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A NEW FORMULA FOR ADOMIAN POLYNOMIALS AND THE ANALYSIS OF ITS TRUNCATED SERIES SOLUTION FOR FRACTIONAL NON-DIFFERENTIABLE INITIAL VALUE PROBLEMS

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Abstract

A new formula for Adomian polynomials is introduced and applied to obtain truncated series solutions for fractional initial value problems with nondifferentiable functions. These kinds of equations contain a fractional single term which is examined using Jumarie fractional derivatives and fractional Taylor series for nondifferentiable functions. The property of nonlocality of these equations is examined, and the existence and uniqueness of solutions are discussed. Convergence and error analysis for the Adomian series solution are also studied. Numerical examples show the accuracy and efficiency of this formula for solving initial value problems for high-order fractional differential equations.

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1. Introduction

Recently, many phenomena in fluid mechanics, biology, physics, engineering and other areas of science have been modelled by the use of fractional differential equations (FDEs). Several analytical and numerical methods have been proposed to solve the FDEs [8, 12–17, 19–22, 25–34]. Since in some of these works the linear operator with fractional derivatives was replaced approximately by a linear operator with integer derivatives, the convergence rate was very low. The modified Riemann–Liouville fractional derivative which was introduced by Jumarie [8] is used here instead of the Riemann–Liouville or Caputo operators. Thus, we can deal with the linear operator

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with fractional derivatives using the properties of the Jumarie fractional derivative without any replacement, which means that it is still fractional, and so the rate of convergence is accelerated. In methods such as the Adomian decomposition method (ADM) [1–3], the solution of the FDEs should be analytical if the fractional derivative is in the Caputo sense. It is known that fractal curves are everywhere continuous but nowhere differentiable. As a result, we cannot employ the Caputo derivative, which requires that the defined functions should be differentiable, to describe motion in fractal timespace. Recently, a modified Riemann–Liouville derivative was proposed by Jumarie which has some properties comparable with those of the classical Caputo derivative. The first is that the definition of the fractional derivative is not required to satisfy higher integer-order derivatives than α. The second one is that the αth derivative of a constant is zero. The modified Riemann–Liouville fractional form was successfully applied in probability calculus [9], fractional Laplace problems [10], fractional variational calculus [4] and fractional Taylor series [8, 11]. The existence and the uniqueness of the solution of nondifferentiable initial value problems are established. A new formulation of Adomian polynomials is introduced which provides a large improvement to the speed of convergence and cancels the calculations of all the inaccurate terms which degrade the convergence especially for higher fractional-order differential equations. This new formula is more efficient and general than the Adomian and El-Kalla formulae [5, 7, 24, 25]. Based on this new formula and its truncated series solution, the convergence and error analysis of the Adomian series solution for a class of nondifferentiable fractional equations are discussed. For these reasons we call this technique the improved fractional Adomian decomposition method (IFADM). In our test examples, we do a comparison between solutions dependent on Adomian's polynomials and the proposed new formula to clarify the efficiency and applicability of this treatment. The procedures which are used to obtain the approximate analytical solutions of our test examples are implemented in Mathematica.

The paper is arranged in the following way. In Section 2, we introduce some basic definitions about fractional derivatives and Jumarie operators. In Section 3, we give the analysis of the decomposition method associated with the new IFADM formula applied to nondifferentiable initial value problems. In Section 4, we present the new formula for Adomian polynomials and clarify how we obtain it and an exponential notation formula equivalent to the proposed new formula is introduced also. In Section 5, we prove the existence of the unique solution for nondifferentiable initial value problems, study the convergence of the proposed method, and estimate the maximum absolute error of the truncated series solution. In Section 6, we present numerical examples in order to clarify the efficiency and the convergence of the proposed new formula. Finally, in Section 7, the paper ends with a brief conclusion and some remarks.

2. Basic definitions

In this section, we present some basic definitions and properties of the fractional calculus that will be useful in the following sections.

DEFINITION 2.1. Let $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto f(x)$, denote a continuous function. Then its fractional derivative of order α is defined by [8]

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)}\int_0^x (x-\xi)^{-\alpha-1} f(\xi)\,d\xi, \quad \alpha < 0.$$

For positive α, one will set

$$f^{(\alpha)}(x) := (f^{(\alpha-1)}(x))', \quad 0<\alpha<1, \tag{2.1}$$

and

$$f^{(\alpha)}(x) := (f^{(\alpha-n)}(x))^{(n)}, \quad n\geq 2,\ n\leq\alpha<n+1.$$

If $f(x)=k$ (constant), using equation (2.1) we find that the αth derivative is $kx^{\alpha}/\Gamma(1-\alpha)$ which is different from zero. To circumvent this defect some authors proposed the following definitions.

DEFINITION 2.2. Let $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto f(x)$ denote a continuous function. Its Riemann–Liouville fractional derivative D^{α} of order α is defined by [8]

$$D^{\alpha}f(x) := \frac{1}{\Gamma(n+1-\alpha)}\int_0^x (x-\xi)^{n-\alpha} f^{(n+1)}(\xi)\,d\xi, \quad n\leq\alpha<n+1.$$

This definition does not apply when $f(x)$ is not differentiable and if we want to get the first derivative of $f(x)$ we must already have its second derivative. So Jumarie modified the Riemann–Liouville fractional derivative via a finite difference scheme.

DEFINITION 2.3 (Modified Riemann–Liouville fractional derivative [11]). Let $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto f(x)$ denote a continuous function. If $f(x)$ is not a constant, then its fractional derivative of order α is defined by

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)}\int_0^x (x-\xi)^{-\alpha-1}(f(\xi)-f(0))\,d\xi, \quad \alpha<0.$$

For positive α, one will set

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\xi)^{-\alpha}(f(\xi)-f(0))\,d\xi, \quad 0<\alpha<1,$$

and

$$f^{(\alpha)}(x) := (f^{(n)}(x))^{(\alpha-n)}, \quad n\geq 1,\ n\leq\alpha<n+1.$$

DEFINITION 2.4. The integral with respect to $(dx)^{\alpha}$ is defined as the solution of the fractional differential equation

$$dy \cong f(x)(dx)^{\alpha}, \quad x\geq 0, \quad y(0)=0, \quad 0<\alpha<1. \tag{2.2}$$

That is, if $f(x)$ denotes a continuous function, then the solution of equation (2.2) is defined as [11]

$$y(x) = \int_0^x f(\xi)(d\xi)^{\alpha} = \alpha\int_0^x (x-\xi)^{\alpha-1} f(\xi)\,d\xi, \quad 0<\alpha\leq 1.$$

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THEOREM 2.5 (Fractional Taylor–Jumarie series). *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *has fractional derivative of order* αk, *for any positive integer* k *and any* α, $0 < \alpha \leq 1$. *Then* $f(x)$ *is expanded in the series form*

$$f(x) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1 + \alpha k)} ((D^{\alpha})^{k}) f(0).$$

DEFINITION 2.6 (Jumarie fractional derivative via fractional difference [8]). Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$, denote a continuous (but not necessarily differentiable) function, and let the partition step be $h > 0$ on the interval $[0, 1]$. Define the forward operator $\mathrm{FW}(h)$ in the form

$$(\mathrm{FW}(h)) f(x) := f(x + h).$$

Then the fractional difference of order α $(0 < \alpha \leq 1)$ of $f(x)$ is defined by the expression

$$\triangle^{\alpha} f(x) := (\mathrm{FW} - 1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^{k} \binom{n}{k} f(x + (\alpha - k)h),$$

and its fractional derivative of order α is

$$f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\triangle^{\alpha} f(x)}{h^{\alpha}}.$$

REMARK. The αth derivative of a constant using this definition is zero.

For more details about the fractional calculus see the books by Miller and Ross [19] and Podlubny [23].

3. The IFADM applied to nondifferentiable IVPs

Consider the homogenous form of the nondifferentiable fractional initial value problems

$$L^{s\alpha} u(x, t) = \beta(t)(Ru(x, t) + Fu(x, t)), \quad 0 < \alpha \leq 1, \tag{3.1}$$

subject to the initial conditions

$$\frac{\partial^{(s-1)\alpha} u(x, 0)}{\partial t^{(s-1)\alpha}} = f_{s-1}(x), \quad s = 1, 2, \ldots, \tag{3.2}$$

where $L^{s\alpha} = \partial^{s\alpha} / \partial t^{s\alpha}$ is the highest fractional partial derivative with respect to t in terms of the Jumarie fractional derivative, and R is a linear bounded operator, which means that it is possible to find a number ϵ such that $\|Ru\| \leq \epsilon \|u\|$ and $|\beta(\tau)| \leq \eta$ for all $0 \leq \tau \leq t \leq T$ such that $t \in I = [0, T]$ and η is a constant. Here, we use any norm. $F(u)$ is the nonlinear term which is Lipschitzian by $|F(u) - F(v)| \leq L|u - v|$ for any positive constant L.

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Define the inverse operator $L^{-s\alpha}$ in terms of the Jumarie derivative in the following formula:

$$L^{-s\alpha}(\cdot) = \frac{1}{\Gamma^{s}(\alpha+1)} \int_0^{(t_s)} \int_0^{(t_{s-1})} \cdots \int_0^{(t_1)} (\cdot)(d\tau_1)^{\alpha} \cdots (d\tau_{s-1})^{\alpha}(d\tau_s)^{\alpha}.$$

Applying the inverse operator $L^{-s\alpha}$ to both sides of equation (3.1) gives (for $0 < \alpha \le 1$)

$$u(x,t) = \sum_{k=0}^{s-1} \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} u^{(\alpha k)}(x,0) + L^{-s\alpha}(\beta(t)(Ru(x,t) + Fu(x,t))), \tag{3.3}$$

where the first part of the right-hand side of equation (3.3) is obtained from the solution of the homogenous fractional differential equation $L^{s\alpha}u(x,t) = 0$ using the Taylor series of fractional order introduced by Jumarie [9].

The property of nonlocality for nondifferentiable IVPs is discussed in the following theorem.

THEOREM 3.1. *The solution of the homogenous form of the nondifferentiable fractional initial value problem* (3.1) *has the property of nonlocality for any* $0 < \alpha \le 1$ *and* $s = 1$, *which means that, for two different values of* t, *say* t_1, t_2 *such that* $t_1 < t_2$, *in order to compute the solution at the point* t_2 *we must know the value of* $u(x,t)$ *at the starting point* t_1.

PROOF. From equation (3.3) the truncated solution at the two points t_1 and t_2 has the form

$$\begin{aligned} u(x,t_1) &= u(x,0) + \frac{1}{\Gamma(1+\alpha)} \int_0^{t_1} (\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))(d\tau)^{\alpha} \\ &= u(x,0) + \frac{\alpha}{\Gamma(1+\alpha)} \int_0^{t_1} (t_1 - \tau)^{\alpha-1}(\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))\, d\tau. \end{aligned}$$

In the same manner,

$$u(x,t_2) = u(x,0) + \frac{\alpha}{\Gamma(1+\alpha)} \int_0^{t_2} (t_2 - \tau)^{\alpha-1}(\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))\, d\tau.$$

Then

$$\begin{aligned} u(x,t_2) - u(x,t_1) &= \frac{\alpha}{\Gamma(1+\alpha)} \int_0^{t_2} (t_2 - \tau)^{\alpha-1}(\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))\, d\tau \\ &\quad - \frac{\alpha}{\Gamma(1+\alpha)} \int_0^{t_1} (t_1 - \tau)^{\alpha-1}(\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))\, d\tau, \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}](\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))\, d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1}(\beta(\tau)(Ru(x,\tau) + Fu(x,\tau)))\, d\tau. \end{aligned} \tag{3.4}$$

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If $\alpha = 1$, then the first integral on the right-hand side of equation (3.4) vanishes. Then, if we already know the solution $u(x, t_1)$ of our given initial value problem (3.1) at the point $t_1 > 0$, we may compute the solution at the point $t_2 > t_1$ on the basis of $u(x, t_1)$ and the function $\beta(t)(Ru(x, t) + Fu(x, t))$, which expresses the locality of integer first-order initial value problems. But, if $0 < \alpha \leq 1$, then the first integral on the right-hand side of equation (3.4) does not vanish, so we must take into account the entire history of $u(x, t)$ from the starting point to the point of interest, which expresses the nonlocality of the homogenous form of nondifferentiable fractional initial value problems. □

According to Theorem 3.1, we are going to approximate the first integral on the right-hand side of equation (3.1) by using the new Adomian polynomials introduced in the next section.

The Adomian decomposition method defines the solution $u(x, t)$ of equation (3.1) as an infinite series in the form [2, 4]

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \tag{3.5}$$

where the components $u_n(x, t)$ can be obtained recursively. The nonlinear term $F(u)$ can be decomposed by an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} \bar{A}_n, \tag{3.6}$$

where the components $\bar{A}_n$ can be obtained using the following formula:

$$\begin{aligned}\bar{A}_n &= \frac{1}{(ns)!}\bigg[\frac{d^{ns}}{d\lambda^{ns}}F\Big(\sum_{i=0}^{\infty}\lambda^i h_i t^{\alpha i}\Big)\bigg]_{\lambda=0} + \frac{1}{(ns+1)!}\bigg[\frac{d^{(ns+1)}}{d\lambda^{(ns+1)}}F\Big(\sum_{i=0}^{\infty}\lambda^i h_i t^{\alpha i}\Big)\bigg]_{\lambda=0} + \cdots \\ &\quad + \frac{1}{(ns+s-1)!}\bigg[\frac{d^{(ns+s-1)}}{d\lambda^{(ns+s-1)}}F\Big(\sum_{i=0}^{\infty}\lambda^i h_i t^{\alpha i}\Big)\bigg]_{\lambda=0}, \qquad n = 0, 1, \ldots, \end{aligned} \tag{3.7}$$

where the h_n are the coefficients of $t^{n\alpha}$ in the components $u_n(x, t)$.

The formula (3.7) is a generalization of the formula introduced by Abassy [1] at $\alpha = 1$. Also there are many approaches which were introduced to improve the accuracy of the Adomian decomposition method [5, 18].

From (3.7) we deduce the standard formula of Adomian's polynomials

$$A_n = \frac{1}{n!}\bigg[\frac{d^n}{d\lambda^n}F\Big(\sum_{i=0}^{\infty}\lambda^i u_i\Big)\bigg]_{\lambda=0}, \qquad n = 0, 1, \ldots. \tag{3.8}$$

Substituting (3.5) and (3.6) into equation (3.3) gives

$$\sum_{n=0}^{\infty} u_n(x, t) = \sum_{k=0}^{s-1} \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} u^{(\alpha k)}(x, 0) + L^{-s\alpha}\bigg[\beta(\tau)\Big(R\Big(\sum_{n=0}^{\infty} u_n\Big) + \sum_{n=0}^{\infty} \bar{A}_n\Big)\bigg].$$

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Substituting the initial conditions (3.2), we can obtain the components $u_n(x,t)$ of the solution by the following formula:

$$u_0(x,t) = f_0(x) + \frac{f_1(x)}{\Gamma(1+\alpha)}t^{\alpha} + \cdots + \frac{f_{s-1}(x)}{\Gamma(1+(s-1)\alpha)}t^{(s-1)\alpha},$$
$$u_{n+1}(x,t) = L^{-s\alpha}[\beta(\tau)(Ru_n + \bar{A}_n)], \quad n \geq 0.$$

4. The derivation of a new formula for Adomian's polynomials

If we take a look at formula (3.7), which is used to decompose the nonlinear term $F(u)$ instead of Adomian's formula (3.8), we can see easily that the two formulae are equivalent at $s = 1$ but formula (3.7) leads to a more accurate and convergent solution than formula (3.8), which degrades the convergence of the solution at $s = 2, 3, \ldots$. By rearranging the terms of Adomian's polynomials at $s = 1$ in formula (3.7) in the form of a Taylor series about $f_0(x)$, we see that it gives the same rearrangement of Adomian's polynomials that El-Kalla [5, 6] deduces from the rearrangement of the old polynomials in formula (3.8) about the first component $u_0(x,t)$ as there is no difference between formula (3.7) at $s = 1$ and the old formula (3.8).

But rearranging the terms of Adomian's polynomials at $s = 2$ in formula (3.7) in the form of a Taylor series about $f_0(x)$ gives another rearrangement, as follows:

$$\begin{aligned}
\hat{A}_0 &= F[f_0(x)] + t^{\alpha} f_1(x)F^{(1)}[f_0(x)] + \frac{1}{2!}t^{2\alpha} f_1^2(x)F^{(2)}[f_0(x)] \\
&\quad + \frac{1}{3!}t^{3\alpha} f_1^3(x)F^{(3)}[f_0(x)] + \cdots, \\
\hat{A}_1 &= (t^{2\alpha} f_2(x) + t^{3\alpha} f_3(x))F^{(1)}[f_0(x)] + \frac{1}{2!}(2f_1(x)f_2(x)t^{3\alpha} + f_2^2(x)t^{4\alpha} \\
&\quad + 2f_1(x)f_3(x)t^{4\alpha} + 2f_2(x)f_3(x)t^{5\alpha} \\
&\quad + f_3^2(x)t^{6\alpha})F^{(2)}[f_0(x)] + \frac{1}{3!}(3f_1^2(x)f_2(x)t^{4\alpha} + 3f_1(x)f_2^2(x)t^{5\alpha} + \cdots \\
&\quad + f_3^3(x)t^{9\alpha})F^{(3)}[f_0(x)] + \cdots, \\
\hat{A}_2 &= (t^{4\alpha} f_4(x) + t^{5\alpha} f_5(x))F^{(1)}[f_0(x)] + \frac{1}{2!}(2f_1(x)f_4(x)t^{5\alpha} + 2f_2(x)f_4(x)t^{6\alpha} + \cdots \\
&\quad + f_5^2(x)t^{10\alpha})F^{(2)}[f_0(x)] \\
&\quad + \frac{1}{3!}(3f_1^2(x)f_4(x)t^{6\alpha} + 6f_1(x)f_2(x)f_4(x)t^{7\alpha} + \cdots + f_5^3(x)t^{15\alpha})F^{(3)}[f_0(x)] + \cdots,
\end{aligned}$$

and so on.

Similarly, rearranging the terms of Adomian's polynomials at $s = 3, 4, \ldots$ in the form of a Taylor series about $f_0(x)$ in formula (3.7) gives other rearrangements.

To deduce the general form for the new formula for Adomian's polynomials which includes all states at $s = 1, 2, \ldots$, we go through the following steps.

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(1) The first case at $s = 1$, using the Taylor series expansion [7]:

$$\begin{aligned}\hat{A}_{1,0} &= F[f_0(x)],\\ \hat{A}_{1,1} &= F[f_0(x) + f_1(x)t^{\alpha}] - \hat{A}_{1,0},\\ \hat{A}_{1,2} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha}] - \hat{A}_{1,1} - \hat{A}_{1,0},\\ \hat{A}_{1,3} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha}] - \hat{A}_{1,2} - \hat{A}_{1,1} - \hat{A}_{1,0},\end{aligned}$$

and so on, then

$$\begin{aligned}\hat{A}_{1,0} &= F[f_0(x)] = F(W_0),\\ \hat{A}_{1,0} + \hat{A}_{1,1} &= F[f_0(x) + f_1(x)t^{\alpha}] = F(W_{0,1}),\\ \hat{A}_{1,0} + \hat{A}_{1,1} + \hat{A}_{1,2} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha}] = F(W_{0,1,2}),\\ \hat{A}_{1,0} + \hat{A}_{1,1} + \hat{A}_{1,2} + \hat{A}_{1,3} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha}] = F(W_{0,1,2,3}),\end{aligned}$$

and so on, then

$$\sum_{i=0}^{n} \hat{A}_{1,i} = F(W_{0,1,\ldots,n}),$$

and

$$\hat{A}_{1,n} = F(W_{0,1,\ldots,n}) - \sum_{i=0}^{n-1} \hat{A}_{1,i} = F(W_{0,1,\ldots,n}) - F(W_{0,1,\ldots,n-1}). \tag{4.1}$$

(2) The second case at $s = 2$, using the Taylor series expansion [7]:

$$\begin{aligned}\hat{A}_{2,0} &= F[f_0(x) + f_1(x)t^{\alpha}],\\ \hat{A}_{2,1} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha}] - \hat{A}_{2,0},\\ \hat{A}_{2,2} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha} + f_4(x)t^{4\alpha} + f_5(x)t^{5\alpha}] - \hat{A}_{2,1} - \hat{A}_{2,0},\\ \hat{A}_{2,3} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + \cdots + f_6(x)t^{6\alpha} + f_7(x)t^{7\alpha}] - \hat{A}_{2,2} - \hat{A}_{2,1} - \hat{A}_{2,0},\end{aligned}$$

and so on, then

$$\begin{aligned}\hat{A}_{2,0} &= F[f_0(x) + f_1(x)t^{\alpha}] = F(W_{0,1}),\\ \hat{A}_{2,0} + \hat{A}_{2,1} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha}] = F(W_{0,1,\ldots,3}),\\ \hat{A}_{2,0} + \hat{A}_{2,1} + \hat{A}_{2,2} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha} + f_4(x)t^{4\alpha} + f_5(x)t^{5\alpha}]\\ &= F(W_{0,1,\ldots,5}),\\ \hat{A}_{2,0} + \hat{A}_{2,1} + \hat{A}_{2,2} + \hat{A}_{2,3} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + \cdots + f_6(x)t^{6\alpha} + f_7(x)t^{7\alpha}]\\ &= F(W_{0,1,\ldots,7}),\end{aligned}$$

and so on, then

$$\sum_{i=0}^{n} \hat{A}_{2,i} = F(W_{0,1,\ldots,2n+1}),$$

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and

$$\hat{A}_{2,n} = F(W_{0,1,\ldots,2n+1}) - \sum_{i=0}^{n-1} \hat{A}_{2,i} = F(W_{0,1,\ldots,2n+1}) - F(W_{0,1,\ldots,2n-1}). \tag{4.2}$$

(3) The third case at $s = 3$, using the Taylor series expansion [7]:

$$\begin{aligned}
\hat{A}_{3,0} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha}],\\
\hat{A}_{3,1} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_4(x)t^{4\alpha} + f_5(x)t^{5\alpha}] - \hat{A}_{3,0},\\
\hat{A}_{3,2} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_7(x)t^{7\alpha} + f_8(x)t^{8\alpha}] - \hat{A}_{3,1} - \hat{A}_{3,0},\\
\hat{A}_{3,3} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{10}(x)t^{10\alpha} + f_{11}(x)t^{11\alpha}] - \hat{A}_{3,2} - \hat{A}_{3,1} - \hat{A}_{3,0},
\end{aligned}$$

and so on, then

$$\begin{aligned}
\hat{A}_{3,0} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha}] = F(W_{0,1,2}),\\
\hat{A}_{3,0} + \hat{A}_{3,1} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_4(x)t^{4\alpha} + f_5(x)t^{5\alpha}] = F(W_{0,1,\ldots,5}),\\
\hat{A}_{3,0} + \hat{A}_{3,1} + \hat{A}_{3,2} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_7(x)t^{7\alpha} + f_8(x)t^{8\alpha}] = F(W_{0,1,\ldots,8}),\\
\hat{A}_{3,0} + \hat{A}_{3,1} + \hat{A}_{3,2} + \hat{A}_{3,3} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{10}(x)t^{10\alpha} + f_{11}(x)t^{11\alpha}]\\
&= F(W_{0,1,\ldots,11}),
\end{aligned}$$

and so on, then

$$\sum_{i=0}^{n} \hat{A}_{3,i} = F(W_{0,1,\ldots,3n+2}),$$

and

$$\hat{A}_{3,n} = F(W_{0,1,\ldots,3n+2}) - \sum_{i=0}^{n-1} \hat{A}_{3,i} = F(W_{0,1,\ldots,3n+2}) - F(W_{0,1,\ldots,3n-1}). \tag{4.3}$$

Thus, we can generalize these cases to deduce the sth formula:

$$\begin{aligned}
\hat{A}_{s,0} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{s-1}(x)t^{(s-1)\alpha}],\\
\hat{A}_{s,1} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + \cdots + f_{2s-1}(x)t^{(2s-1)\alpha}],\\
\hat{A}_{s,2} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha} + \cdots + f_{3s-1}(x)t^{(3s-1)\alpha}],\\
\hat{A}_{s,3} &= F[f_0(x) + f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha} + f_4(x)t^{4\alpha} + \cdots + f_{4s-1}(x)t^{(4s-1)\alpha}],
\end{aligned}$$

and so on, then

$$\begin{aligned}
\hat{A}_{s,0} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{s-1}(x)t^{(s-1)\alpha}] = F(W_{0,1,\ldots,s-1}),\\
\hat{A}_{s,0} + \hat{A}_{s,1} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{2s-1}(x)t^{(2s-1)\alpha}] = F(W_{0,1,\ldots,2s-1}),\\
\hat{A}_{s,0} + \hat{A}_{s,1} + \hat{A}_{s,2} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{3s-1}(x)t^{(3s-1)\alpha}] = F(W_{0,1,\ldots,3s-1}),\\
\hat{A}_{s,0} + \hat{A}_{s,1} + \hat{A}_{s,2} + \hat{A}_{s,3} &= F[f_0(x) + f_1(x)t^{\alpha} + \cdots + f_{4s-1}(x)t^{(4s-1)\alpha}] = F(W_{0,1,\ldots,4s-1}),
\end{aligned}$$

and so on, then

$$\sum_{i=0}^{n} \hat{A}_{s,i} = F(W_{0,1,\ldots,ns+s-1}),$$

and

$$\hat{A}_{s,n} = F(W_{0,1,\ldots,ns+s-1}) - \sum_{i=0}^{n-1} \hat{A}_{s,i} = F(W_{0,1,\ldots,ns+s-1}) - F(W_{0,1,\ldots,(n-1)s+s-1}). \tag{4.4}$$

Equation (4.4) gives us a new formula for Adomian's polynomials. This formula is equivalent to equation (4.1), which is similar to El-Kalla's formula [5, 6] at $s = 1$ and equivalent to equations (4.2) and (4.3) at $s = 2$ and $s = 3$, respectively. More focus on this new formula is given in the following sections.

In the following theorem, we introduce a new exponential notation formula for the proposed new formula for Adomian's polynomials (4.4).

Theorem 4.1. *The exponential notation formula which is equivalent to the new formula (4.4) has the form*

$$\begin{aligned}\theta_{s,1} &= F(W_{0,1,\ldots,s-1}),\\ \theta_{s,2} &= e^{(W_{0,1,\ldots,s-1}-f_0(x))(\partial/\partial f_0)} F[f_0(x)], \ldots,\\ \theta_{s,n+1} &= e^{u_{s,n}(\partial/\partial f_0)}\theta_{s,n}, \quad n > 1,\end{aligned}$$

where, $u_{s,n}$ is the nth recurrent term at any s.

Proof. Assume that $\theta_{s,n+1} = \hat{A}_{s,0} + \hat{A}_{s,1} + \cdots + \hat{A}_{s,n}$. For the first case $s = 1$,

$$\begin{aligned}\theta_{1,1} &= \hat{A}_{1,0} = F[f_0(x)] = F(W_0),\\ \theta_{1,2} &= \hat{A}_{1,0} + \hat{A}_{1,1} = \sum_{v=0}^{\infty} \frac{(f_1(x)t^{\alpha})^v}{v!} \frac{\partial^v}{\partial(f_0(x))^v} F[f_0(x)]\\ &= e^{(W_{0,1}-f_0(x))(\partial/\partial f_0(x))} F[f_0(x)] = e^{(W_{0,1}-f_0(x))(\partial/\partial f_0(x))}\theta_{1,1}.\end{aligned}$$

In the same manner,

$$\begin{aligned}\theta_{1,3} &= \hat{A}_{1,0} + \hat{A}_{1,1} + \hat{A}_{1,2} = \sum_{v=0}^{\infty} \frac{(f_1(x)t^{\alpha} + f_2(x)t^{2\alpha})^v}{v!} \frac{\partial^v}{\partial(f_0(x))^v} F[f_0(x)]\\ &= e^{(W_{0,1,2}-f_0(x))(\partial/\partial f_0(x))} F[f_0(x)] = e^{(f_1(x)t^{\alpha}+f_2(x)t^{2\alpha})(\partial/\partial f_0(x))}\theta_{1,1}\\ &= e^{(f_2(x)t^{2\alpha})(\partial/\partial f_0(x))}\theta_{1,2} = e^{u_{1,2}(\partial/\partial f_0(x))}\theta_{1,2}.\end{aligned}$$

In a similar way, we obtain the general formula at $s = 1$:

$$\theta_{1,n+1} = e^{u_{1,n}(\partial/\partial f_0(x))}\theta_{1,n}, \quad n > 1.$$

For the second case at $s = 2$,

$$\begin{aligned}\theta_{2,1} &= \hat{A}_{2,0} = F(W_{0,1}),\\ \theta_{2,2} &= \hat{A}_{2,0} + \hat{A}_{2,1} = F(W_{0,1,\ldots,3})\\ &= \sum_{v=0}^{\infty} \frac{(f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + f_3(x)t^{3\alpha})^v}{v!} \frac{\partial^v}{\partial(f_0(x))^v} F[f_0(x)]\\ &= e^{(W_{0,1,\ldots,3}-f_0(x))(\partial/\partial f_0(x))} F[f_0(x)],\end{aligned}$$

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$$
\begin{aligned}
\theta_{2,3} &= \hat{A}_{2,0} + \hat{A}_{2,1} + \hat{A}_{2,2} = F(W_{0,1,\dots,5}) \\
&= \sum_{\nu=0}^{\infty} \frac{(f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + \cdots + f_5(x)t^{5\alpha})^{\nu}}{\nu!} \frac{\partial^{\nu}}{\partial(f_0(x))^{\nu}} F[f_0(x)] \\
&= e^{(W_{0,1,\dots,5} - f_0(x))(\partial/\partial f_0(x))} F[f_0(x)] = e^{u_{2,2}(\partial/\partial f_0(x))} \theta_{2,2}.
\end{aligned}
$$

In a similar way, we obtain the general formula at $s = 2$:

$$
\theta_{2,n+1} = e^{u_{2,n}(\partial/\partial f_0(x))} \theta_{2,n}, \qquad n > 1.
$$

For the third case at $s = 3$,

$$
\begin{aligned}
\theta_{3,1} &= \hat{A}_{3,0} = F(W_{0,1,2}), \\
\theta_{3,2} &= \hat{A}_{3,0} + \hat{A}_{3,1} = F(W_{0,1,\dots,5}) \\
&= \sum_{\nu=0}^{\infty} \frac{(f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + \cdots + f_5(x)t^{5\alpha})^{\nu}}{\nu!} \frac{\partial^{\nu}}{\partial(f_0(x))^{\nu}} F[f_0(x)] \\
&= e^{(W_{0,1,\dots,5} - f_0(x))(\partial/\partial f_0(x))} F[f_0(x)].
\end{aligned}
$$

In the same manner,

$$
\begin{aligned}
\theta_{3,3} &= \hat{A}_{3,0} + \hat{A}_{3,1} + \hat{A}_{3,2} = F(W_{0,1,\dots,8}) \\
&= \sum_{\nu=0}^{\infty} \frac{(f_1(x)t^{\alpha} + f_2(x)t^{2\alpha} + \cdots + f_8(x)t^{8\alpha})^{\nu}}{\nu!} \frac{\partial^{\nu}}{\partial(f_0(x))^{\nu}} F[f_0(x)] \\
&= e^{(W_{0,1,\dots,8} - f_0(x))(\partial/\partial f_0(x))} F[f_0(x)] = e^{u_{3,2}(\partial/\partial f_0(x))}\, \theta_{3,2}.
\end{aligned}
$$

Similarly, we obtain the general formula at $s = 3$:

$$
\theta_{3,n+1} = e^{u_{3,n}(\partial/\partial f_0(x))} \theta_{3,n}, \qquad n > 1.
$$

Similarly, we can prove the exponential notation formula at any s which is equivalent to the proposed new formula (4.4), which completes the proof. □

5. Convergence analysis

In this section, we introduce the condition that guarantees the existence of a unique solution in Theorems 5.1 and 5.2, prove the convergence of the series solution in Theorem 5.3, and estimate the maximum of the absolute error of the truncated series in Theorem 5.4.

Denote by $\varphi = (C[I], \|\cdot\|)$ the Banach space of all continuous functions on $I = [0, T]$ with the norm $\|u(x,t)\| = \max_{t\in I} e^{-kt}|u(x,t)|$, $k > 0$, for any constant k.

THEOREM 5.1. *The problem of the form* (3.1) *at* $s = 1$ *has a unique solution for* $0 < \gamma < 1$ *where* $\gamma = (L + \epsilon)\eta/k^{\alpha}$ *where the constants* L, ϵ, η *and* k *are defined after equation* (3.2).

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PROOF. Define a mapping $\psi : \varphi \to \varphi$ such that

$$\begin{aligned}\psi u &= f_0(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \beta(\tau)(R(u) + F(u))(d\tau)^\alpha \\ &= f_0(x) + \frac{\alpha}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1}\beta(\tau)(R(u) + F(u))\, d\tau.\end{aligned}$$

Let $u, v \in \varphi$ be two different solutions of equation (3.1). Then

$$\begin{aligned}\|\psi u - \psi v\| &= \max_{t\in I} e^{-kt}\left|\frac{\alpha}{\Gamma(1+\alpha)}\left[\int_0^t (t-\tau)^{\alpha-1}\beta(\tau)((F(u)-F(v)) + (R(u)-R(v)))\, d\tau\right]\right| \\ &\leq \frac{1}{\Gamma(\alpha)} \max_{t\in I} e^{-kt}\left[\int_0^t (t-\tau)^{\alpha-1}|\beta(\tau)|\, |(F(u)-F(v)) + (R(u)-R(v))|\, d\tau\right] \\ &\leq \frac{(L+\epsilon)\eta}{\Gamma(\alpha)} \max_{t\in I} e^{-kt}\left[\int_0^t (t-\tau)^{\alpha-1}|u-v|\, d\tau\right],\end{aligned}$$

where $e^{-kt} = e^{-k\tau}e^{-k(t-\tau)}$, and hence

$$\begin{aligned}\|\psi u - \psi v\| &\leq \frac{(L+\epsilon)\eta}{\Gamma(\alpha)} \max_{t\in I}\left[\int_0^t (e^{-k\tau}|u-v|)e^{-k(t-\tau)}(t-\tau)^{\alpha-1}\, d\tau\right] \\ &\leq \frac{(L+\epsilon)\eta}{\Gamma(\alpha)} \max_{t\in I}(e^{-k\tau}|u-v|)\left[\int_0^t e^{-k(t-\tau)}(t-\tau)^{\alpha-1}\, d\tau\right].\end{aligned}$$

Let $t - \tau = s$. Then the integration on the right-hand side of the above inequality becomes less than $I = \int_0^\infty e^{-ks}s^{\alpha-1}\, ds$, so

$$\|\psi u - \psi v\| \leq \frac{(L+\epsilon)\eta}{\Gamma(\alpha)}\|u - v\| \times I.$$

The change of variables $ks = u$ in the integration I yields $I = k^{-\alpha}\,\Gamma(\alpha)$. Then

$$\|\psi u - \psi v\| \leq \frac{(L+\epsilon)\eta}{k^\alpha}\|u - v\| = \gamma\,\|u - v\|.$$

Under the criterion $0 < \gamma < 1$, the mapping $\psi : \varphi \to \varphi$ is a contraction. Therefore, by the Banach fixed-point theorem, there exists a unique solution to the problem (3.1), which completes the proof. □

In the following theorem, we discuss the existence of a unique solution for the problem (3.1) at any s.

THEOREM 5.2. *The problem of the form* (3.1) *has a unique solution for* $0 < \delta < 1$, *where*

$$\delta = \frac{(L+\epsilon)\eta\, \alpha^s\Gamma(\alpha)}{\Gamma^s(\alpha+1)k^\alpha}\frac{T^{\alpha(s-1)}}{\alpha^{s-1}}.$$

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PROOF. Define a mapping $\psi : \varphi \to \varphi$ such that

$$\psi u = \sum_{k=0}^{s-1} \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} u^{\alpha k}(x,0) + L^{-s\alpha}(\beta(\tau)(R(u)+F(u))).$$

Let $u, v \in \varphi$ be two different solutions of equation (3.1). Then

$$\begin{aligned}
\|\psi u - \psi v\| &= \max_{t\in I} e^{-kt} |L^{-s\alpha}(\beta(\tau)((R(u)-R(v)) + (F(u)-F(v))))| \\
&\leq \max_{t\in I} e^{-kt} [L^{-s\alpha} |\beta(\tau)|\, |(R(u)-R(v)) + (F(u)-F(v))|] \\
&\leq (L+\epsilon)\eta \max_{t\in I} e^{-kt} L^{-s\alpha} |u-v| \\
&\leq (L+\epsilon)\eta \max_{t\in I} [L^{-s\alpha}(e^{-k\tau}|u-v|) e^{-k(t-\tau)}] \\
&\leq (L+\epsilon)\eta \max_{t\in I} (e^{-k\tau}|u-v|)[L^{-s\alpha} e^{-k(t-\tau)}].
\end{aligned}$$

Then $\|\psi u - \psi v\| \leq (L+\epsilon)\eta \max_{t\in I}(e^{-k\tau}|u-v|) \times I$, where

$$\begin{aligned}
I &= L^{-s\alpha}(e^{-k(t-\tau)}) \\
&= \frac{1}{\Gamma^s(\alpha+1)} \int_0^t \int_0^t \cdots \int_0^t (e^{-k(t-\tau)})(d\tau)^\alpha \cdots (d\tau)^\alpha (d\tau)^\alpha \\
&= \frac{\alpha^s}{\Gamma^s(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \int_0^t (t-\tau)^{\alpha-1} \cdots \int_0^t (t-\tau)^{\alpha-1} (e^{-k(t-\tau)})\, d\tau \cdots d\tau\, d\tau.
\end{aligned}$$

Let $t-\tau = s$. Then the inner integration $\int_0^t (t-\tau)^{\alpha-1}(e^{-k(t-\tau)})\, d\tau$ will be less than $\int_0^\infty s^{\alpha-1}(e^{-ks})\, ds$. The change of variables $ks = u$ yields $k^{-\alpha} \int_0^\infty s^{\alpha-1} e^{-s}\, ds = k^{-\alpha}\Gamma(\alpha)$. Then

$$I \leq \frac{\alpha^s \Gamma(\alpha)}{\Gamma^s(\alpha+1) k^\alpha} \frac{T^{\alpha(s-1)}}{\alpha^{s-1}}.$$

Therefore

$$\begin{aligned}
\|\psi u - \psi v\| &\leq (L+\epsilon)\eta \max_{t\in I}(e^{-k\tau}|u-v|) \times \frac{\alpha^s \Gamma(\alpha)}{\Gamma^s(\alpha+1) k^\alpha} \frac{T^{\alpha(s-1)}}{\alpha^{s-1}} \\
&\leq \frac{(L+\epsilon)\eta\alpha^s \Gamma(\alpha)}{\Gamma^s(\alpha+1) k^\alpha} \frac{T^{\alpha(s-1)}}{\alpha^{s-1}} \|u-v\| = \delta\, \|u-v\|.
\end{aligned}$$

Under the criterion $0 < \delta < 1$, the mapping $\psi : \varphi \to \varphi$ is a contraction. Therefore, by the Banach fixed-point theorem, there exists a unique solution to the problem (3.1), which completes the proof. □

If we put $s = 1$ in Theorem 5.2, we obtain the condition which leads to the existence of a unique solution in Theorem 5.1, which means that Theorem 5.1 is a special case of Theorem 5.2 at $s = 1$.

In the following theorem, the convergence for the series solution (3.5) is discussed.

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THEOREM 5.3. *The series solution (3.5) of the problem (3.1) is convergent whenever* $0 < \delta < 1$,

$$\delta = (L + \epsilon)\eta \frac{\alpha^s \Gamma(\alpha)}{\Gamma^s(\alpha + 1)k^\alpha} \frac{T^{\alpha(s-1)}}{\alpha^{s-1}},$$

and the linear operator R is bounded.

PROOF. Let $W_{0,1,\ldots,ns+s-1}$ and $W_{0,1,\ldots,ms+s-1}$ be two partial sums with $n > m$. Now we prove that $W_{0,1,\ldots,ns+s-1}$ is a Cauchy sequence in the Banach space φ:

$$\begin{aligned}\|W_{0,1,\ldots,ns+s-1} - W_{0,1,\ldots,ms+s-1}\| &= \max\nolimits_{t\in I} e^{-kt}|W_{0,1,\ldots,ns+s-1} - W_{0,1,\ldots,ms+s-1}| \\ &= \max\nolimits_{t\in I} e^{-kt}\Big|\sum_{i=m+1}^{n} u_{s,i}(x,t)\Big| \\ &= \max\nolimits_{t\in I} e^{-kt}\Big|\sum_{i=m+1}^{n} L^{-s\alpha}[\beta(\tau)(Ru_{s,i-1} + \hat{A}_{s,i-1})]\Big| \\ &= \max\nolimits_{t\in I} e^{-kt}\Big|\sum_{i=m}^{n-1} L^{-s\alpha}[\beta(\tau)(Ru_{s,i} + \hat{A}_{s,i})]\Big|.\end{aligned}$$

Using equation (4.4),

$$\sum_{i=m}^{n-1} \hat{A}_{s,i} = F(W_{0,1,\ldots,(n-1)s+s-1}) - F(W_{0,1,\ldots,(m-1)s+s-1}),$$

$$\sum_{i=m}^{n-1} u_{s,i} = W_{0,1,\ldots,(n-1)s+s-1} - W_{0,1,\ldots,(m-1)s+s-1}.$$

Then

$$\begin{aligned}&\|W_{0,1,\ldots,ns+s-1} - W_{0,1,\ldots,ms+s-1}\| \\ &\quad = \max_{t\in I} e^{-kt}\Big|L^{-s\alpha}\Big[\beta(\tau)\Big(\sum_{i=m}^{n-1} Ru_{s,i} + (F(W_{0,1,\ldots,(n-1)s+s-1}) - F(W_{0,1,\ldots,(m-1)s+s-1}))\Big)\Big]\Big|.\end{aligned}$$

But

$$|F(W_{0,1,\ldots,(n-1)s+s-1}) - F(W_{0,1,\ldots,(m-1)s+s-1})| \le L\,\|W_{0,1,\ldots,(n-1)s+s-1} - W_{0,1,\ldots,(m-1)s+s-1}\|,$$

$$\sum_{i=m}^{n-1} Ru_{s,i} \le \sum_{i=m}^{n-1} \epsilon|u_{s,i}| = \epsilon(W_{0,1,\ldots,(n-1)s+s-1} - W_{0,1,\ldots,(m-1)s+s-1}).$$

Thus

$$\begin{aligned}&\|W_{0,1,\ldots,ns+s-1} - W_{0,1,\ldots,ms+s-1}\| \\ &\quad \le \max\nolimits_{t\in I} \eta(L+\epsilon)|L^{-s\alpha}[e^{-k\tau}(W_{0,1,\ldots,(n-1)s+s-1} - W_{0,1,\ldots,(m-1)s+s-1}) \times e^{-k(t-\tau)}]| \\ &\quad \le \frac{\eta(L+\epsilon)\alpha^s}{\Gamma^s(1+\alpha)}\|W_{0,1,\ldots,(n-1)s+s-1} - W_{0,1,\ldots,(m-1)s+s-1}\| \times I,\end{aligned}$$

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where $I = L^{-s\alpha}(e^{-k(t-\tau)})$. Therefore

$$\|W_{0,1,\dots,ns+s-1} - W_{0,1,\dots,ms+s-1}\| \le \delta \|W_{0,1,\dots,(n-1)s+s-1} - W_{0,1,\dots,(m-1)s+s-1}\|,$$

where

$$\delta = (L+\epsilon)\eta \frac{\alpha^s \Gamma(\alpha)}{\Gamma^s(\alpha+1)k^\alpha} \frac{T^{\alpha(s-1)}}{\alpha^{s-1}}.$$

Let $n = m+1$. Then

$$\begin{aligned}\|W_{0,1,\dots,(m+1)s+s-1} - W_{0,1,\dots,ms+s-1}\| &\le \delta \|W_{0,1,\dots,ms+s-1} - W_{0,1,\dots,(m-1)s+s-1}\| \\ &\le \delta^2 \|W_{0,1,\dots,(m-1)s+s-1} - W_{0,1,\dots,(m-2)s+s-1}\| \\ &\vdots \\ &\le \delta^m \|W_{0,1,\dots,2s-1} - W_{0,1,\dots,s-1}\|.\end{aligned}$$

By using the triangle inequality,

$$\begin{aligned}\|W_{0,1,\dots,ns+s-1} - W_{0,1,\dots,ms+s-1}\| \le [&\|W_{0,1,\dots,(m+1)s+s-1} - W_{0,1,\dots,ms+s-1}\| \\ &+ \cdots + \|W_{0,1,\dots,ns+s-1} - W_{0,1,\dots,(n-1)s+s-1}\|].\end{aligned}$$

Then

$$\begin{aligned}\|W_{0,1,\dots,ns+s-1} - W_{0,1,\dots,ms+s-1}\| &\le [\delta^m + \delta^{m+1} + \cdots + \delta^{n-1}]\|W_{0,1,\dots,2s-1} - W_{0,1,\dots,s-1}\| \\ &\le \delta^m[1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}]\|W_{0,1,\dots,2s-1} - W_{0,1,\dots,s-1}\| \\ &\le \delta^m\Big[\frac{1-\delta^{n-m}}{1-\delta}\Big]\|u_{s,1}(x,t)\|.\end{aligned}$$

As $0 < \delta < 1$, $1 - \delta^{n-m} < 1$, which leads to

$$\|W_{0,1,\dots,ns+s-1} - W_{0,1,\dots,ms+s-1}\| \le \frac{\delta^m}{1-\delta} \max_{t\in I} e^{-kt}|u_{s,1}(x,t)|.$$

So $W_{0,1,\dots,ns+s-1}$ is a Cauchy sequence in the space φ as $\max_{t\in I} e^{-kt}|u_{s,1}(x,t)| < \infty$ and $\|W_{0,1,\dots,ns+s-1} - W_{0,1,\dots,ms+s-1}\| \to 0$ if $m \to \infty$, which completes the proof. □

In the following theorem, the estimation of the maximum absolute truncation error is presented.

Theorem 5.4. *The maximum absolute truncation error of the series solution* (3.5) *is bounded as follows:*

$$\max_{t\in I}\Big|u(x,t) - \sum_{i=0}^{m} u_{s,i}(x,t)\Big| \le \frac{\delta^m}{1-\delta} \max_{t\in I} |u_{s,1}(x,t)|.$$

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Proof. From the previous theorem, we find that

$$\|W_{0,1,\ldots,ns+s-1} - W_{0,1,\ldots,ms+s-1}\| \le \frac{\delta^m}{1-\delta} \max_{t\in I} e^{-kt} |u_{s,1}(x,t)|.$$

If $n \to \infty$, then we observe that $W_{0,1,\ldots,ns+s-1} \to u(x,t)$. Then

$$\|u(x,t) - W_{0,1,\ldots,ms+s-1}\| \le \frac{\delta^m}{1-\delta} \max_{t\in I} e^{-kt} |u_{s,1}(x,t)|.$$

So the maximum absolute error in the interval I is computed by

$$\max_{t\in I} \left| u(x,t) - \sum_{i=0}^{m} u_{s,i}(x,t) \right| \le \frac{\delta^m}{1-\delta} \max_{t\in I} |u_{s,1}(x,t)|,$$

which completes the proof. □

6. Applications and numerical results

In this section, we apply the decomposition method with our new Adomian polynomials (IFADM) for nondifferentiable fractional initial value problems. We use Mathematica to perform numerical algorithms to obtain the approximate analytical solutions for three different fractional IVPs.

Example 6.1. Consider the initial value problem for the single-term Jumarie nonlinear fractional Fisher differential equation

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + 6u(x,t)[1 - u(x,t)], \quad t > 0, \ \ 0 < \alpha \le 1, \tag{6.1}$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{6.2}$$

The exact solution of this problem is $u(x,t) = (1 + e^{(x-5t)})^{-2}$.

In order to obtain the truncated approximate solutions, we take the following steps.

(1) Equation (6.1) can be rewritten in the operator form

$$L^\alpha u(x,t) = Ru(x,t) + Fu(x,t), \tag{6.3}$$

where $L^\alpha = (\partial^\alpha/\partial t^\alpha)$, $Ru = u_{xx} + 6\,u$ and $F(u) = -6u^2$.

(2) Applying the inverse operator $L^{-\alpha}$, which is defined by

$$L^{-\alpha}(\cdot) = \frac{1}{\Gamma(1+\alpha)} \int_0^t (\cdot)(d\tau)^\alpha,$$

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to both sides of (6.3) gives

$$u(x,t) = u(x,0) + L^{-\alpha}[Ru(x,t) + Fu(x,t)].$$

Substituting from equations (3.5) and (3.6) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L^{-\alpha}\Big[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} \bar{A}_n\Big],$$

and substituting the initial condition (6.2) the components $u_n(x,t)$ of the solution $u(x,t)$ can be written as

$$\begin{aligned} u_0(x,t) &= \frac{1}{(1+e^x)^2}, \\ u_{n+1}(x,t) &= L^{-\alpha}[Ru_n + \bar{A}_n], \quad n \geq 0, \end{aligned} \tag{6.4}$$

where A_n can be obtained from formula (3.7), (3.8) and (4.4) at $s = 1$.

(3) To obtain the components $u_n(x,t)$ of the solution using the iteration formula (6.4) in the IFADM in which we use formula (3.7), we proceed as follows:

$$u_0(x,t) = \frac{1}{(1+e^x)^2} = h_0(x) \quad \text{and} \quad A_0 = -6(h_0(x))^2,$$

$$u_1(x,t) = L^{-\alpha}\Big[6u_0(x,t) + \frac{\partial^2 u_0(x,t)}{\partial x^2} + A_0\Big] = \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^{\alpha} = h_1(x)t^{\alpha},$$

that is, $h_1(x) = 10e^x/(1+e^x)^3\Gamma(1+\alpha)$ and $A_1 = -12(h_0(x))h_1(x)t^{\alpha}$,

$$u_2(x,t) = L^{-\alpha}\Big[6u_1(x,t) + \frac{\partial^2 u_1(x,t)}{\partial x^2} + A_1\Big] = \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha} = h_2(x)t^{2\alpha},$$

and, following the same procedure,

$$\begin{aligned} u_3(x,t) = {} & \frac{50\alpha e^x((5-6e^x-15e^{2x}+20e^{3x})\Gamma^2(1+\alpha) - 12e^x\Gamma(1+2\alpha))}{(1+e^x)^6\Gamma^3(1+\alpha)\Gamma(1+2\alpha)} \\ & \times \Gamma(1+\alpha)\Gamma(1+2\alpha)\,{}_2F_1[1, 1+2\alpha, 2+3\alpha, 1]t^{3\alpha}. \end{aligned}$$

So the solution can be approximated by $\phi_3(x,t)$ and defined as follows:

$$\begin{aligned} \phi_3(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \\ &= \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^{\alpha} + \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha} \\ &\quad + \frac{50\alpha e^x((5-6e^x-15e^{2x}+20e^{3x})\Gamma^2(1+\alpha) - 12e^x\Gamma(1+2\alpha))}{(1+e^x)^6\Gamma^3(1+\alpha)\Gamma(1+2\alpha)} \\ &\quad \times \Gamma(1+\alpha)\Gamma(1+2\alpha)\,{}_2F_1[1, 1+2\alpha, 2+3\alpha, 1]t^{3\alpha}. \end{aligned}$$

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(4) To obtain the components $u_{\diamond n}(x,t)$ of the solution we use the iteration formula (6.4) in the IFADM, in which we use the formula (4.4) such that $f_n(x) = h_n(x)$, $n = 0, 1, \ldots,$ which gives

$$u_{\diamond 0}(x,t) = \frac{1}{(1+e^x)^2} \quad \text{and} \quad \hat{A}_0 = F[f_0(x)] = \frac{-6}{(1+e^x)^4},$$
$$u_{\diamond 1}(x,t) = L^{-\alpha}\Big[6u_0(x,t) + \frac{\partial^2 u_0(x,t)}{\partial x^2} + \hat{A}_0\Big] = \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)} t^\alpha,$$

and also

$$\hat{A}_1 = F[f_0(x) + f_1(x)t^\alpha] - \hat{A}_0 = \frac{-600e^{2x}t^{2\alpha}}{(1+e^x)^6\Gamma^2(1+\alpha)} - \frac{120e^x t^\alpha}{(1+e^x)^5\Gamma(1+\alpha)},$$
$$\begin{aligned} u_{\diamond 2}(x,t) &= L^{-\alpha}\Big[6u_1(x,t) + \frac{\partial^2 u_1(x,t)}{\partial x^2} + \hat{A}_1\Big] \\ &= \frac{1}{(1+e^x)^6\Gamma(1+\alpha)^3} 50e^x t^{2\alpha}\Gamma(1+\alpha)\Big(\frac{-12t^\alpha\Gamma(1+2\alpha)(\cosh(x)+\sinh(x))}{\Gamma(1+3\alpha)} \\ &\quad + \frac{4^{-\alpha}\sqrt{\pi}\Gamma(1+\alpha)(1+\cosh(x)+\sinh(x))^2(-1+2\cosh(x)+2\sinh(x))}{\Gamma(\frac{1}{2}+\alpha)}\Big). \end{aligned}$$

Following the same procedure,

$$\begin{aligned} &u_{\diamond 3}(x,t) \\ &= \frac{50e^x t^{3\alpha}\Gamma(1+\alpha)}{(1+e^x)^8\Gamma(1+\alpha)^2\Gamma(1+2\alpha)^2} \\ &\quad \times \Big(\frac{-300t^{3\alpha}\Gamma(1+\alpha)\Gamma(1+4\alpha)(\cosh(x)+\sinh(x))(-1+2\cosh(x)+2\sinh(x))^2}{\Gamma(1+5\alpha)} + \cdots\Big). \end{aligned}$$

So the solution can be approximated by $\phi_3(x,t)$ as follows:

$$\begin{aligned} &\phi_3(x,t) \\ &= u_{\diamond 0}(x,t) + u_{\diamond 1}(x,t) + u_{\diamond 2}(x,t) + u_{\diamond 3}(x,t) \\ &= \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)} t^\alpha + \frac{1}{(1+e^x)^6}\Gamma(1+\alpha)^3 50e^x t^{2\alpha}\Gamma(1+\alpha) \\ &\quad \times \Big(\frac{-12t^\alpha\Gamma(1+2\alpha)(\cosh(x)+\sinh(x))}{\Gamma(1+3\alpha)} \\ &\quad + \frac{4^{-\alpha}\sqrt{\pi}\Gamma(1+\alpha)(1+\cosh(x)+\sinh(x))^2(-1+2\cosh(x)+2\sinh(x))}{\Gamma(\frac{1}{2}+\alpha)}\Big) \\ &\quad + \frac{50e^x t^{3\alpha}\Gamma(1+\alpha)}{(1+e^x)^8\Gamma(1+\alpha)^2\Gamma(1+2\alpha)^2} \\ &\quad \times \Big(\frac{-300t^{3\alpha}\Gamma(1+\alpha)\Gamma(1+4\alpha)(\cosh(x)+\sinh(x))(-1+2\cosh(x)+2\sinh(x))^2}{\Gamma(1+5\alpha)} + \cdots\Big). \end{aligned}$$

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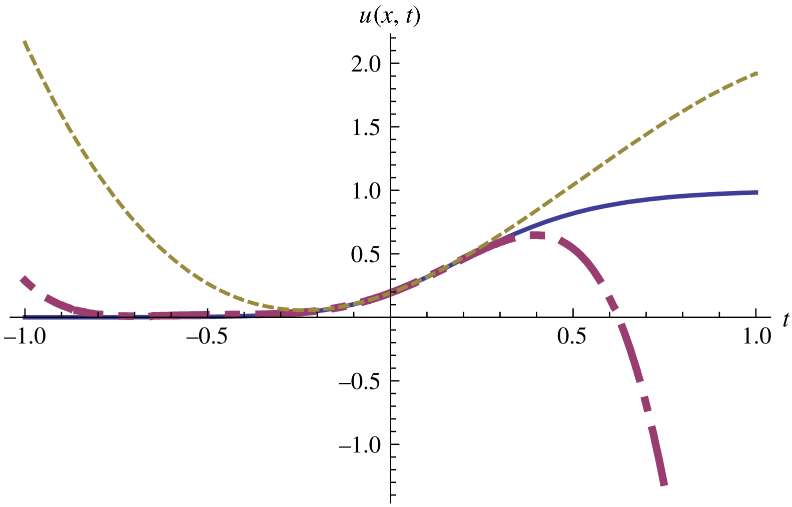

FIGURE 1. The IFADM solution (dot–dashed curve), the ADM solution (dashed curve) and the exact solution (solid curve) at $\alpha = 1$.

In order to obtain the approximate solution using the standard ADM, we compute the first components of the solution as follows:

$$\begin{aligned}
u_0(x,t) &= \frac{1}{(1+e^x)^2},\\
u_1(x,t) &= \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^{\alpha},\\
u_2(x,t) &= \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha},\\
u_3(x,t) &= \frac{50e^x((5-6e^x-15e^{2x}+20e^{3x})\Gamma^2(1+\alpha))-12e^x\Gamma(1+2\alpha)}{(1+e^x)^6\Gamma^3(1+\alpha)\Gamma(1+3\alpha)}t^{3\alpha}.
\end{aligned}$$

So the solution can be approximated by

$$\begin{aligned}
\phi_3(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)\\
&= \frac{1}{(1+e^x)^2}+\frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^{\alpha}+\frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha}\\
&\quad+\frac{50e^x((5-6e^x-15e^{2x}+20e^{3x})\Gamma^2(1+\alpha))-12e^x\Gamma(1+2\alpha)}{(1+e^x)^6\Gamma^3(1+\alpha)\Gamma(1+3\alpha)}t^{3\alpha}.
\end{aligned}$$

A comparison between the behaviour of the exact solution, the approximate solution using the IFADM, and the approximate solution using the ADM in which we use formula (3.8) and Riemann–Liouville derivatives is given in Figures 1 and 2 at $x = 0.25$ with $\alpha = 1$ and $\alpha = 0.75$, respectively.

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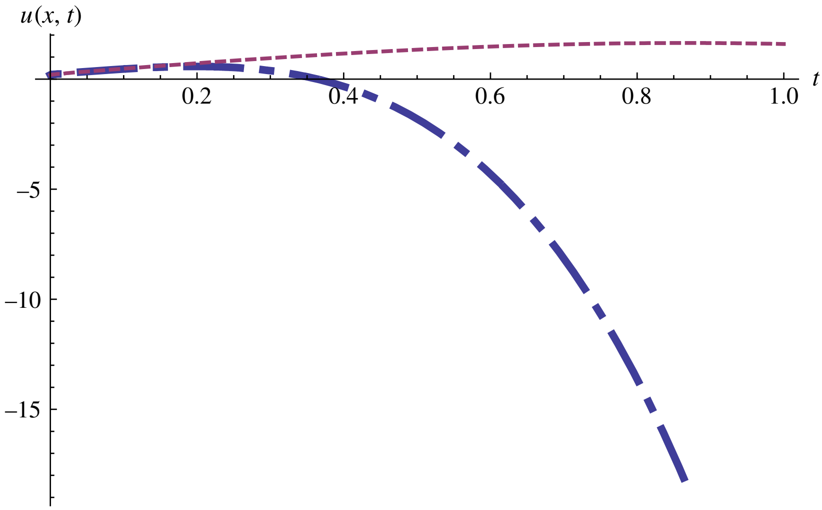

FIGURE 2. The IFADM solution (dot–dashed curve) and the ADM solution (dashed curve) at $\alpha = 0.75$.

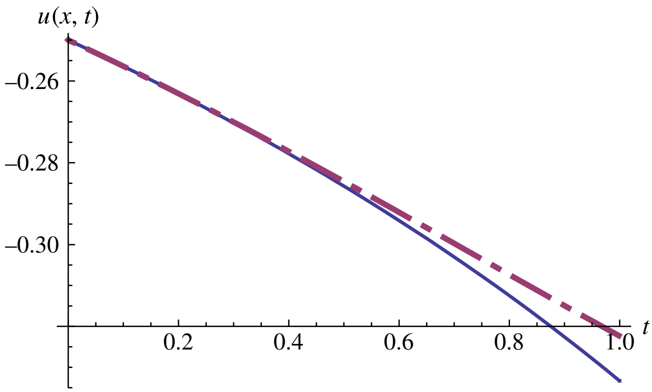

FIGURE 3. The IFADM solution (dot–dashed curve) and the exact solution (solid curve) at $\alpha = 1$.

EXAMPLE 6.2. Consider the nonlinear fractional initial value problem

$$D_t^{3\alpha}u(x,t) + u_x(x,t) - 2x(u(x,t))^2 + 6(u(x,t))^4 = 0, \quad t > 0, \ \ 0 < \alpha \le 1, \tag{6.5}$$

under the initial conditions

$$u(x,0) = \frac{-1}{x^2}, \quad u_t^{(\alpha)}(x,0) = \frac{1}{x^4} \quad \text{and} \quad u_t^{(2\alpha)}(x,0) = \frac{-1}{x^6}.$$

The exact solution of equation (6.5) at $\alpha = 1$ is $u(x,t) = 1/(t - x^2)$.

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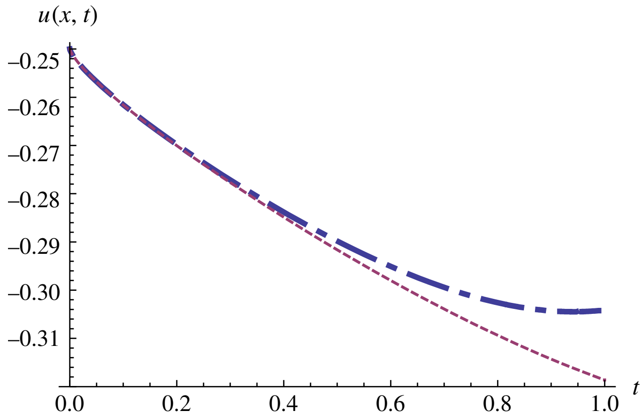

FIGURE 4. The IFADM solution (dot–dashed curve) and the ADM solution (dashed curve) at $\alpha = 0.75$.

A comparison between the behaviour of the exact solution, the approximate solution using the IFADM, and the approximate solution using the ADM in which we use formula (3.8) and Riemann–Liouville derivatives is given in Figures 3 and 4 at $x = 0.25$ with $\alpha = 1$ and $\alpha = 0.75$, respectively.

EXAMPLE 6.3. Consider the initial value problem for the nonlinear higher fractional-order equation

$$u^{(4\alpha)}(t) = 16u(t) + 40u^3(t) + 24u^5(t), \quad 0 < \alpha \le 1, \tag{6.6}$$

subject to the following initial conditions:

$$u(0) = 0, \quad u^{(\alpha)}(0) = \frac{1}{\Gamma(1+\alpha)}, \quad u^{(2\alpha)}(0) = 0 \quad \text{and} \quad u^{(3\alpha)}(0) = \frac{2}{\Gamma(1+3\alpha)}.$$

The exact solution of equation (6.6) at $\alpha = 1$ is $u(t) = \tan(t)$.

A comparison between the behaviour of the exact solution, the approximate solution using the IFADM, and the approximate solution using the ADM in which we use formula (3.8) and Riemann–Liouville derivatives is given in Figures 5 and 6 with $\alpha = 1$ and $\alpha = 0.75$, respectively.

7. Conclusion and remarks

In this work, a new formula for analytical approximate solutions using the Adomian decomposition method is introduced. An exponential notation formulation of this new formula is also introduced. Both the new formula and its exponential notation formulation are more general than El-Kalla's formula and its equivalent exponential notation formulation which was introduced by Adomian. From the numerical results obtained for the problems introduced, we can conclude that the proposed formula

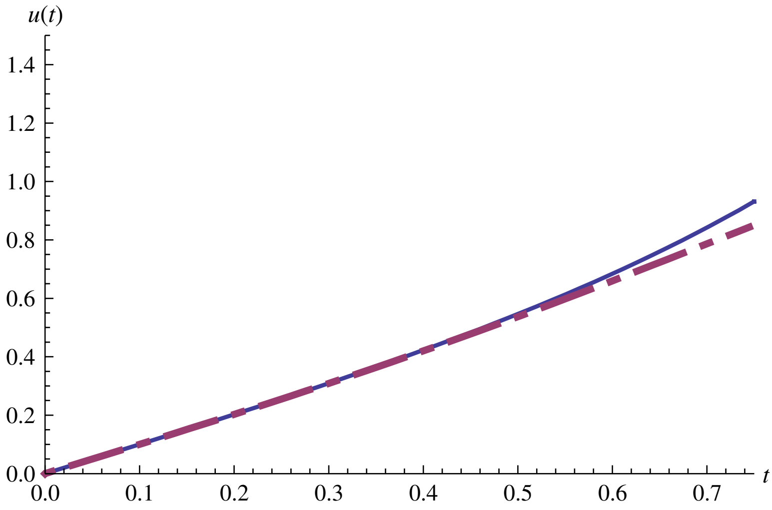

FIGURE 5. The IFADM solution (dot–dashed curve) and the exact solution (solid curve) at $\alpha = 1$.

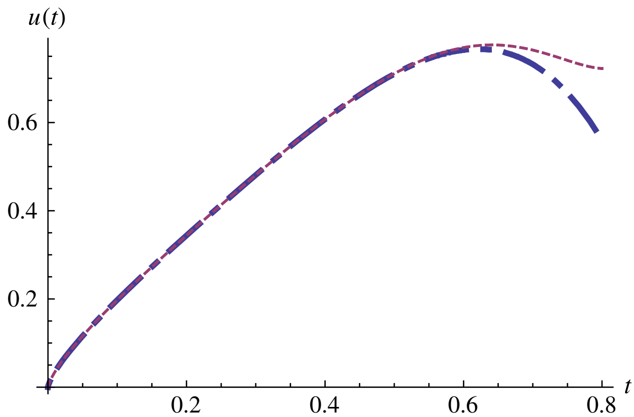

FIGURE 6. The IFADM solution (dot–dashed curve) and the ADM solution (dashed curve) at $\alpha = 0.75$.

can be applied to solve a wide class of nonlinear fractional differential equations. Also, from these results we found that there is excellent agreement with the exact solution. Finally, numerical examples were given to clarify the efficiency based on some procedures implemented in Mathematica.

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