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FORMS WITH O-ORTHOGONAL LIE ALGEBRAS

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A form P of degree r is a homogeneous polynomial in $k[Y_l, \ldots, Y_n]$ on k^n, k a field; Y_i are the coordinate functions on k^n. Let $V(n, r)$ denote the k-vector space of forms of degree r. $M_n(k) = \mathrm{End}_k(k^n)$ has canonical Lie algebra structure with $[A, B] = AB - BA$ and it acts as a k-Lie Algebra of k-derivations of degree 0 on $k[Y_l, \ldots, Y_n]$ defined by setting $D(A)Y = Y \circ (-A)$ for $A \in \mathrm{End}_k(k^n)$, $Y \in V(n, 1) = \mathrm{Hom}_k(k^n, k)$ and extending as a k-derivation. Define the *orthogonal Lie Algebra*, $LO(P)$, *of* P by $LO(P) = \{A \in \mathrm{End}_k(k^n) \mid D(A)P = 0\}$.

THEOREM: *If* $r \geq 3$, *then* $\{P \in V(n, r) \mid LO(P) = 0\}$ *is Zariski open dense in* $V(n, r)$.

Proof: The proof uses the diagram below. Let $d(n, r) = \text{dimension } V(n, r)$; $d(n, r) = \binom{n+r-1}{r}$. For $r \geq 3$, $d(n, r) \geq n^2$. Let Grass $(\mathrm{End}_k(V(n, r)); d(n, r)^2 - d(n, r))$ be the Grassman variety of $d(n, r)^2 - d(n, r)$ dimensional subspaces of $\mathrm{End}_k(V(n, r))$; denote it simply by Grass[1]. Grass is a projective variety. Isograss $= \{W \in \text{Grass} \mid W = \{C \in \mathrm{End}_k(V(n, r)) \mid CP = 0\}$ for some $P \neq 0$ in $V(n, r)\}$; Isograss is a closed subvariety of Grass. Let $\mathfrak{G} = \{W \in \text{Grass} \mid W \cap D(\mathrm{End}_k(k^n)) = \{0\}\}$; the fact that $d(n, r) \geq n^2$ implies that $\mathfrak{G}$ is a Zariski open dense subset of Grass. Let $\mathbb{P}(V(n, r))$ denote the projective variety associated to $V(n, r)$; $\mathbb{P}(V(n, r)) = \{kP \mid P \neq 0, P \in V(n, r)\}$. We have an isomorphism of projective varieties

$$\mathbb{P}(V(n, r)) \overset{i}{\rightarrowtail} \text{Isograss}$$

$kP \longmapsto \{C \in \mathrm{End}_k(V(n, r)) \mid CP = 0\}$ and a diagram of inclusions

$$\begin{array}{ccccc}
\mathbb{P}(V(n, r)) & \overset{i}{\rightarrowtail} & \text{Isograss} & \xhookrightarrow[\text{inclusion}]{\text{closed}} & \text{Grass} \\
\uparrow{\scriptstyle \text{open inclusion}} & & \uparrow{\scriptstyle \text{open inclusion}} & & \uparrow{\scriptstyle \text{open inclusion}} \\
i^{-1}(\mathfrak{G} \cap \text{Isograss}) & \rightarrowtail & \mathfrak{G} \cap \text{Isograss} & \xhookrightarrow[\text{inclusion}]{\text{closed}} & \mathfrak{G}
\end{array}$$

with $\mathfrak{G} \cap \text{Isograss}$ easily shown non-empty. $LO(P) = \{0\}$ if and only if $kP \in i^{-1}(\mathfrak{G} \cap \text{Isograss})$. Clearly, the set of such P is Zariski open, dense, in $V(n, r)$.

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This theorem has the following nice corollary for orthogonal groups. $GL(n, k)$ acts on $V(n, r)$ by $\lambda(g)P = P \circ g^{-1}$ as a polynomial function for all $P \in V(n, r)$, $g \in GL(n, k)$. *The orthogonal group of P* is defined as $O(P) = \{g \in GL(n, k) \mid \lambda(g)P = P\}$, a linear algebraic group defined over k. If k is algebraically closed and of characteristic p with $p = 0$ or $p > r =$ degree P, then $LO(P)$ is isomorphic to the Lie Algebra of $O(P)$, since λ is a separable morphism[2]. Hence $O(P)$ is finite if and only if $LO(P) = \{0\}$.

COROLLARY: *Let $r \geq 3$, let k be an algebraically closed field of characteristic O or p greater than r. Then $\{P \in V(n, r) \mid O(P)$ is finite$\}$ is Zariski open dense and $GL(n, k)$ invariant.*

Proof. The result follows from the theorem and the remarks above. The $\lambda GL(n, k)$-invariance of $\{P \in V(n, r) \mid O(P)$ is finite$\}$ follows from the identity

$$O(\lambda(g)P) = gO(P)g^{-1}.$$

Alternatively, the morphism i and the inclusions in the proof of the theorem are compatible with the action of $GL(n, k)$ on $\mathbb{P}(V(n, r))$ and on Grass via λ and the composite of λ with the adjoint representation of $GL(\mathrm{End}_k(k^n))$.

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