partial^{(\alpha_n)}(\mathcal{F}_{\{n\}}) = (\partial^{(\alpha_n)}(\mathcal{F}))_{\{n\}} = \{\varnothing\}, \tag{3.1}$$

where $\alpha_n = \iota(\mathcal{F}_{\{n\}})$ is such that $(\alpha_n)_n$ is increasing and with limit α. By (3.1), $\varnothing \in \partial^{(\alpha)}\mathcal{F}$. If there were some $s \in \partial^{(\alpha)}\mathcal{F}$, $s \neq \varnothing$, then $s \in \partial^{(\alpha_n+1)}\mathcal{F}$ for every n, and hence $\partial^{(\alpha_n)}(\mathcal{F}_{\{\min s\}}) \neq \{\varnothing\}$, a contradiction.

(iv) This follows easily by induction on α using (i).

(v) This is shown by induction on $\iota(\mathcal{G})$.

(vi) By Proposition 3.8, there is some $N \subseteq M$ such that either $\mathcal{F}\restriction N \subseteq \mathcal{G}\restriction N$ or else $\mathcal{G}\restriction N \subseteq \mathcal{F}\restriction N$. The second alternative is impossible since it implies that $\iota(\mathcal{F}) = \iota(\mathcal{F}\restriction N) \ge \iota(\mathcal{G}\restriction N) = \iota(\mathcal{G})$.

Finally suppose that we are dealing with regular families.

(i) First, note that there must be some $s \in \mathcal{F}$ with $|s| = \iota(\mathcal{F})$, since otherwise, $\mathcal{F} \subseteq [M]^{<\iota(\mathcal{F})}$ and so, $\iota(\mathcal{F}) < \iota(\mathcal{F})$ is absurd. In a similar way one shows that $\mathcal{F} \subseteq [M]^{\le\iota(\mathcal{F})}$. All of this shows that $\mathcal{F}\restriction(M/s) = [M/s]^{\le\iota(\mathcal{F})}$.

(ii) This is clear.

(iv) Fix $N \subseteq M$, and let $\Theta\colon M \to N$ be the unique order-preserving onto mapping between these two sets. Since $\mathcal{F}$ is spreading on M, we obtain that $\{\Theta''s : s \in \mathcal{F}\} \subseteq \mathcal{F}\restriction N$. Using that $\Theta''s \neq \Theta''t$ is $s \neq t$ we obtain that $\iota(\mathcal{F}\restriction N) \ge \iota(\mathcal{F}) \ge i(\mathcal{F}\restriction N)$, as desired.

(vi) This follows from (iv), while (v) for regular families is a consequence of Theorem 3.5 and (v) for homogeneous families. ■

The following result is a weaker form of [6, Theorem II.3.22].

Theorem 3.5 (**[6]**) *Suppose that $\mathcal{F}$ is a non-empty compact and hereditary family. Then for every infinite subset M of $\mathbb{N}$ there is some infinite $N \subseteq M$ such that $\mathcal{F}\restriction N$ is homogeneous on N.*

Proposition 3.6 *Suppose that $\mathcal{F}$ is a family of finite sets of integers. Then for every countable ordinal α the following conditions are equivalent:*

(i) *$\mathcal{F}$ is α-homogeneous on M.*
(ii) *$\mathcal{F}$ is the topological closure of an α-uniform family on M.*
(iii) *$\mathcal{F}$ is compact and the set $\mathcal{F}^{\sqsubseteq-\max}$ of $\sqsubseteq$-maximal elements of $\mathcal{F}$ is α-uniform on M. Moreover*

$$\mathcal{F} = \{s \sqsubseteq t : t \in \mathcal{F}^{\sqsubseteq-\max}\}, \tag{3.2}$$

hence $\mathcal{F}$ is $\sqsubseteq$-hereditary, i.e., *if $s \sqsubseteq t \in \mathcal{F}$, then $s \in \mathcal{F}$.*

Proof (i) implies (ii). The proof is by induction on α. If $\alpha = 0$, the result is clear. Suppose that $\alpha = \beta + 1$. Then for every $n \in M$, $\mathcal{F}_{\{n\}}$ is β-homogeneous on M/n. Choose β-uniform families $\mathcal{G}_n$ on M/n ($n \in M$) such that for every $n \in M$, $\mathcal{F}_{\{n\}} = \overline{\mathcal{G}_n}$. Set $\mathcal{G} = \{\{n\} \cup s : s \in \mathcal{G}_n\}$. It follows readily that $\mathcal{G}_{\{n\}} = \mathcal{G}_n$ which yields that $\mathcal{G}$ is an α-uniform family on M. To finish the proof we show that $\overline{\mathcal{G}} = \mathcal{F}$. First observe that if $s \in \mathcal{F}$, $n = \min s$, then ${}_*s \in \mathcal{F}_{\{n\}}$. So, ${}_*s \in \overline{\mathcal{G}_n}$, and hence

$$s = \{n\} \cup {}_*s \in \overline{\mathcal{G}_n \oplus \{\{n\}\}} \subseteq \overline{\mathcal{G}}.$$

Now suppose that $(s_k) \subseteq \mathcal{G}$, $s_k \to_k s \in \overline{\mathcal{G}}$. Going to a subsequence if necessary, we may assume that (s_k) is a Δ-sequence with root s, *i.e.,* $s \sqsubseteq s_k$ for every k, and $(s_k \setminus s)$ is a block sequence. If $s = \varnothing$, then $s \in \mathcal{F}$ by hypothesis. Otherwise, let $n = \min s$. Then $\min s_k = n$ for every k, and hence ${}_*s_k \in \mathcal{G}_{\{n\}}$. Hence ${}_*s \in \overline{\mathcal{G}_{\{n\}}} = \mathcal{F}_{\{n\}}$, and so $s \in \mathcal{F}$. The proof if α is limit is similar.

(ii) implies (iii). Suppose that $\mathcal{F} = \overline{\mathcal{G}}$, where $\mathcal{G}$ is α-uniform on M. It is not difficult to show by induction on α that $\mathcal{G}$ is a *front on M* (see [6]), *i.e.,* for every infinite $N \subseteq M$ there is some $s \in \mathcal{G}$ such that $s \sqsubseteq N$, and if $s, t \in \mathcal{G}$ and $s \sqsubseteq t$, then $s = t$. Observe that the topological closure of a front is its $\sqsubseteq$-downwards closure. Indeed, suppose that s is a strict initial part of some $t \in \mathcal{G}$. For every $m > s$ consider the set $M_m = s \cup M/m$. Using that $\mathcal{G}$ is a front on M, we find $t_m \sqsubseteq M_m$ such that $t_m \in \mathcal{G}$, moreover s has to be initial segment of every t_m. This implies that t_m converges to s.

So, we have that $\mathcal{F} = \{s \sqsubseteq t : t \in \mathcal{G}\}$. It is clear that this implies that $\mathcal{F}^{\sqsubseteq-\max} = \mathcal{G}$.

(iii) implies (i). Suppose that $\mathcal{F}$ is compact and $\mathcal{F}^{\sqsubseteq-\max}$ is α-uniform on M. The proof is an easy induction on α using that for every $m \in M$, by (3.2), $\mathcal{F}_{\{m\}} = \{s \sqsubseteq t : t \in \mathcal{G}_{\{m\}}\}$, where $\mathcal{G} = \mathcal{F}^{\sqsubseteq-\max}$. ∎

The next result is the well-known *Ramsey* property of uniform families (see [6] for a more complete explanation of the Ramsey property).

Proposition 3.7 (**Ramsey Property**) *Suppose that $\mathcal{B}$ is a uniform family on M, and suppose that $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$. Then there is an infinite $N \subseteq M$ and $i = 0, 1$ such that $\mathcal{B}\restriction N = \mathcal{B}_i\restriction N$.*

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Proof We use induction on α. Given $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, using inductive hypothesis we can find a decreasing sequence $(M_k)_k$ of infinite subsets of M, such that setting $m_k = \min M_k$ for every k, $M_{k+1} \subseteq {}_* M_k$ and there is an $i_k \in \{0, 1\}$ such that $\mathcal{B}_{\{m_k\}} \restriction M_{k+1} = (\mathcal{B}_{i_k})_{\{m_k\}} \restriction M_{k+1}$. Then every $N \subseteq \{m_k\}_k$ for which i_k is constant has the desired property. ■

As an application of this Ramsey property we obtain the following two facts.

Proposition 3.8 (i) *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two compact and hereditary families. Then there is some infinite set M such that either $\mathcal{F}\restriction M \subseteq \mathcal{G}\restriction M$ or $\mathcal{G}\restriction M \subseteq \mathcal{F}\restriction M$* (see [6, 12]).

(ii) *Suppose that $\mathcal{F}$ is homogeneous on M. Then there is some $N \subseteq M$ such that $\mathcal{F}\restriction N$ is hereditary.*

Proof (ii) Set $\mathcal{B} = \mathcal{F}^{\sqsubseteq-\max}$, and let $\mathcal{B}_0 = \{s \in \mathcal{B} : \mathcal{P}(s) \not\subseteq \mathcal{F}\}$, $\mathcal{B}_1 = \mathcal{B} \setminus \mathcal{B}_0$. By Ramsey, there is $N \subseteq M$ and $i = 0, 1$ such that $\mathcal{B}\restriction N = \mathcal{B}_i\restriction N$. If $i = 1$, then we are done. Otherwise, fix $s \in \mathcal{B}\restriction N$ and $t \subseteq s$ such that $t \notin \mathcal{F}$. Using that $\mathcal{F}^{\sqsubseteq-\max}$ is a front on M, we get $u \in \mathcal{B}\restriction N$ such that $u \sqsubseteq t \cup (N/s)$. If $t \sqsubseteq u$ then $t \in \mathcal{F}$, which is impossible. So, $u \sqsubset t \subsetneq s$. This means that for every $s \in \mathcal{B}\restriction N$ there is some $t \subsetneq s$, $t \in \mathcal{B}\restriction N$. Hence $\varnothing \in \mathcal{B}\restriction N$, which implies that $\mathcal{B}\restriction N = \{\varnothing\}$ and so $\mathcal{F}\restriction N = \{\varnothing\}$ is hereditary. ■

3.2 The Basic Combinatorial Results

The families $\mathcal{F}$ and $\mathrm{Ad}(\mathcal{F})$ are in general different, unless $\mathcal{F}$ is spreading. Nevertheless, as is shown in the next result, they are not so far from the topological point of view.

Proposition 3.9 *Suppose that $\mathcal{F}$ is a compact hereditary family. Then for every infinite set M of integers such that $\mathrm{Ad}(\mathcal{F})\restriction M$ is homogeneous on M,*

$$\iota(\mathcal{F}) \le \iota(\mathrm{Ad}(\mathcal{F})\restriction M) \le 2\iota(\mathcal{F}).$$

Proof The proof is done by induction on $\iota(\mathcal{F}) = \lambda + r$, λ limit ordinal (including $\lambda = 0$), and $r \in \mathbb{N}$. Set $\mathcal{B} = \mathrm{Ad}(\mathcal{F})\restriction M$. Suppose that $r = 0$. So $\iota(\mathcal{F}) = \lambda$ is limit. If $\lambda = 0$, there is noting to prove. We suppose then that $\lambda > 0$. By Proposition 3.1(vii), we can find an infinite set $N \subseteq \mathbb{N}$ and a sequence of ordinals $\lambda_n \uparrow_{n\in N} \lambda$ such that

$$\lambda_n \le \iota(\mathcal{F}_{\{n\}}\restriction \mathbb{N}/k) < \lambda$$

for every $n < k$ in N. For a given n, we fix $m = m_n \in M/n$, $k \in N/m$ and $P_n \subseteq M/k$ such that $(\mathrm{Ad}(\mathcal{F}_{\{n\}}\restriction(\mathbb{N}/k)))\restriction P_n$ and $(\mathrm{Ad}(\mathcal{F}_{\{l\}}))\restriction P_n$ are homogeneous on P_n for every $l \le m$. By inductive hypothesis,

$$\iota(\mathrm{Ad}(\mathcal{F}_{\{n\}}\restriction(\mathbb{N}/k))\restriction P_n) \ge \iota(\mathcal{F}_n\restriction(\mathbb{N}/k)) \ge \lambda_n.$$

Now using that

$$\mathcal{B}_{\{m\}} = (\mathrm{Ad}(\mathcal{F})\restriction M)_{\{m\}} \supseteq \mathrm{Ad}(\mathcal{F}_{\{n\}}\restriction(\mathbb{N}/k))\restriction P_n,$$
$$\mathcal{B}_{\{m\}}\restriction P_n \subseteq \bigcup_{l\le m} (\mathrm{Ad}(\mathcal{F}_{\{l\}}))\restriction P_n,$$

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we obtain,

$$\lambda_n \le \iota(\mathcal{B}_{\{m\}}\restriction P_n) \le \iota\Big(\bigcup_{l\le m}(\mathrm{Ad}(\mathcal{F}_{\{l\}}))\restriction P_n\Big) = \max_{l\le m}\iota((\mathrm{Ad}(\mathcal{F}_{\{l\}}))\restriction P_n)$$
$$\le 2\max_{l\le m}\iota(\mathcal{F}_{\{l\}}) < \lambda,$$

the last inequality holding because λ is a limit ordinal. Since $\mathcal{B}_{\{m\}}$ is homogeneous on M/m, and n was arbitrary, it readily follows, by the definition of homogeneous families, that $\iota(\mathcal{B}) = \lambda$, as desired.

Suppose now that $\iota(\mathcal{F}) = \lambda + r + 1$. We now use Proposition 3.1(viii) to find two integers $n < p$ such that $\iota(\mathcal{F}_{\{n\}}\restriction(\mathbb{N}/q)) = \lambda + r$ for every $q \ge p$. A similar argument to the one for $\iota(\mathcal{F}) = \lambda$ shows that for infinitely many $m \in M$ we obtain that $\iota(\mathcal{B}_{\{m\}}) \ge \lambda + r$, and this implies that $\iota(\mathcal{B}) \ge \lambda + r + 1$, as desired.

Now we work to show the other inequality $\iota(\mathcal{B}) \le 2\iota(\mathcal{F}) = \lambda + 2r + 2$. We proceed by contradiction assuming that $\iota(\mathcal{B}) \ge \lambda + 2r + 3$. By Proposition 3.4(iv) we may assume that $\iota(\mathcal{B}_{\{m\}}) \ge \lambda + 2r + 2$ for every $m \in M$. Let $\mathcal{G} = \mathcal{B}^{\sqsubseteq-\max}$. Fix $m_0 \in M$, and define the finite coloring $\Theta\colon \mathcal{G}_{\{m_0\}} \to \{0,\dots,m_0\}$ by $\Theta(s) = k$, if and only if there is some $t \in \mathcal{F}_{\{k\}}$ such that $\{k\}\cup t$ interpolates $\{m_0\}\cup s$. By the Ramsey property of $\mathcal{G}_{\{m_0\}}$, we may assume, going to a subset if necessary, that Θ is constant with value $k_0 \in \{0,\dots,m_0\}$. Suppose first that $\iota(\mathcal{F}_{\{k_0\}}) \le \lambda + r$; then by inductive hypothesis, $\iota(\mathcal{B}_{\{m_0\}}) \le \lambda + 2r$, a contradiction with our assumption. So, $\iota(\mathcal{F}_{\{k_0\}}) = \lambda + r + 1$. Moreover, $\partial^{(\lambda+r+1)}(\mathcal{F}_{\{k_0\}}) = \varnothing$. Since $\mathcal{F}$ is hereditary, this means that $\{\varnothing\} \subsetneq \partial^{(\lambda+r)}\mathcal{F}_{\{k_0\}}$ is finite. Let $l = \max(\partial^{(\lambda+r)}(\mathcal{F}_{\{k_0\}}))$. Observe that by Proposition 3.1(v),

$$\partial^{(\lambda+r)}(\mathcal{F}_{\{k_0\}}\restriction(\mathbb{N}/p)) = \{\varnothing\} \quad \text{for every } p \ge l. \tag{3.3}$$

By the Ramsey property of $\mathcal{G}_{\{m_0\}}$ we may assume that either for every $s \in \mathcal{G}_{\{m_0\}}$ there is some $t \in \mathcal{F}_{\{k_0\}}\restriction(\mathbb{N}/l)$ that interpolates s, or else there is some $k_1 \in \{m_0 + 1,\dots,l\}$ such that for every $s \in \mathcal{G}_{\{m_0\}}$ there is some $t \in \mathcal{F}_{\{k_0,k_1\}}$ such that $\{k_0,k_1\}\cup t$ interpolates $\{m_0\}\cup s$. In the first case, $\mathcal{B}_{\{m_0\}} \subseteq \mathrm{Ad}(\mathcal{F}_{\{k_0\}}\restriction(\mathbb{N}/l))$; since from (3.3) we know that $\iota(\mathcal{F}_{\{k_0\}}\restriction(\mathbb{N}/l)) = \lambda + r$, we arrive, by inductive hypothesis, to the contradiction $\iota(\mathcal{B}_{\{m_0\}}) \le \lambda + 2r$.

In the second case, consider $m_1 \in M/l$ such that the homogeneous family $\mathcal{B}_{\{m_0,m_1\}}$ has index at least $\lambda + 2r + 1$. Note that $\mathcal{B}_{\{m_0,m_1\}} \subseteq \mathrm{Ad}(\mathcal{F}_{\{k_0,k_1\}}\restriction(\mathbb{N}/l))$. Since from (3.3) we know that $\iota(\mathcal{F}_{\{k_0,k_1\}}\restriction(\mathbb{N}/m_1)) \le \lambda + r$, we obtain by inductive hypothesis that $\iota(\mathcal{B}_{\{m_0,m_1\}}) \le \lambda + 2r$, a contradiction. ■

Remark 3.10. The previous result is the best possible. For every limit ordinal λ and $n \in \mathbb{N}$ there is some compact hereditary family $\mathcal{F}$ such that $\iota(\mathcal{F}) = \lambda + k$ and $\iota(\mathrm{Ad}(\mathcal{F})\restriction M) = \lambda + 2k$ for every M, hence $\iota(\mathrm{Ad}(\mathcal{F})\restriction M) = 2\iota(\mathcal{F})$. The families are closely related to [8, Example 3.10]. Consider a regular family $\mathcal{B}$ on $\{2n\}$ of index $\iota(\mathcal{B}) = \lambda + k$, and let $\mathcal{F}$ be the downwards closure of $\mathcal{F} = \{s \cup \{n+1\}_{n\in s} : s \in \mathcal{B}\}$. It is not difficult to prove that $\iota(\mathcal{F}) = \iota(\mathcal{B}) = \lambda + k$ and that $\iota((\mathrm{Ad}(\mathcal{F}))\restriction M) = \lambda + 2k$ for every M.

The next result tells us that we may assume that the given family $\mathcal{F}$ is indeed spreading.

Proposition 3.11 *Fix an arbitrary compact hereditary family $\mathcal{F}$, and an infinite set M.*

(i) *There is some regular family $\mathcal{B}$ with the same index as $\mathcal{F}$ and some $N \subseteq M$ such that every $\mathcal{B}$-admissible sequence of subsets of N is also $\mathcal{F}$-admissible.*

(ii) *For every regular family $\mathcal{B}$ on M with $\iota(\mathcal{B}) > \iota(\mathcal{F})$ there is some $N \subseteq M$ such that no sequence (s_i) of subsets of N with $\{\min s_i\} \in (\mathcal{B}{\upharpoonright}N)^{\sqsubseteq - \max}$ is $\mathcal{F}$-admissible.*

Proof Fix M and $\mathcal{F}$. Consider the unique decomposition $\iota(\mathcal{F}) = \lambda+r$ with $\lambda = \lambda(\mathcal{F})$ a limit ordinal (including 0) and $r = r(\mathcal{F})$ an integer. Let $\mathcal{C}$ be an arbitrary regular and homogeneous family on M with $\iota(\mathcal{C}) \geq \iota(\mathcal{F})$. Now let $\mathcal{G} = \mathcal{G}(\mathcal{C}) = \mathcal{C}^{\sqsubseteq - \max}$. It follows, by Proposition 3.6, that $\mathcal{G}$ is a uniform family on M, as well as $[M]^2 \otimes \mathcal{G}$. Observe that every $s \in [M]^2 \otimes \mathcal{G}$ has a unique "canonical" decomposition $s = s[0] \cup \cdots \cup s[l(s)]$ with $s[0] < \cdots < s[l(s)]$, $\#s[i] = 2$, and $\{\min s[i]\}_{i=0}^{l(s)} \in \mathcal{G}$. Consider the following coloring $h_{\mathcal{F},\mathcal{C}} \colon [M]^2 \otimes \mathcal{G} \to \{0, \dots, r, \infty\}$ defined for $s \in [M]^2 \otimes \mathcal{G}$ by $h_{\mathcal{F},\mathcal{C}}(s) = k \in \{0, \dots r\}$ if and only if $l(s) \geq r-1$ and k is minimal with the property that $(s[k], s[k+1], \dots, s[r-1], s[r+1], \dots, s[l])$ is $\mathcal{F}$-admissible, and $h_{\mathcal{F},\mathcal{C}}(s) = \infty$ otherwise.

Claim *The following are equivalent.*

(i) *There is an infinite $N \subseteq M$ such that $h_{\mathcal{F},\mathcal{C}} \upharpoonright ([N]^2 \otimes \mathcal{G}{\upharpoonright}N)$ is constant with value 0.*

(ii) $\iota(\mathcal{F}) = \iota(\mathcal{C})$.

Proof of Claim The proof is by induction on $\iota(\mathcal{F})$. Suppose first that (i) holds but $\iota(\mathcal{C}) > \iota(\mathcal{F})$. Fix $N \subseteq M$ such that $h_{\mathcal{F},\mathcal{C}}{\upharpoonright}([N]^2 \otimes \mathcal{G}{\upharpoonright}N)$ is constant with value 0. By Proposition 3.4, we may assume, going to an infinite subset if needed, that for every $n \in N$, $\iota(\mathcal{C}_{\{n\}}) \geq \iota(\mathcal{F})$. Fix $n \in N$ and consider the new coloring

$$d \colon ([N]^2 \otimes (\mathcal{G}_{\{n\}}){\upharpoonright}N) \oplus ([N/n]^1) \to \{0, \dots, n\},$$

defined for $s = \{k\} \cup s[1] \cup s[2] \cup \cdots \cup s[l] \in ([N]^2 \otimes (\mathcal{G}_{\{n\}}){\upharpoonright}N) \oplus ([N/n]^1)$ in its canonical form by $d(s) = j$ if and only if there is some $t \in \mathcal{F}$ such that $\min t = j$ and t interpolates, $(\{n,k\}, s[1], \dots, s[r-1], s[r+1], \dots, s[l])$. Observe that d is well defined since we are assuming that $h_{\mathcal{F},\mathcal{G}}{\upharpoonright}[N]^2 \otimes \mathcal{G}{\upharpoonright}N$ is constant with value 0. By the Ramsey property of the uniform family considered as domain of d there is some infinite set $P \subseteq N/n$ such that d is constant on $([P]^2 \otimes (\mathcal{G}_{\{n\}}){\upharpoonright}P) \oplus ([P/n]^1)$ with value $j_0 \in \{0, \dots, n\}$. Pick some $p \in P$ such that $\iota(\mathcal{F}_{\{j_0\}}{\upharpoonright}(\mathbb{N}/p)) < \iota(\mathcal{F})$ (See Proposition 3.1). Then $h_{\mathcal{F}_{\{j_0\}}{\upharpoonright}(\mathbb{N}/p), \mathcal{C}_{\{n\}}}$ is constant when restricted to $[P]^2 \otimes (\mathcal{G}_{\{n\}}){\upharpoonright}P$ with value 0. Observe that $\mathcal{G}_{\{n\}} = \mathcal{G}(\mathcal{C}_{\{n\}})$, so, by inductive hypothesis, $\iota(\mathcal{C}_{\{n\}}) = \iota(\mathcal{F}_{\{j_0\}}{\upharpoonright}(\mathbb{N}/p))) < \iota(\mathcal{F})$, a contradiction.

Now suppose (ii) holds, *i.e.,* $\iota(\mathcal{C}) = \iota(\mathcal{F}) = \lambda + r$. The coloring $h_{\mathcal{F},\mathcal{C}}$ is finite, so we fix $N \subseteq M$ such that $h_{\mathcal{F},\mathcal{C}}$ is constant with value k_0, when restricted to $[N]^2 \otimes \mathcal{G}{\upharpoonright}N$, and also that $\mathrm{Ad}(\mathcal{F}){\upharpoonright}N$ is homogeneous on N. Our intention is to show that $k_0 = 0$. Set $\mathcal{C}_0 = \mathcal{C}$ and for every $1 \leq i \leq r$, $\mathcal{C}_i = {}_*\mathcal{C}_{i-1} = \{{}_*s : s \in \mathcal{C}_{i-1}\}$. Since $\mathcal{C}$ is regular, it follows easily that $\mathcal{C}_r$ is regular with index λ. Consider the regular

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family $\mathcal{D} = [N]^{\leq 2} \otimes (\mathcal{C}_r \upharpoonright N)$ on N with index $\iota(\mathcal{D}) = \lambda$. By (ii) and Proposition 3.9 $\iota(\mathrm{Ad}(\mathcal{F})\upharpoonright N) \geq \lambda$, so there is some P such that $\mathcal{D}\upharpoonright P \subseteq \mathrm{Ad}(\mathcal{F}) \oplus [P]^{\leq 1}$. It readily follows that ${}_*(\mathcal{D}\upharpoonright P) \subseteq \mathrm{Ad}(\mathcal{F})$. This shows that $k_0 \in \{0, \dots, r\}$. If $r = 0$, then we are done. Now suppose that $r > 0$. Let $q \in \mathbb{N}$ and $n_0, n_1 \in N$ be such that $q < n_0 < n_1$ and that $\iota(\mathcal{F}_{\{q\}}\upharpoonright(\mathbb{N}/n_1)) = \lambda + r - 1$ (see Proposition 3.1). Since $\mathcal{C}$ is regular, we have that $\iota(\mathcal{C}_{\{n_0\}}) = \lambda + r - 1 = \iota(\mathcal{F}_{\{q\}}\upharpoonright(\mathbb{N}/n_1))$. So by inductive hypothesis there is some $P \subseteq N/n_1$ such that $h_{\mathcal{F}_{\{q\}}\upharpoonright(\mathbb{N}/n_1), \mathcal{C}_{\{n_0\}}}$ is constant with value 0 when restricted to $[P]^2 \otimes \mathcal{G}_{\{n_0\}}\upharpoonright P$. Take arbitrary $s \in [P]^2 \otimes (\mathcal{G}_{\{n_0\}})\upharpoonright P$. Then $(s[0], \dots, s[r-2], s[r], \dots, s[l])$ is $\mathcal{F}_{\{q\}}\upharpoonright(\mathbb{N}/n_1)$-admissible, so

$$(\{n_0, n_1\}, s[0], \dots, s[r-2], s[r], \dots, s[l])$$

is $\mathcal{F}$-admissible. Since $\{n_0, n_1\} \cup s \in [N]^2 \otimes \mathcal{G}\upharpoonright N$ we obtain that $k_0 = 0$, as desired. ■

We work now to show (i). Suppose that $\iota(\mathcal{F}) = \lambda + r$, λ limit and $r \in \mathbb{N}$. Fix a regular and homogeneous family $\mathcal{D}$ with index λ, and let $\mathcal{C} = \mathcal{D} \oplus [\mathbb{N}]^{\leq r}$. This is a regular and homogeneous family whose index is $\lambda + r$. Since $h_{\mathcal{F},\mathcal{C}}$ is a finite coloring, and since $\iota(\mathcal{C}) = \iota(\mathcal{F})$, by the Claim, we can find $P \subseteq M$ such that

(iii) $h_{\mathcal{F},\mathcal{C}}$ is constant on $[P]^2 \otimes \mathcal{G}\upharpoonright P$ with value 0, where $\mathcal{G}$ is the set of $\sqsubseteq$-maximal nodes of $\mathcal{C}$, and

(iv) $[P]^{\leq 2} \otimes \mathcal{C}\upharpoonright P$ is homogeneous on P, and hence

$$[P]^{\leq 2} \otimes \mathcal{C}\upharpoonright P = \{s \sqsubseteq t : t \in [P]^2 \otimes \mathcal{G}\upharpoonright P\}$$

(see Proposition 3.6 (iii)).

Let $\mathcal{E} = {}_*(\mathcal{D}\upharpoonright N) = \{{}_*s : s \in \mathcal{D}\upharpoonright N\}$, where $N = \{p_{4k}\}_k$ and $\{p_k\}_k$ is the increasing enumeration of P. It is not difficult to see that $\mathcal{E}$ is a regular family whose index is λ, so we leave the details to the reader.

Let $\mathcal{B}$ be an arbitrary regular family such that $\mathcal{B}\upharpoonright N = \mathcal{E} \oplus [N]^{\leq r}$ (see Proposition 1.7). We claim that $\mathcal{B}$ and N fulfill the conditions required in (i). First, by the permanence property of the index of regular families, $\iota(\mathcal{B}) = \iota(\mathcal{F})$. Next, suppose that $(s_i)_{i=0}^k$ is a $\mathcal{B}$-admissible sequence of subsets of N. Since $\mathcal{B}$ is regular, we have that $\{\min s_i\}_{i=0}^k \in \mathcal{B}\upharpoonright N = \mathcal{E} \oplus [N]^{\leq r}$. For every $i < k$ let $n_i = \min((P \setminus N) \cap (\max s_i, \min s_{i+1}))$, and let $n_k > s_k$, $n_k \in P$. Observe that if $k \geq r$, then $u = \{\min s_i\}_{r \leq i \leq k} \in \mathcal{E} = {}_*(\mathcal{D}\upharpoonright N)$, so there is $q < u$ in N such that $\{q\} \cup u \in \mathcal{D}\upharpoonright N$. Since $\mathcal{D}$ is spreading, and by the choice of N out of P, we may assume that there is some $q' \in P \setminus N$ such that $s_{r-1} < n_{r-1} < q < q' < s_r$. Then

$$t = \{q, q'\} \cup \{\min s_i\}_{i=0}^k \cup \{n_i\}_{i=0}^k \in [P]^{\leq 2} \otimes \mathcal{C}\upharpoonright P.$$

By (iv) there is some $v \in [P]^2 \otimes \mathcal{G}\upharpoonright N$ such that $t \sqsubseteq v$. Let $v = v[0] \cup \cdots \cup v[l]$ be the canonical decomposition of v as element of $[P]^2 \otimes \mathcal{G}\upharpoonright N$. By construction we obtain that for every $i \leq \min\{r-1, k\}$, $\{\min s_i, n_i\} = v[i]$, while (if defined) $v[r] = \{q, q'\}$ and $\{\min s_i, n_i\} = v[i]$ for every $i > r$. By (iii) and as $\mathcal{F}$ is hereditary, $(\{\min s_i, n_i\})_{i=0}^k$ is $\mathcal{F}$-admissible. Since for every $i \leq k$, $\min s_i \leq \max s_i < n_i$, we obtain that our sequence $(s_i)_{i=0}^k$ is also $\mathcal{F}$-admissible, as desired.

Finally, we proceed to prove (ii): Suppose that $\mathcal{B}$ is an arbitrary regular family on M with $\iota(\mathcal{B}) \geq \iota(\mathcal{F})$. Find $M' \subseteq M$ such that $\mathcal{B}\restriction M'$ is in addition homogeneous on M', and let $N \subseteq M'$ be such that $h_{\mathcal{F},\mathcal{B}\restriction M'}$ is constant when restricted to $[N]^2 \otimes \mathcal{G}\restriction N$ with value k_0, where $\mathcal{G} = (\mathcal{B}\restriction M')^{\sqsubseteq-\max}$. Suppose that $(s_i)_i$ is a sequence of subsets of N, $\# s_i \geq 2$, with $\{\min s_i\} \in (\mathcal{B}\restriction N)^{\sqsubseteq-\max}$. Then $t = (\{\min s_i, \max s_i\})_i \in [N]^2 \otimes \mathcal{B}^{\sqsubseteq-\max}\restriction N$. As $\mathcal{B}$ is spreading on M it follows that $\mathcal{B}^{\sqsubseteq-\max}\restriction N = \mathcal{G}\restriction N$. Now suppose that indeed $(s_i)_i$ is $\mathcal{F}$-admissible. Then t is also $\mathcal{F}$-admissible, so, since $\mathcal{F}$ is hereditary, $h_{\mathcal{F},\mathcal{B}\restriction M'}(t) = 0$, hence $k_0 = 0$. By the claim applied to $\mathcal{F}$, $\mathcal{B}\restriction M'$ and M' we obtain that $\iota(\mathcal{B}) = \iota(\mathcal{B}\restriction M') = \iota(\mathcal{F})$, as desired. ■

4 Block Sequences of $T[((\mathcal{F}_i, \theta_i)_{i=1}^r)]$

In this last section we show that for a given finite sequence $(\mathcal{F}_i, \theta_i)_{i\in I}$ with at least one $\mathcal{F}_i$ having infinite index, there is $i_0 \in I$ such that every normalized block sequence in the space $T[(\mathcal{F}_i, \theta_i)_{i\in I}]$ has a subsequence equivalent to a subsequence of the basis of the space $T(\mathcal{F}_{i_0}, \theta_{i_0})$. We first obtain this result for the subsequences of the basis of $T[(\mathcal{F}_i, \theta_i)_{i\in I}]$ by applying the result of the previous section, and in the sequel we extend this result for block sequences.

To obtain the result for a given block sequence $(x_n)_n$ we show first that we can pass to a subsequence $(x_n)_{n\in M}$ which is equivalent to the subsequence $(e_{p_n})_{n\in M}$, $p_n = \min\operatorname{supp} x_n$, of the basis of the space $T([\mathbb{N}]^{\leq 2} \otimes \operatorname{Ad}(\mathcal{F}_{i_0}), \theta_{i_0})$, for appropriate fixed $1 \leq i_0 \leq r$. Using the results for the regular families we pass to a space $T(\mathcal{B}, \theta_{i_0})$ where $\mathcal{B}$ is a regular family with $\iota(\mathcal{B}) = \iota(\mathcal{F}_{i_0})$ and moreover the subsequence $(e_{p_n})_{n\in M}$ is equivalent in the two spaces.

Restricting the study to the families $\mathcal{S}_\xi$, we obtain that if $(x_n)_n$, $(y_n)_n$ are normalized block sequences in the space $T(\mathcal{S}_\xi, \theta)$ such that $x_n < y_n < x_{n+1}$ ($n \in \mathbb{N}$), then the two sequences are equivalent.

Proposition 4.1 *Fix $(\mathcal{F}_i, \theta_i)_{i=1}^r$ with at least one of the families with infinite index. Let i_0 be such that $(\mathcal{F}_{i_0}, \theta_{i_0}) = \max_{\leq_{\mathrm{T}}}\{(\mathcal{F}_i, \theta_i)\}_{i=1}^r$ (See Definition* 2.10*). Then for every M there is some $N \subseteq M$ and a regular family $\mathcal{B}$ with the same index as $\mathcal{F}_{i_0}$ such that for every sequence $(a_n)_{n\in N}$ of scalars,*

$$\Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{B},\theta_{i_0})} \leq \Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} \leq 2C\Big\|\sum_{n\in N} a_n e_n\Big\|_{(\mathcal{B},\theta_{i_0})}.$$

where the constant C is given in Theorem 2.13.

Proof By Proposition 3.11 we get $N_0 \subseteq M$ and regular families $\mathcal{B}_i$, with $\iota(\mathcal{B}_i) = \iota(\mathcal{F}_i)$ ($1 \leq i \leq r$), such that every $\mathcal{B}_i$-admissible sequence of subsets of N_0 is also $\mathcal{F}_i$-admissible. By Fact 2 it follows that for every sequence $(a_n)_{n\in N_0}$ of scalars,

$$\Big\|\sum_{n\in N_0} a_n e_n\Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r} \leq \Big\|\sum_{n\in N_0} a_n e_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r}.$$

Counting the corresponding indices we can find now $N_1 \subseteq N_0$ such that

$$[N_1]^{\leq 2} \otimes \operatorname{Ad}(\mathcal{F}_i)\restriction N_1 \subseteq (\mathcal{C}_i\restriction N_1) \otimes [N_1]^{\leq 2} \quad \text{for every } i \leq r,$$

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where $\mathcal{C}_i = \mathcal{B}_i \oplus [N_1]^{\leq 1}$ if $\iota(\mathcal{F}_i) < \omega$, $\mathcal{C}_i = \mathcal{B}_i$ otherwise. It follows from Proposition 2.1 that

$$\Big\| \sum_{n\in N_1} a_n e_n \Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} \leq 2 \| \sum_{n\in N_1} a_n e_n \|_{(\mathcal{C}_i,\theta_i)_{i=1}^r}.$$

By Theorem 2.13, using that $\mathcal{F}_{i_0}$ has infinite index, there exist $N \subseteq N_1$ such that

$$\Big\| \sum_{n\in N} a_n e_n \Big\|_{(\mathcal{C}_i,\theta_i)_{i=1}^r} \sim \Big\| \sum_{n\in N} a_n e_n \Big\|_{(\mathcal{B}_{i_0},\theta_{i_0})}.$$

Since $\| \sum_{n\in N} a_n e_n \|_{(\mathcal{B}_{i_0},\theta_{i_0})} \leq \| \sum_{n\in N} a_n e_n \|_{(\mathcal{B}_i,\theta_i)_{i=1}^r}$, we get the result. ■

Remark 4.2. It is worth mentioning that the conclusion of the above theorem does not hold in the case that all families $\mathcal{F}_i$ have finite index (see [8]).

To extend the above result to block sequences, we shall need some preparatory work. The following notion is descendant of the definition, introduced in [3], of initial and final part of a vector with respect to a tree analysis.

Definition 4.3 Fix compact and hereditary families $\mathcal{F}_i$ and real numbers $0 < \theta_i < 1$, $i \leq r$. Let $x \in c_{00}$, $f \in K((\mathcal{F}_i,\theta_i)_{i=1}^r)$ and $(f_t)_{t\in\mathcal{T}}$ a tree-analysis for f. Suppose that $\operatorname{supp} f \cap \operatorname{ran} x \neq \varnothing$. Let $t \in \mathcal{T}$ be a $\preceq$-maximal node with respect to the property that $\operatorname{supp} f_t \cap \operatorname{ran} x = \operatorname{supp} f \cap \operatorname{ran} x$. It is clear that such t exists and it is unique. Let us call it $t(x)$. Note that if $t(x)$ is not a maximal node of $\mathcal{T}$, then, by maximality of $t(x)$, there are $s_1 \neq s_2 \in S_{t(x)}$ such that $\operatorname{supp} f_{s_i} \cap \operatorname{ran} x \neq \varnothing$, for $i = 1, 2$. Observe that the set S_t of immediate $\preceq$-successors of t is naturally ordered according to $s < t$ if and only if $f_s < f_t$. Now for $t = t(x)$ not a maximal node, let

$$s_L(x) = \min\{s \in S_t : \operatorname{supp} f_s \cap \operatorname{ran} x \neq \varnothing\},$$
$$s_R(x) = \max\{s \in S_t : \operatorname{supp} f_s \cap \operatorname{ran} x \neq \varnothing\},$$

where both minimum and maximum are with respect to the relation $<$ on S_t.

Now fix a block sequence $(x_n)_n$. For a given n, let $t(n) = t(x_n)$, $s_L(n) = s_L(x_n)$ and $s_R(n) = s_R(x_n)$. For $t \in \mathcal{T}$, we define recursively

$$\begin{aligned} D_t = & \bigcup_{t \preceq_{\mathcal{T}} u} \{n : u = t(n)\} \\ = & \{n \in \mathbb{N} : \operatorname{supp} f_t \cap \operatorname{ran} x_n = \operatorname{supp} f \cap \operatorname{ran} x_n \neq \varnothing\}, \\ E_t = & D_t \setminus \bigcup_{s\in S_t} D_s = \{n : t = t(n)\}. \end{aligned}$$

For each n, set $q_n = \max \operatorname{supp} x_n$, $Q = \{q_n\}_{n\in\mathbb{N}}$. Define recursively on $t \in \mathcal{T}$

$$g_t = \theta_i \Big(\sum_{n\in E_t} \frac{f_t(x_n)}{\theta_i} e^*_{q_n} + \sum_{s\in S_t} g_s \Big)$$

if $f_t = \theta_i \sum_{s\in S_t} f_s$, where $(f_s)_{s\in S_t}$ is $\mathcal{F}_i$-admissible.

Proposition 4.4 (i) $\operatorname{supp} g_t = \{q_n : n \in D_t\}$ *for every* $t \in \mathcal{T}$.
(ii) $\{e^*_{q_n}\}_{n\in E_t} \cup \{g_s\}_{s\in S_t}$ *is a block family and*

$$\{\min q_n : n \in E_t\} \cup \{\min \operatorname{supp} g_s : s \in S_t,\ g_s \neq 0\} \in [Q]^{\leq 2} \otimes \operatorname{Ad}(\mathcal{F}_i)\restriction Q \tag{4.1}$$

for every $t \in \mathcal{T}$ *such that* $f_t = \theta_i \sum_{s\in S_t} f_s$.

Proof (i) follows readily from the definitions.

(ii) Suppose that $\#S_t > 1$, otherwise the result is trivial. Let us observe that for every $s \in S_t$, s not being the $<$-maximal element of S_t and with $g_s \neq 0$,

$$\min \operatorname{supp} f_s \leq \operatorname{supp} g_s < \min \operatorname{supp} f_{s^+}. \tag{4.2}$$

The first inequality follows readily from (i). Let us show now the last inequality. Assume otherwise that $\min \operatorname{supp} f_{s^+} \leq \max \operatorname{supp} g_s = \max\{q_n : n \in D_s\}$. Then there exists $n \in D_s$ such that $\min \operatorname{supp} f_{s^+} \leq q_n = \max \operatorname{supp} x_n$, hence $\operatorname{supp} f_{s^+} \cap \operatorname{ran} x_n \neq \emptyset$, a contradiction since $n \in D_s$. It is clear that for every $n \in E_t$, it holds that

$$\max \operatorname{supp} f_{s_L(n)} < q_n \leq \max \operatorname{supp} f_{s_R(n)} < \min \operatorname{supp} f_{s_R(n)^+}, \tag{4.3}$$

and moreover,

$$q_n < g_{s_R(n)}, \text{ if } g_{s_R(n)} \neq 0. \tag{4.4}$$

The definition of g_t together with (4.2), (4.3) and (4.4) gives that $H = \{e^*_{q_n}\}_{n\in E_t} \cup \{g_s : s \in S_t,\ g_s \neq 0\}$ is a block family. Let us prove now that the set $\{q_n\}_{n\in E_t} \cup \{\min g_s\}_{s\in S_t}$ belongs to $[Q]^{\leq 2} \otimes (\operatorname{Ad}(\mathcal{F}_i))\restriction Q$. First we order $H = \{h_j\}_{j=0}^k$ according to the block order, *i.e.*, $j < j'$ implies $\operatorname{supp} h_j < \operatorname{supp} h_{j'}$. For every $0 \leq j \leq k$, let $s_j \in S_t$ be such that either $h_j = g_{s_j}$ or $h_j = e^*_{q_n}$ and $s_j = s_R(n)$. Observe that for every $0 \leq j < k$, either $s_j < s_{j+1}$ or else $s_j = s_{j+1}$ and this can only occur if $h_j = e^*_{q_n}$ and $h_{j+1} = g_{s_{j+1}}$ with $s_{j+1} = s_R(n)$. Fix $t = \{m_s\}_{s\in S_t} \in \mathcal{F}_i$ that interpolates $(\operatorname{supp} f_s)_{s\in S_t}$. We claim that for every even $0 < j \leq k$ we have that

$$\operatorname{supp} h_{j-2} < m_{s_j} \leq \operatorname{supp} h_j. \tag{4.5}$$

From this we have that $\{m_{s_j}\}_{j \text{ even}} \in \mathcal{F}_i$ interpolates the set $\{\operatorname{supp} h_j : 0 \leq j \leq k,\ j \text{ even}\}$, hence $\{\min \operatorname{supp} h_j : 0 \leq j \leq k,\ j \text{ even}\} \in \operatorname{Ad}(\mathcal{F}_i)\restriction Q$, and so we easily get (4.1). Let us then show (4.5). Suppose first that $s_{j-2} = s_{j-1} < s_j$. Then $h_{j-2} = e^*_{q_n}$ and $h_{j-1} = g_{s_{j-1}}$ with $s_R(n) = s_{j-1}$. Hence

$$\max \operatorname{supp} h_{j-2} = q_n < \max f_{s_{j-2}} < m_{s_j} \leq \min \operatorname{supp} f_{s_j} \leq \min \operatorname{supp} h_j,$$

as desired. Suppose now that $s_{j-2} < s_{j-1} < s_j$. Then

$$\max \operatorname{supp} h_{j-2} < \min \operatorname{supp} f_{s^+_{j-2}} \leq \min \operatorname{supp} f_{s_{j-1}} \leq \max \operatorname{supp} f_{s_{j-1}},$$

so

$$\max \operatorname{supp} h_{j-2} < \max \operatorname{supp} f_{s_{j-1}} < m_{s_j} \leq \min \operatorname{supp} f_{s_j} \leq \min \operatorname{supp} h_j.$$

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Finally, suppose that $s_{j-2} < s_{j-1} = s_j$. Then $h_{j-1} = e^*_{q_n}$ and $h_j = g_{s_j}$ with $s_R(n) = s_j$. So

$$\begin{aligned}\max\operatorname{supp} h_{j-2} < \max\operatorname{supp} f_{s_L(n)} < m_{s_L(n)^+} \le m_{s_R(n)} = m_{s_j} \le \min\operatorname{supp} f_{s_j}\\ \le \min\operatorname{supp} h_j,\end{aligned}$$

and we are done. ■

Proposition 4.5 *Suppose that $(x_n)_{n\in\mathbb{N}}$ is in addition normalized.*

(i) *For every sequence (a_n) of scalars and every $t \in \mathcal{T}$,*

$$f_t\Big(\sum_{n\in D_t} a_n x_n\Big) = g_t\Big(\sum_{n\in D_t} a_n e_{q_n}\Big).$$

In particular, $f(\sum_n a_n x_n) = g_\varnothing(\sum_{n\in D_\varnothing} a_n e_{q_n})$.

(ii) *For every $t \in \mathcal{T}$, we have $g_t \in \frac{1}{\theta_0} B(T[([Q]^{\le 2} \otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q, \theta_i)_{i=1}^r]^*)$, where $\theta_0 = \min_{1\le i\le r}\theta_i$.*

(iii) *For every sequence (a_n) of scalars,*

$$\Big\|\sum_n a_n x_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} \le \frac{1}{\theta_0}\Big\|\sum_n a_n e_{q_n}\Big\|_{([Q]^{\le 2}\otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q,\theta_i)_{i=1}^r}.$$

Proof (i) can be shown easily by downwards induction on $t \in \mathcal{T}$. (ii) follows from Proposition 4.4(ii) and the fact that the dual ball of $T[([Q]^{\le 2} \otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q, \theta_i)_{i=1}^r]$ is closed on the $([Q]^{\le 2} \otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q, \theta_i)$-operation (see Remark 1.4).

(iii) Follows from (i) and (ii). ■

Remark 4.6. It is worth pointing out that Proposition 4.5(iii) gives that for every $M \subseteq \mathbb{N}$ and every sequence $(a_n)_{n\in M}$ of scalars

$$\Big\|\sum_{n\in M} a_n x_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} \le \frac{1}{\theta_0}\Big\|\sum_{n\in M} a_n e_{q_n}\Big\|_{([Q_M]^{\le 2}\otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q_M,\theta_i)_{i=1}^r},$$

where $Q_M = \{q_n\}_{n\in M} = \{\max\operatorname{supp} x_n\}_{n\in M}$.

Before we give the proof of the main result of the section we need one more auxiliary lemma.

Lemma 4.7 *Fix $(\mathcal{F}_i, \theta_i)_{i=1}^r$ with at least one of the families with infinite index and a normalized block sequence $(x_n)_n$ in the space $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$. Then for every i_0 such that $\iota(\mathcal{F}_{i_0}) \ge \omega$, there exists an infinite set M such that*

$$\Big\|\sum_{n\in M} a_n e_{p_n}\Big\|_{(\mathcal{F}_{i_0},\theta_{i_0})} \le 2\Big\|\sum_{n\in M} a_n x_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r},$$

where $p_n = \min\operatorname{supp} x_n$ for every n.

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Proof Set $P_0 = \{p_n\}$. Let $M_0 \subseteq \mathbb{N}$ be infinite and let $(\mathcal{B}_i)$ be a sequence of regular families on $\mathbb{N}$ with $\iota(\mathcal{B}_i) = \iota(\mathcal{F}_i)$ and such that

(4.6) every $\mathcal{B}_i$-admissible block sequence of subsets of $\{p_n\}_{n\in M_0}$ is $\mathcal{F}_i$-admissible for $1 \le i \le r$.

Let $M_1 = \{m_{2i}\}$, where $\{m_i\}$ is the increasing enumeration of M_0.

Claim *For every sequence of scalars $(a_n)_{n\in M_1}$,*

$$\Big\| \sum_{n\in M_1} a_n e_{p_n} \Big\|_{(\mathcal{B}_i,\theta_i)_{i=1}^r} \le \Big\| \sum_{n\in M_1} a_n x_n \Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r}.$$

Proof of Claim For every n, choose $\phi_n \in K((\mathcal{F}_i,\theta_i)_{i=1}^r)$ such that $\phi_n x_n \approx 1$ and $\operatorname{supp}\phi_n \subsetneq \operatorname{supp} x_n$. Let $P_1 = \{p_n\}_{n\in M_1}$, and define now $F\colon K^{P_1}((\mathcal{B}_i,\theta_i)_{i=1}^r) \to K((\mathcal{F}_i,\theta_i)_{i=1}^r)$ by $F(e_{p_n}^*) = \phi_n$, and extend it by

$$F(\theta_i(\psi_0+\cdots+\psi_n)) = \theta_i(F(\psi_0)+\cdots+F(\psi_n))$$

if $(\psi_i)_{i=0}^n \subseteq K^{P_1}((\mathcal{B}_i,\theta_i)_{i=1}^r)$ is a $\mathcal{B}_i$-admissible block sequence ($1 \le i \le r$). To see that F is well defined, suppose that $(\psi_i)_{i=0}^n \subseteq K^{P_1}((\mathcal{B}_i,\theta_i)_{i=1}^r)$ is $\mathcal{B}_i$-admissible block sequence and set $\min\operatorname{supp}\psi_i = p_{m_{2k_i}}$, $\max\operatorname{supp}\psi_i = p_{m_{2l_i}}$ ($0 \le i \le n$). Then we have that for every $0 \le i \le n$,

$$\operatorname{supp} F(\psi_i) \subseteq [p_{m_{2k_i}}, p_{m_{2l_i+1}}]. \tag{4.7}$$

Since, by (4.6), $(\{p_{m_{2k_i}}, p_{m_{2l_i+1}}\})_{i=0}^n$ is $\mathcal{F}_i$-admissible, condition (4.7) yields that $(F(\psi_i))_{i=0}^n$ is $\mathcal{F}_i$-admissible. It is clear now that the existence of F shows the desired result. ∎

Let i_0 be such that $\mathcal{F}_{i_0}$ has infinite index $\lambda+r$, with $\lambda > 0$ a limit ordinal and $r \in \mathbb{N}$. Let $M_2 \subseteq M_1$ be such that $\mathrm{Ad}(\mathcal{F}_{i_0})\restriction P_2$ is homogeneous on P_2, where $P_2 = \{p_n\}_{n\in M_2}$. Then by Proposition 3.9 and the properties of the homogeneous families we know that

$$\iota([P_2]^{\le 2}\otimes \mathrm{Ad}(\mathcal{F}_{i_0})\restriction P_2) = 2\iota(\mathrm{Ad}(\mathcal{F}_{i_0})\restriction P_2) \le 2(2(\lambda+r)) = \lambda + 4r < (\lambda+r)2,$$

so by Proposition 3.8 we can find $M \subseteq M_2$ such that, setting $P = \{p_n\}_{n\in M}$,

$$[P]^{\le 2}\otimes \mathrm{Ad}(\mathcal{F}_{i_0})\restriction P \subseteq \mathcal{B}_{i_0}\otimes [P]^{\le 2}.$$

It follows, by Proposition 2.1 that for every sequence of scalars $(a_n)_{n\in M}$,

$$\Big\| \sum_{n\in M} a_n e_{p_n} \Big\|_{(\mathcal{F}_{i_0},\theta_{i_0})} \le 2\Big\| \sum_{n\in M} a_n e_{p_n} \Big\|_{(\mathcal{B}_{i_0},\theta_{i_0})}.$$

This, combined with the previous claim, completes the proof. ∎

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Theorem 4.8 *Fix a finite sequence $(\mathcal{F}_i,\theta_i)_{i=1}^r$ of compact hereditary families and real numbers such that there is some $1\le i\le r$ with $\iota(\mathcal{F}_i)$ infinite. Then there is $1\le i_0\le r$ such that every normalized block sequence $(x_n)\subseteq T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ has a subsequence $(x_n)_{n\in N}$ which is equivalent to the subsequence $(e_{p_n})_{n\in N}$ of the natural basis $(e_n)_{n\in\mathbb{N}}$ of $T(\mathcal{F}_{i_0},\theta_{i_0})$, and where $p_n=\min\operatorname{supp} x_n$, for every n.*

Proof Let $(x_n)\subseteq T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ be a normalized block sequence. Let

$$(x_n)\subseteq T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$$

be a normalized block sequence. By Proposition 4.5 (see also Remark 4.6) we get that for every $W\subseteq\mathbb{N}$,

$$\Big\|\sum_{n\in W}a_nx_n\Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r}\le C\Big\|\sum_{n\in W}a_ne_{q_n}\Big\|_{([Q_W]^{\le 2}\otimes\operatorname{Ad}(\mathcal{F}_i)\restriction Q_W,\theta_i)_{i=1}^r},$$

where $C=\max_{1\le i\le r}\theta_i^{-1}$, $q_n=\max\operatorname{supp} x_n$ for each n, and $Q_W=\{q_n\}_{n\in W}$. Find an infinite set M of integers and a sequence $(\mathcal{G}_i)_{i=1}^r$ of regular families such that for every $1\le i\le r$

(i) $\operatorname{Ad}(\mathcal{F}_i)\restriction Q_M$ is homogeneous on Q_M,
(ii) $\iota(\mathcal{G}_i)=\iota(\operatorname{Ad}([Q_M]^{\le 2}\otimes\operatorname{Ad}(\mathcal{F}_i)\restriction Q_M))+1$,
(iii) $\operatorname{Ad}([Q_M]^{\le 2}\otimes\operatorname{Ad}(\mathcal{F}_i)\restriction Q_M)\restriction Q_M\subseteq\mathcal{G}_i\restriction Q_M$.

By Theorem 2.13 there is some $N\subseteq M$ and $D\ge 1$ such that

$$\Big\|\sum_{n\in N}a_ne_{q_n}\Big\|_{(\mathcal{G}_i,\theta_i)_{i=1}^r}\le D\Big\|\sum_{n\in N}a_ne_{q_n}\Big\|_{(\mathcal{G}_{i_0},\theta_{i_0})},$$

where i_0 is such that $(\iota(\mathcal{G}_{i_0}),\theta_{i_0})=\max_{<_{\mathrm{T}}}\{(\iota(\mathcal{G}_i),\theta_i):1\le i\le r\}$. Notice that $\iota(\mathcal{G}_{i_0})$ and $\iota(\mathcal{F}_{i_0})$ are both infinite. By Corollary 2.3 we can find $R\subseteq N$ such that

$$\Big\|\sum_{n\in R}a_ne_{q_n}\Big\|_{(\mathcal{G}_{i_0},\theta_{i_0})}\le 2\Big\|\sum_{n\in R}a_ne_{p_n}\Big\|_{(\mathcal{G}_{i_0},\theta_{i_0})},$$

where $p_n=\min\operatorname{supp} x_n$ for every n. Now use Proposition 3.11 to find an infinite subset S of R and a regular family $\mathcal{B}$ with the same index as $\mathcal{F}_{i_0}$ such that

(4.8) every $\mathcal{B}$-admissible sequence of subsets of $\{p_n\}_{n\in R}$ is $\mathcal{F}_{i_0}$-admissible.

Since, by the choice of M, the family $\operatorname{Ad}(\mathcal{F}_{i_0})\restriction Q_M$ is homogeneous on Q_M we obtain by Proposition 3.9 that

$$\begin{aligned}(4.9)\quad \iota(\mathcal{G}_{i_0})&\le 2\iota([Q_M]^{\le 2}\otimes\operatorname{Ad}(\mathcal{F}_{i_0})\restriction Q_M)+1\\ &=2\iota([Q_M]^{\le 2})\iota(\operatorname{Ad}(\mathcal{F}_{i_0})\restriction Q_M)+1\le 8\iota(\mathcal{F}_{i_0})+1.\end{aligned}$$

As $\mathcal{G}_{i_0}$ and $\mathcal{B}$ are both regular and $\iota(\mathcal{F}_{i_0})$ is infinite, (4.9) implies that

$$\iota([\mathbb{N}]^{\le 2}\otimes\mathcal{G}_{i_0})=2\iota(\mathcal{G}_{i_0})\le 16\iota(\mathcal{F}_{i_0})+2<\iota(\mathcal{F}_{i_0})2=\iota(\mathcal{B}\otimes[\mathbb{N}]^{\le 2}),$$

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so we can find an infinite $V \subseteq S$ such that

$$[\{p_n\}_{n\in V}]^{\le 2} \otimes \mathcal{G}_{i_0} \restriction \{p_n\}_{n\in V} \subseteq \mathcal{B} \otimes [\mathbb{N}]^{\le 2}.$$

Hence, by Proposition 2.1 ($\mathcal{G}_{i_0}$ is regular),

$$\Big\| \sum_{n\in V} a_n e_{p_n} \Big\|_{(\mathcal{G}_{i_0},\theta_{i_0})} \le 2 \Big\| \sum_{n\in V} a_n e_{p_n} \Big\|_{(\mathcal{B},\theta_{i_0})},$$

while by Lemma 4.7 we can find $W \subseteq V$ such that

$$\Big\| \sum_{n\in W} a_n e_{p_n} \Big\|_{(\mathcal{F}_{i_0},\theta_{i_0})} \le 2 \Big\| \sum_{n\in W} a_n x_n \Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r}.$$

Finally, from (iii) and Proposition 2.1 we get that

$$\Big\| \sum_{n\in W} a_n e_{q_n} \Big\|_{([Q_W]^{\le 2}\otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q_W,\theta_i)_{i=1}^r} \le \Big\| \sum_{n\in W} a_n e_{q_n} \Big\|_{(\mathcal{G}_i,\theta_i)_{i=1}^r}.$$

Putting all these inequalities together and also using (4.8), we obtain

$$\begin{aligned}
\frac{1}{4CD} \Big\| \sum_{n\in W} a_n x_n \Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r} &\le \frac{1}{4D} \Big\| \sum_{n\in W} a_n e_{q_n} \Big\|_{([Q_W]^{\le 2}\otimes \mathrm{Ad}(\mathcal{F}_i)\restriction Q_W,\theta_i)_{i=1}^r} \\
&\le \frac{1}{4D} \Big\| \sum_{n\in W} a_n e_{q_n} \Big\|_{(\mathcal{G}_i,\theta_i)_{i=1}^r} \le \frac{1}{4} \Big\| \sum_{n\in W} a_n e_{q_n} \Big\|_{(\mathcal{G}_{i_0},\theta_{i_0})} \\
&\le \frac{1}{2} \Big\| \sum_{n\in W} a_n e_{p_n} \Big\|_{(\mathcal{G}_{i_0},\theta_{i_0})} \le \Big\| \sum_{n\in W} a_n e_{p_n} \Big\|_{(\mathcal{B},\theta_{i_0})} \\
&\le \Big\| \sum_{n\in W} a_n e_{p_n} \Big\|_{(\mathcal{F}_{i_0},\theta_{i_0})} \le 2 \Big\| \sum_{n\in W} a_n x_n \Big\|_{(\mathcal{F}_i,\theta_i)_{i=1}^r}.
\end{aligned}$$

So, $(x_n)_{n\in W} \subseteq T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ and $(e_{p_n})_{n\in W} \subseteq T(\mathcal{F}_{i_0},\theta_{i_0})$ are equivalent, as desired. ■

We recall from Definition 2.9 that for a given compact and hereditary family $\mathcal{F}$ we define $\gamma(\mathcal{F}) = \iota(\mathcal{F})$ and $n(\mathcal{F}) = 0$ if $\mathcal{F}$ has finite index, and $\gamma(\mathcal{F}) = \omega^{\omega^\gamma}$ and $n(\mathcal{F}) = n$ satisfying that $\omega^{\omega^\gamma n} \le \alpha < \omega^{\omega^\gamma (n+1)}$, if $\mathcal{F}$ has infinite index. Using this terminology we can reformulate a result from [8, 9] as follows.

Theorem *Suppose that $(\mathcal{F}_i,\theta_i)_{i=1}^r$ is such that all $\mathcal{F}_i$ have finite index. Let i_0 be such that $(\mathcal{F}_{i_0},\theta_{i_0}) = \max_{\le_T}\{(\mathcal{F}_i,\theta_i)\}_{i=1}^r$, and $\mathcal{G}$ be an arbitrary regular family such that $\gamma(\mathcal{G}) = \gamma(\mathcal{F}_{i_0})$. Then every normalized block sequence (x_n) of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ has a block subsequence $(y_n)_n$ equivalent to the basis of $T(\mathcal{G},\theta_{i_0})$.*

Notice that in that case, the family $\mathcal{G}$ is on a tail equal to $[\mathbb{N}]^{\le \iota(\mathcal{F}_{i_0})}$. We present now the natural generalization of this theorem.

Corollary 4.9 *Fix $(\mathcal{F}_i, \theta_i)_{i=1}^r$. Let i_0 be such that $(\mathcal{F}_{i_0}, \theta_{i_0}) = \max_{\leq_T}\{(\mathcal{F}_i, \theta_i)\}_{i=1}^r$. Suppose that $\mathcal{G}$ is an arbitrary compact and hereditary family. If $\gamma(\mathcal{G}) = \gamma(\mathcal{F}_{i_0})$, then every normalized block sequence (x_n) of $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$ has a subsequence $(x_n)_{n\in M}$ equivalent to the subsequence $(e_{\min \operatorname{supp} x_n})_{n\in M}$ of the basis of $T(\mathcal{G}, \theta_{i_0}^{n(\mathcal{B})/n(\mathcal{F}_{i_0})})$, where we use the convention $0/0 = 1$.*

Proof We may assume that at least one of the families $\mathcal{F}_i$ has infinite index. By Theorem 4.8, it is enough to have the conclusion for subsequences of the basis of $T(\mathcal{F}_{i_0}, \theta_{i_0})$, and by Proposition 4.1 we may assume that $\mathcal{F}_{i_0}$ and $\mathcal{G}$ are both regular families. Let

$$\iota(\mathcal{F}_{i_0}) = \omega^{\omega^{\alpha} m+\beta} n + \delta, \quad \iota(\mathcal{G}) = \omega^{\omega^{\bar{\alpha}} \bar{m}+\bar{\beta}} \bar{n} + \bar{\delta}$$

be canonical decompositions. This is possible since $\gamma(\mathcal{F}_{i_0}) = \gamma(\mathcal{B})$ is infinite. Moreover $\bar{\alpha} = \alpha$. Using

$$\omega^{\omega^{\alpha} m} \leq \iota(\mathcal{F}_{i_0}) = \omega^{\omega^{\alpha} m+\beta} n + \delta < \omega^{\omega^{\alpha} m+\beta+1}$$

and the corresponding inequality for $\mathcal{G}$, by Theorem 2.8 we may assume that $\iota(\mathcal{F}_{i_0}) = \omega^{\omega^{\alpha} m}$, and $\iota(\mathcal{G}) = \omega^{\omega^{\alpha} \bar{m}}$ Now the result follows from the application of Proposition 2.6 to the families $\mathcal{F}_{i_0}$ and $\mathcal{G}$. ■

In particular for Schreier families we obtain the following.

Corollary 4.10 *Fix $(\mathcal{F}_i, \theta_i)_{i=1}^r$ such that at least one of the families has infinite index. Let i_0 be such that $(\mathcal{F}_{i_0}, \theta_{i_0}) = \max_{\leq_T}\{(\mathcal{F}_i, \theta_i)\}_{i=1}^r$, and set $\iota(\mathcal{F}_{i_0}) = \omega^{\omega^{\alpha} k+\delta} m + \gamma$ in canonical form. Then every normalized block sequence (x_n) of $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$ has a subsequence $(x_n)_{n\in M}$ equivalent to the subsequence $(e_{\min \operatorname{supp} x_n})_{n\in M}$ of the basis of $T(\mathcal{S}_{\omega^\alpha}, \theta_{i_0}^{1/k})$.*

The last result of the section concerns equivalence of block sequences in the spaces $T(\mathcal{S}_\xi, \theta)$.

Proposition 4.11 *Let (x_n), (y_n) be two normalized block sequences in the space $T(\mathcal{S}_\xi, \theta)$ such that $x_n < y_n < x_{n+1}$ for every n. Then (x_n) and (y_n) are $24\theta^{-2}$-equivalent.*

Proof For the proof we shall use the following two relations concerning the Schreier families $\mathcal{S}_\xi$, and infinite subsets N of integers with $\min N \geq 3$.

$$[N]^{\leq 3} \otimes \mathcal{S}_\xi \subseteq \mathcal{S}_\xi \otimes [N]^{\leq 2}, \tag{4.10}$$

$$[N]^{\leq 3} \otimes (\mathcal{S}_\xi \oplus [N]^{\leq 1}) \subseteq \mathcal{S}_\xi \otimes [N]^{\leq 3}. \tag{4.11}$$

The proofs of these two relations follow easily by induction on ξ. Now we show that a normalized block sequence (x_n) is equivalent to the subsequence $(e_{p_n})_n$ of the basis, $p_n = \min \operatorname{supp} x_n$, and this implies the result. Without loss of generality we may assume that $p_n \geq 3$ for every n. It follows easily from the spreading property of the families $\mathcal{S}_\xi$ that

$$\Big\| \sum_n a_n e_{p_n} \Big\|_{(\mathcal{S}_\xi, \theta)} \leq \Big\| \sum_n a_n x_n \Big\|_{(\mathcal{S}_\xi, \theta)}.$$

For the reverse inequality, by Proposition 4.5 we get

$$\Big\|\sum_n a_n x_n\Big\|_{(\mathcal{S}_\xi,\theta)} \le \theta^{-1}\Big\|\sum_n a_n e_{q_n}\Big\|_{([\mathbb{N}]^{\le 2}\otimes\mathcal{S}_\xi,\theta)},$$

where $q_n = \max\operatorname{supp} x_n$ for each n. By (4.10) and Proposition 2.1, we get

$$\Big\|\sum_n a_n e_{q_n}\Big\|_{([\mathbb{N}]^{\le 2}\otimes\mathcal{S}_\xi,\theta)} \le 2\Big\|\sum_n a_n e_{q_n}\Big\|_{(\mathcal{S}_\xi,\theta)}.$$

As in the proof of Corollary 2.3, we get that

$$\Big\|\sum_n a_n e_{q_n}\Big\|_{(\mathcal{S}_\xi,\theta)} \le \Big\|\sum_n a_n e_{p_n}\Big\|_{(\mathcal{S}_\xi\oplus[\mathbb{N}]^{\le 1},\theta)}.$$

Now by (4.11) and again Proposition 2.1, we get that

$$\Big\|\sum_n a_n e_{p_n}\Big\|_{(\mathcal{S}_\xi\oplus[\mathbb{N}]^{\le 1},\theta)} \le 3\Big\|\sum_n a_n e_{p_n}\Big\|_{(\mathcal{S}_\xi,\theta)}.$$

and this completes the proof. ■

4.1 Incomparability

The goal here is to turn the implication presented in Corollary 4.9 into an equivalence. So we are now going to deal with the incomparability of the Tsirelson-type spaces. The main tools to distinguish two such spaces are the special convex combinations, introduced in [3]. The following lemma provides the existence of the special convex combinations, in a more general setting than the one in [3], and it is a version of the well-known Pták's lemma (see [6] for a proof).

Lemma 4.12 *Suppose that $\mathcal{F}_0$ and $\mathcal{F}_1$ are two regular families with indices $\iota(\mathcal{F}_i) = \omega^{\alpha_i} n_i + \beta_i$, $\alpha_i > 0$, $n_i \in \mathbb{N}$ $\beta_i < \omega^{\alpha_i}$ $(i = 0, 1)$. If $\alpha_0 < \alpha_1$, then for every $\varepsilon > 0$ there is a convex mean μ such that* $\operatorname{supp}\mu \in \mathcal{F}_1$ *and such that* $\sup_{t\in\mathcal{F}_0}\sum_{n\in t}\mu(n) < \varepsilon$.

The first case where the spaces are going to be totally incomparable is if the index of one of the families is at least the ω-power of the other.

Lemma 4.13 *Suppose that $\mathcal{F}_0, \mathcal{F}_1$ are two regular families such that $\iota(\mathcal{F}_0)^\omega \le \iota(\mathcal{F}_1)$. Then $T(\mathcal{F}_0, \theta_0)$ and $T(\mathcal{F}_1, \theta_1)$ are totally incomparable.*

Proof Suppose that the desired result does not hold. By standard arguments we may assume that there exists a normalized block sequence $(x_n)_n \in T(\mathcal{F}_i, \theta_i)$ equivalent to a normalized block sequence $(z_n)_n$ of $T(\mathcal{F}_j, \theta_j)$, $j \ne i$. By Theorem 4.8, passing to subsequences if necessary, we may assume that $(x_n)_n$ is equivalent to a subsequence $(e_n)_{n\in M_i}$ of the natural basis (e_n) of $T(\mathcal{F}_i, \theta_i)$ and that (z_n) is equivalent to a subsequence $(e_n)_{n\in M_j}$ of the natural basis (e_n) of $T(\mathcal{F}_j, \theta_j)$.

For $k = 0, 1$, let $\varphi_k \colon M_k \to \mathbb{N}$ be the unique order-preserving onto mapping between M_k and $\mathbb{N}$. Note that for $k = 0, 1$ the family $\phi_k^{-1}\mathcal{F}_k$ is regular on M_k, $\iota(\varphi_k^{-1}\mathcal{F}_k) = \iota(\mathcal{F}_k)$ and $(e_n)_{n\in M_k} \subseteq T(\mathcal{F}_k, \theta_k)$ is 1-equivalent to

$$e_n)_{n\in\mathbb{N}} \subseteq T(\varphi_k^{-1}\mathcal{F}_k, \theta_k).$$

So, without loss of generality, we may assume that $M_1 = M_2 = \mathbb{N}$. So, we are supposing that $(e_n) \subseteq T(\mathcal{F}_0, \theta_0)$ is, say, C-equivalent to $(e_n) \subseteq T(\mathcal{F}_1, \theta_1)$ *i.e.*, for all scalars (a_n),

$$\frac{1}{C}\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{F}_0,\theta_0)} \leq \Big\|\sum_n a_n e_n\Big\|_{(\mathcal{F}_1,\theta_1)} \leq C\Big\|\sum_n a_n e_n\Big\|_{(\mathcal{F}_0,\theta_0)}.$$

Let $l \in \mathbb{N}$ be such that $\theta_0^l < \theta_1/(2C)$. By our hypothesis over the indices, $\iota(\mathcal{F}_0)^l < \iota(\mathcal{F}_1)$. So, by Lemma 4.12 there is some convex mean μ such that $\operatorname{supp}\mu \in \mathcal{F}_1$, and $\sum_{n\in t}\mu(n) < \frac{\theta_1}{2C}$ for every $t \in \mathcal{F}_0^{\otimes(l-1)}$. Observe that every $\phi \in K(\mathcal{F}_0, \theta_0)$ has a decomposition $\phi = \phi_0 + \phi_1$, where $\operatorname{supp}\phi_0 \in \mathcal{F}_0^{\otimes(l-1)}$, $\|\phi_1\|_\infty \leq \theta^l$ and $\operatorname{supp}\phi_0 \cap \operatorname{supp}\phi_1 = \varnothing$. So, for every $\phi \in K(\mathcal{F}_0, \theta_0)$,

$$\begin{aligned}\Big|\phi\Big(\sum_{n\in s}\mu(n)e_n\Big)\Big| &= \Big|\phi_0\Big(\sum_{n\in s}\mu(n)e_n\Big) + \phi_1\Big(\sum_{n\in s}\mu(n)e_n\Big)\Big| \\ &\leq \sum_{n\in \operatorname{supp}\phi_0\cap s}\mu(n) + \|\phi_1\|_\infty\sum_{n\in s}\mu(n) < \frac{\theta_1}{2C} + \theta_0^l < \frac{\theta_1}{C},\end{aligned}$$

while

$$\Big\|\sum_{n\in s}\mu(n)e_n\Big\|_{(\mathcal{F}_1,\theta_1)} \geq \theta_1\sum_{n\in s}\mu(n) = \theta_1,$$

and so, by (4.1),

$$\theta_1 \leq \Big\|\sum_{n\in s}\mu(n)e_n\Big\|_{(\mathcal{F}_1,\theta_1)} \leq C\Big\|\sum_{n\in s}\mu(n)e_n\Big\|_{(\mathcal{F}_0,\theta_0)} < C\frac{\theta_1}{C},$$

a contradiction. ■

The second case of total incomparability we consider is when the two families have the same index, but the corresponding θ's are different.

Lemma 4.14 *Suppose that $\mathcal{F}_0$ and $\mathcal{F}_1$ are two regular families with the same index, and suppose that $\theta_0 \neq \theta_1$, and $\max\{\theta_0, \theta_1\} > 1/\iota(\mathcal{F}_0)$, where by convention, $1/\alpha = 0$ if α is an infinite ordinal. Then the corresponding spaces $T(\mathcal{F}_0, \theta_0)$ and $T(\mathcal{F}_1, \theta_1)$ are totally incomparable.*

Proof Suppose first that $\iota(\mathcal{F}_0) = \iota(\mathcal{F}_1)$ is finite. Then $T(\mathcal{F}_0, \theta_0)$ and $T(\mathcal{F}_1)$ are isomorphic to different classical spaces c_0 or ℓ_p ($p > 1$), and the conclusion of the lemma trivially holds.

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Suppose that $\iota(\mathcal{F}_0)=\iota(\mathcal{F}_1)$ is infinite. As in the previous lemma, we may assume that $\mathcal{F}_0=\mathcal{F}_1=\mathcal{F}$ and that $(e_n)\subseteq T(\mathcal{F},\theta_0)$ is C-equivalent to $(e_n)\subseteq T(\mathcal{F},\theta_1)$, *i.e.*, for all scalars (a_n),

$$\frac{1}{C}\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{F},\theta_0)}\le\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{F},\theta_1)}\le C\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{F},\theta_0)}. \tag{4.12}$$

Suppose that $\theta_0<\theta_1$. Let $l\in\mathbb{N}$, $l>1$ be such that $(\theta_1/\theta_0)^l>2C$. Let $(a_n)_{n\in s}$ be a convex mean such that $s\in\mathcal{F}^{\otimes(l)}$ and $\sum_{n\in t}a_n<\theta_1^l/(2C)$ for every $t\in\mathcal{F}^{\otimes(l-1)}$. As before, any functional $\phi\in K(\mathcal{F},\theta_0)$ is decomposed $\phi=\phi_0+\phi_1$, $\operatorname{supp}\phi_0\cap\operatorname{supp}\phi_1=\varnothing$, $\operatorname{supp}\phi_0\in\mathcal{F}^{\otimes(l-1)}$ and $\|\phi_1\|_\infty\le\theta_0^l$. Then

$$\Big|\phi\Big(\sum_{n\in s}a_ne_n\Big)\Big|=\Big|\phi_0\Big(\sum_{n\in s}a_ne_n\Big)+\phi_1\Big(\sum_{n\in s}a_ne_n\Big)\Big|<\frac{\theta_1^l}{2C}+\theta_0^l<\frac{\theta_1^l}{C}.$$

Finally, by (4.12),

$$\theta_1^l\le\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{F},\theta_1)}\le C\Big\|\sum_n a_ne_n\Big\|_{(\mathcal{F},\theta_0)}<C\frac{\theta_1^l}{C},$$

a contradiction. ∎

4.2 Main Result

We collect in a single result the facts we have so far. It is written in terms of the ordinal invariants γ and n introduced in Definition 2.9, and using the convention $0/0=1$.

Theorem 4.15 **(Classification theorem)** *Fix two sequences $(\mathcal{F}_i,\theta_i)_{i=1}^r$ and $(\mathcal{G}_i,\eta_i)_{i=1}^s$ of pairs of compact and hereditary families and real numbers in $(0,1)$. Let $1\le i_0\le r$ and $1\le j_0\le s$ be such that $(\mathcal{F}_{i_0},\theta_{i_0})=\max_{\le_{\mathrm{T}}}\{(\mathcal{F}_i,\theta_i):1\le i\le r\}$, and $(\mathcal{G}_{j_0},\eta_{j_0})=\max_{\le_{\mathrm{T}}}\{(\mathcal{G}_i,\eta_i):1\le i\le s\}$. The following are equivalent:*

- (i) *Either*
  - (a) $\gamma(\mathcal{F}_{i_0}),\gamma(\mathcal{G}_{j_0})\ge\omega$, $\gamma(\mathcal{F}_{i_0})=\gamma(\mathcal{G}_{j_0})$ *and* $\theta_{i_0}^{n(\mathcal{G}_{j_0})}=\eta_{j_0}^{n(\mathcal{F}_{i_0})}$, *or else*
  - (b) *both $\mathcal{F}_{i_0},\mathcal{G}_{j_0}$ have finite index, and either*
    - (1) $\theta_{i_0}\gamma(\mathcal{F}_{i_0}),\eta_{i_0}\gamma(\mathcal{G}_{j_0})\le 1$, *or else*
    - (2) $\log_{\gamma(\mathcal{F}_{i_0})}\theta_{i_0}=\log_{\gamma(\mathcal{G}_{j_0})}\eta_{j_0}$.
- (ii) *Every closed infinite dimensional subspace of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ contains a subspace isomorphic to a subspace of $T[(\mathcal{G}_i,\eta_i)_{i=1}^s]$.*
- (iii) *For every regular family $\mathcal{B}$ such $\gamma(\mathcal{B})=\gamma(\mathcal{G}_{j_0})$ and every normalized block sequence of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$ there is a block subsequence (subsequence if $\mathcal{G}_{i_0}$ has infinite index) equivalent to a subsequence of the natural basis of $T(\mathcal{B},\eta_{j_0}^{n(\mathcal{B})/n(\mathcal{G}_{j_0})})$.*

Proof (ii) implies (iii). Fix a regular family $\mathcal{B}$ with same index as $\mathcal{G}_{j_0}$, and fix a normalized block sequence (x_n) of $T[(\mathcal{F}_i,\theta_i)_{i=1}^r]$. By (b), there is some block sequence (y_n) of (x_n) which is equivalent to a semi normalized block sequence (z_n) of

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$T[(\mathcal{G}_i, \eta_i)_{i=1}^s]$. By Corollary 4.9, we can find a further block subsequence (w_n) of (z_n), which is equivalent to a subsequence of the natural basis of $T(\mathcal{B}, \eta_{j_0}^{1/n(\mathcal{G}_{j_0})})$, as desired.

(iii) implies (i). First, fix a regular family $\mathcal{C}$ with index $\gamma(\mathcal{F}_{i_0})$. By Corollary 4.9 we know that $T((\mathcal{F}_i, \theta_i)_{i=1}^r)$ is saturated by subsequences of the basis of $T(\mathcal{C}, \theta_{i_0}^{1/n(\mathcal{F}_{i_0})})$. Notice that (iii) implies that $T(\mathcal{B}, \eta_{j_0}^{1/n(\mathcal{G}_{j_0})})$ and $T(\mathcal{C}, \theta_{i_0}^{1/n(\mathcal{F}_{i_0})})$ are not totally incomparable. Suppose first that $\mathcal{G}_{j_0}$ has finite index. Lemma 4.13 gives that $\mathcal{F}_{i_0}$ has also finite index, and in particular $n(\mathcal{F}_{i_0}) = 1$. Now (b) follows from the properties of ℓ_p's and c_0.

Assume now that $\mathcal{G}_{j_0}$ is infinite. In this case Lemma 4.13 implies that $\gamma(\mathcal{F}_{i_0}) = \gamma(\mathcal{G}_{j_0})$. It follows, by Corollary 4.9 that $T[(\mathcal{F}_i, \theta_i)_{i=1}^r]$ is saturated by subsequences of $T(\mathcal{B}, \theta_{i_0}^{1/n(\mathcal{F}_{i_0})})$. Hence $T(\mathcal{B}, \theta_{i_0}^{1/n(\mathcal{F}_{i_0})})$ and $T(\mathcal{B}, \eta_{j_0}^{1/n(\mathcal{G}_{j_0})})$ are not totally incomparable, so by Lemma 4.14, $\theta_{i_0}^{1/n(\mathcal{F}_{i_0})} = \eta_{i_0}^{1/n(\mathcal{G}_{j_0})}$.

(i) implies (ii) follows from Corollary 4.9. ■

Remark 4.16. (i) If the families $\mathcal{F}$ are compact but not necessarily hereditary, Theorem 4.15 remains true. The main observation is that if $\mathcal{F}$ is an arbitrary compact family, then there is some infinite set M such that $\mathcal{F}[M] = \{s \cap M : s \in \mathcal{F}\}$ is hereditary (see [5]). This fact, when applied to the family $\mathrm{Ad}(\mathcal{F})$ of $\mathcal{F}$-admissible sets, guarantees we can follow the same arguments we use for the case of hereditary families, starting with Proposition 4.5.

(ii) The problem of classification of full mixed Tsirelson spaces $T[(\mathcal{F}_i, \theta_i)_{i=0}^{\infty}]$ seems rather unclear. There are several obstacles for someone wanting to extend the techniques presented in this paper to the general case.

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