HOMOTOPY AND ISOTOPY PROPERTIES OF TOPOLOGICAL SPACES

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1. Introduction. In 1961, S. T. Hu published a paper **(1)** in which he discussed the desirability of discovering those topological properties which are preserved under homotopy and isotopy equivalences. In that paper he gave general tests in terms of weakly hereditary and hereditary topological properties for homotopy and isotopy properties.

In this paper, general tests for homotopy and isotopy properties in terms of weakly hereditary properties and of a class of properties which the author calls open properties are given. In the last sections, we shall show the strong role played by the notions of dimension and separating subsets in forming isotopy properties.

The following notations will be used: I = unit interval and 1_X will stand for the identity map of X.

The definition of homotopy equivalence, etc., can be found in **(1)**, but for convenience we shall give the definition for isotopy equivalence here.

An imbedding $f:X \to Y$ is said to be an *isotopy equivalence* if there exists an imbedding $g:Y \to X$ such that the composite imbeddings $g \circ f$ and $f \circ g$ are isotopic to 1_X and 1_Y respectively. We shall call g an *isotopy inverse* of f. Two spaces, X and Y, are said to be *isotopically equivalent* if there is an isotopy equivalence $f:X \to Y$. A property P of topological spaces is called an *isotopy property* provided that it is preserved by all isotopy equivalences. Precisely, P is an *isotopy property* provided that, for an arbitrary isotopy equivalence $f:X \to Y$, X has P implies that Y also has P.

2. Weakly hereditary properties. A *non-trivial* topological property is one which is enjoyed by at least one non-empty topological space and which is not enjoyed by all non-empty topological spaces. A non-trivial property P is said to be hereditary if every subspace of a space with P enjoys P; it is said to be *weakly hereditary* if every closed subspace of a space with P also possesses P. Several examples of these properties are mentioned in **(1)**.

Hu, in **(1)**, has shown the following theorem.

THEOREM 2.1. *Let P be a weakly hereditary property which is possessed by the singleton space; then P is not a homotopy property.*

Hu has also shown in **(1)** the following theorem.

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THEOREM 2.2. *Every hereditary topological property is an isotopy property.*

The purpose of this section is to investigate weakly hereditary properties which are not hereditary properties (henceforth these will be called *properly weakly hereditary*) to find which are isotopy properties.

First we note that if we restrict ourselves to the class of compact Hausdorff spaces, then any weakly hereditary property is an isotopy property. This follows as a corollary to the following proposition.

PROPOSITION 2.3. *If X is compact and Hausdorff and if Y has a weakly hereditary property P, then X has property P if X is isotopically equivalent to Y.*

Proof. Since the property of being Hausdorff is hereditary, it is an isotopy property and so Y is Hausdorff.

Now the image of any mapping of a compact space into a Hausdorff space must be closed, so since X is isotopically equivalent to Y, X may be considered as a closed subspace of Y. Thus X must inherit P.

In spite of this theorem, it turns out that most of the elementary properly weakly hereditary properties are not isotopy properties, as the following theorem will show. In particular, compactness is not an isotopy property and so the previous theorem loses much of its force.

THEOREM 2.4. *Let P be a weakly hereditary property which holds on some space X and on $X \times I$, but not on $S \subset X$. Then P is not an isotopy property.*

Proof. Consider the subspace Y of $X \times I$ such that

$$Y = \{(x, t) | 0 \leqslant t < 1 \text{ or } x \in S\}.$$

Y does not possess P since $S \times 1 \equiv \{(x, 1) | x \in S\}$ is closed in Y and homeomorphic to S, so if we assume that Y possesses P we see that $S \times 1$ possesses P and thus S possesses P, a contradiction.

However, Y is isotopically equivalent to $X \times I$ and this proves that P cannot be an isotopy property.

To show that Y is isotopically equivalent to $X \times I$ consider the following maps:

$$\begin{aligned}
&i: Y \to X \times I && \text{such that } i(x, t) = (x, t),\\
&j: X \times I \to Y && \text{such that } j(x, t) = (x, t/2),\\
&h_s: X \times I \to X \times I && \text{such that } h_s(x, t) = (x, (1 + s)t/2),\\
&k_s: Y \to Y && \text{such that } k_s(x, t) = (x, (1 + s)t/2).
\end{aligned}$$

It can easily be shown that i and j are imbeddings and that h_s and k_s are isotopies. Now we have

$$\begin{aligned}
h_0 &= i \circ j \quad \text{and} \quad h_1 = \text{the identity map of } X \times I,\\
k_0 &= j \circ i \quad \text{and} \quad k_1 = \text{the identity map of } Y.
\end{aligned}$$

So we have $X \times I$ isotopically equivalent to Y.

COROLLARY 2.5. *The following properties are not isotopy properties:*
(1) *normality,*
(2) *compactness,*
(3) *the property of being a Lindelöf space,*
(4) *local compactness,*
(5) *paracompactness.*

Proof. All these properties are properly weakly hereditary. The Tychonoff plank, T, enjoys all the above properties and contains a subspace which does not enjoy any of the above properties, except for local compactness, and $T \times I$ supports all the properties. For the case of local compactness, I and $I \times I$ are locally compact, but the rationals are not. Applying the previous theorem proves the corollary.

COROLLARY 2.6. *Let P be a weakly hereditary property such that X has P implies $X \times I$ has P. Then P is an isotopy property if and only if P is hereditary.*

Proof. If P is hereditary it must be an isotopy property. If P is not hereditary, then there exists a space X possessing P and a subspace S not possessing P. Since $X \times I$ has P, by Theorem 2.4, P is not an isotopy property.

There is a large class of properly weakly hereditary properties which are isotopy properties.

THEOREM 2.7. *If P is a weakly hereditary property not enjoyed by the unit interval I, and if X is Hausdorff and enjoys P, then any isotopy equivalence $f: X \to Y$ is a homeomorphism.*

Proof. Let $f: X \to Y$ and $g: Y \to X$ and let $h_t: X \to X$ and $k_t: Y \to Y$ be isotopies such that $h_0 = 1_X$ and $h_1 = g \circ f$, and $k_0 = 1_Y$ and $k_1 = f \circ g$.

Let $x \in X$. Let $z = g \circ f(x)$. We can define a path $\sigma: I \to X$ between x and z by $\sigma(t) = h_t(x)$. Now $\sigma(I)$ is a compact connected and locally connected metric space by the Hahn-Mazurkiewicz theorem and hence there is an arc (an imbedding of I) with end points x and z if x and z are distinct. But this implies that I possesses P. Hence, $z = x$ or $g \circ f = 1_X$.

Similarly, let $y \in Y$. Let $w = f \circ g(y)$. We can define a path $\gamma: I \to Y$ such that $\gamma(t) = k_t(y)$. Now Y is Hausdorff by Theorem 2.2 and so there exists an arc α in $\gamma(I)$ with y and w as end points if $y \neq w$. But then $g \circ \alpha$ is an arc of X and since it must be closed in X, I has P. Therefore, $f \circ g = 1_Y$. Thus f has a two-sided inverse and so is a homeomorphism.

COROLLARY 2.8. *If P is a weakly hereditary property not possessed by I, then P plus Hausdorff is an isotopy property.*

3. Open properties. There is a class of topological properties for which analogous theorems to those for weakly hereditary properties hold.

DEFINITION. A non-trivial topological property which is inherited by open sets and preserved by open maps will be called an *open property*. That is, if P is an open property and if S is a topological space with P, then any open set of S possesses P and $f(S)$ possesses P if f is an open map.

Some examples of open properties are:

(1) separability,
(2) local connectedness,
(3) local pathwise connectedness,
(4) first countability,
(5) second countability.

First we have the analogous theorem to 2.1 for homotopy properties.

THEOREM 3.1. *Open properties are not homotopy properties.*

Proof. Let P be an open property.

The singleton space $\{v\}$ must have P, for there exists a space X with P and the constant map $C:X \to \{v\}$ is open.

Let S be a space without P.

$S \times J$, where J is the open unit interval, cannot have P since $S = \pi(S \times J)$, where π is the projection of $S \times J$ onto S. Since π is an open map, S without P implies that $S \times J$ does not possess P.

Consider $C(S)$, the cone over S. $S \times J$ is homeomorphic to an open subset of $C(S)$, so $C(S)$ does not possess P since P is open.

Since $C(S)$ is homotopically equivalent to $\{v\}$, P cannot be a homotopy property.

We can get a somewhat more general result by using the same method of proof.

THEOREM 3.2. *Let P be a property such that the singleton space, $\{v\}$, has P, P is inherited by open sets, and such that there exists a space S where $S \times J$ does not possess P. Then P is not a homotopy property.*

With regard to isotopy properties, the next theorem is analogous to Theorem 2.4.

THEOREM 3.3. *Any open property P is not an isotopy property if there exists a space X and a subspace S such that $X \times I$ enjoys P but S does not.*

Proof. Let S and X be chosen as in the hypothesis.

Consider the subspace Y of $X \times I$ such that

$$Y = \{(x, t) | 0 \leqslant t \leqslant 3/4 \text{ or } x \in S\}.$$

Now $S \times (3/4, 1]$ is open relative to Y and would have P if Y had P. But $S \times (3/4, 1]$ is also a product space and since S is the image of an open projection map from $S \times (3/4, 1]$, we see that S would have P if $S \times (3/4, 1]$ had P. Since S does not have P, Y does not possess P.

We now prove the theorem by showing that Y is isotopically equivalent to $X \times I$. Consider the following maps:

$$\begin{aligned} &i: Y \to X \times I && \text{such that } i(x, t) = (x, t), \\ &j: X \times I \to Y && \text{such that } j(x, t) = (x, t/2), \\ &h_s: X \times I \to X \times I && \text{such that } h_s(x, t) = (x, (1 + s)t/2), \\ &k_s: Y \to Y && \text{such that } k_s(s, t) = (x, (1 + s)t/2). \end{aligned}$$

It can easily be shown that i and j are imbeddings and h_s and k_s are isotopies and that

$$\begin{aligned} h_0 &= i \circ j, \qquad h_1 = \text{identity map on } X \times I; \\ k_0 &= j \circ i, \qquad h_1 = \text{identity map on } Y. \end{aligned}$$

This shows that $X \times I$ is isotopically equivalent to Y, thus proving the theorem.

COROLLARY 3.4. *Let P be an open property such that X has P implies $X \times I$ has P. Then P is an isotopy property if and only if P is hereditary.*

Proof. If P is hereditary, it is an isotopy property by Theorem 2.2.

If P is not hereditary, there exists a space X enjoying P with a subspace S not possessing P. Since $X \times I$ has P, all the conditions of the preceding theorem are satisfied and so P is not an isotopy property.

COROLLARY 3.5. *Separability, local connectedness, and local pathwise connectedness are not isotopy properties.*

Proof. If X is a space possessing one of the above properties, $X \times I$ must also possess it. Since none of the above properties is hereditary, they are not isotopy properties.

4. Dimension. Thus far, aside from hereditary properties, we have shown that large groups of elementary topological properties are not isotopy properties. The next sections will demonstrate many isotopy properties related to concepts of dimension and separating properties. There are many other isotopy properties lurking about in this area which are not mentioned here.

The word dimension in this section will refer to *inductive dimension*, defined as follows: $\dim X = -1$ if X is empty and $\dim X \leqslant n$ if for every point point $P \in X$ and every open neighbourhood U of P there exists an open neighbourhood $V \subset U$ of P such that $\dim \partial V \leqslant n - 1$, where ∂V denotes the boundary $\bar{V} - V$ of V in X **(2**, p. 153**)**.

THEOREM 4.1. *Let F be a family of topological spaces and let Q be any partially ordered class of elements. Let Φ be a function from F to Q satisfying the following two properties:*

(1) *if X is homeomorphic to Y, then $\Phi(X) = \Phi(Y)$.*

(2) *if X can be imbedded in Y, then $\Phi(X) \leqslant \Phi(Y)$.*

Then $\Phi(X)$ is an isotopy invariant of the family F.

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Proof. Let $f:X \to Y$ be an isotopy equivalence between two spaces belonging to F. We must show that $\Phi(X) = \Phi(Y)$.

Now, by (1), $\Phi(X) = \Phi(f(X))$ and since $f(X) \subseteq Y$, (2) states that $\Phi(f(X)) \leqslant \Phi(Y)$ or $\Phi(X) \leqslant \Phi(Y)$.

Let $g: Y \to X$ be an isotopy inverse for f. Then $\Phi(Y) = \Phi(g(Y)) \leqslant \Phi(X)$. Hence $\Phi(X) = \Phi(Y)$.

COROLLARY 4.2. *Cardinality is an isotopy invariant.*

Proof. Cardinality is a function from the class of all topological spaces to the class of cardinal numbers; hence by 4.1 cardinality is an isotopy invariant.

COROLLARY 4.3. *Inductive dimension of a topological space is an isotopy invariant.*

Proof. Inductive dimension is a function from the class of all topological spaces to the integers satisfying (1) and (2); hence dim X is an isotopy invariant.

COROLLARY 4.4. *For the class C of compact Hausdorff spaces, the covering dimension,* Dim X **(1**, p. 170**)**, *is an isotopy invariant.*

Proof. Let $F = C$ and $Q =$ the integers. The covering dimension certainly satisfies (1) and for C it satisfies (2) since it is weakly hereditary.

Corollary 4.3 is the main result of this section and it was shown by Hu in **(1)**.

5. Separating subsets and dimension. The next two theorems show how inductive dimension and separating subsets combine to form isotopy properties of exceedingly general topological spaces.

THEOREM 5.1. *Let X be a topological space. Suppose that there is an open set C of inductive dimension $\leqslant n$ which separates X into two subsets, each of whose complements has dimension greater than n. Then any Y isotopically equivalent to X will have the same property.*

Proof. We first note the following general fact:

(1) If X has a point of inductive dimension n at $x \in X$, then any open set U containing x must have dimension at least n:

If $\{V_\alpha\}$ is a basis of open neighbourhoods of x in X, then $\{V_\alpha | V_\alpha \subseteq U\}$ is a basis of open neighbourhoods of x both in X and in U. If there exists no basis of open neighbourhoods of x in X whose boundaries have dimension $n-2$, then there cannot be one of x in U. Hence, U has dimension greater than $n-1$ at x.

(2) We proceed to prove the theorem. By the hypothesis, let $X - C = X_1 \cup X_2$, where $X_1 \cap X_2 = \emptyset$ and $X_1 \cup C$ and $X_2 \cup C$ are open and have dimensions m_1 and m_2, respectively, greater than n.

(3) Let $h_t:X \to X$ be an isotopy with $h_0 = 1_X$; then there is a point $x_i \in h_t(X_i) \cap X_i$ of dimension m_i $(i = 1, 2)$ for any $t \in I$. Let $x_i \in X_i \cup C$ be a point of dimension m_i. Suppose there is a $t \in I$ such that $h_t(x_i) \notin X_i$. Then there must be an $s \in I$ such that $h_s(x_i) \in C$. Now since h_s is an imbedding, X must have dimension $\geqslant m_i$ at $h_s(x_i)$, so by (1), C has dimension at least $m_i > n$. This is a contradiction.

(4) Let $f:X \to Y$ be an isotopy equivalence with $g:Y \to X$ an isotopy inverse and $h_t:X \to X$ an isotopy such that $h_0 = 1_X$ and $h_1 = g \circ f$. From (1) and (3), we know that $g \circ f(X_i) \cap (X_i \cup C)$ has dimension at least m_i and so $g(Y) \cap (X_i \cup C)$ has dimension at least $m_i > n$. Hence $h_t(x_i) \in X_i$.

(5) $C \cap g(Y)$ is open relative to $g(Y)$ and has dimension less than n. Furthermore,

$$g(Y) - (C \cap g(Y)) = (X_1 \cap g(Y)) \cup (X_2 \cap g(Y))$$

and

$$[X_1 \cap g(Y)] \cap [X_2 \cap g(Y)] = \emptyset.$$

(6) We have just shown that the space $g(Y)$ can be separated by an open set $C \cap g(Y)$ of dimension at most n into two subsets each of whose complement has dimension greater than n. Since Y is homeomorphic to $g(Y)$, our proof is complete.

This theorem tells us that the following two-dimensional subsets of the plane are not isotopically equivalent: X = two disks of radius one tangent to each other, Y = two disks of radius one which do not intersect and the line joining their centre.

Definition. A compact n-space, $n > 1$, is called an *n-dimensional Cantor manifold* if it cannot be separated by a subset of dimension $\leqslant n - 2$.

The property of being an n-Cantor manifold is not an isotopy property even for the family of compact spaces; but on the other hand, a somewhat similar property remains invariant under isotopy equivalences.

THEOREM 5.2. *Let X be a Hausdorff space separated by a closed subspace C of dimension less than or equal to $n - 2$ into two sets, each of whose complement contains an n-Cantor manifold. Then if Y is isotopically equivalent to X, Y has the same separation property.*

Proof. (1) $X - C = X_1 \cup X_2$, where $X_1 \cap X_2 = \emptyset$. X_1 and X_2 are both open in X since C is closed.

(2) Let A be an n-Cantor manifold contained in $X_1 \cup C$. Let $h_t:X \to X$ be an isotopy such that $h_0 = 1_X$; then $h_t(A) \subseteq X_1 \cup C$ for all $t \in I$: Define $J:A \times I \to X$ by $J(x, t) = h_t(x)$ and let $p:A \times I \to I$ be the projection, i.e. $p(x, t) = t$.

Clearly, $pJ^{-1}(X_1)$ and $pJ^{-1}(X_2)$ are both open sets in I.

We wish to show that $pJ^{-1}(X_1) \cup pJ^{-1}(X_2) = I$. If this were not the case, then there would be a $t' \in I$ such that $J(A, t') = h_{t'}(A) \subseteq C$. But $h_{t'}(A)$ is

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a subspace of C of dimension n since $h_{t'}$ is an imbedding. Since C has dimension $\leqslant n-2$, this is a contradiction.

So I is the union of two open sets which must intersect if they are both non-empty. That is, there must be a $t' \in pJ^{-1}(X_1) \cap pJ^{-1}(X_2)$. Hence, $h_{t'}(A) \cap X_1 \neq \emptyset$ and $h_{t'}(A) \cap X_2 \neq \emptyset$. But $h_{t'}(A)$ is an n-Cantor manifold which is separated by a subset $h_{t'}(A) \cap C$ of dimension $\leqslant n-2$. This contradiction forces us to assume that $pJ^{-1}(X_2) = \emptyset$ or, equivalently, that $h_t(A) \subseteq X_1 \cup C$ for all $t \in I$.

(3) Similarly, if B is an n-Cantor manifold contained in $X_2 \cup C$, $h_t(B) \subseteq X_2 \cup C$ for all t.

(4) Let $f: X \to Y$ be an isotopy equivalence with $g: Y \to X$ an isotopy inverse. By (2) and (3), $g \circ f(A) \subseteq X_1 \cup C$ and $g \circ f(B) \subseteq X_2 \cup C$. Therefore, $g \circ f(A) \subseteq (X_1 \cup C) \cap g(Y)$ and $g \circ f(B) \subseteq (X_2 \cup C) \cap g(Y)$.

(5) We see that $C \cap g(Y)$ is a closed set relative to $g(Y)$ of dimension less than or equal to $n-2$ which separates $g(Y)$ into two subsets $g(Y) \cap X_1$ and $g(Y) \cap X_2$ each of whose complement, $g(Y) \cap (X_2 \cup C)$ and $g(Y) \cap (X_1 \cup C)$ respectively, contains an n-dimensional Cantor manifold. Since $g(Y)$ is homeomorphic to Y, Y has the required property.

COROLLARY 5.3. *Let X be a locally compact metric space which can be separated by a closed subspace C of dimension $\leqslant n-2$ into two sets each of whose complement is a set of dimension at least n. Then, if Y is isotopically equivalent to X, it can be separated by a closed set of dimension $\leqslant n-2$ into two subsets, each of whose complement is of dimension at least n.*

Proof. Suppose C separates X into X_1 and X_2. Now $X_i \cup C$ $(i = 1, 2)$ is closed in X and hence is locally compact.

$X_i \cup C$ has dimension $m_i \geqslant n$ at some point x_i. We can find a compact neighbourhood V_i about x_i. V_i must be of dimension at least m_i, and so it contains an m_i-Cantor manifold by the following theorem **(2**, p. 94**)**: Any compact n-dimensional metric space contains an n-dimensional Cantor manifold.

If Y is isotopically equivalent to X, Theorem 5.2 states that Y can be separated by a closed subset of dimension $\leqslant n-2$ in the required way.

6. Branch points.

Definition.

(i) Let S be a pathwise connected space. A point x of S is called a *branch point* of S if $S - \{x\}$ has at least three path-components. Note that this is not the same concept as that of a cut point.

(ii) For any general topological space, a point will be called a *branch point* if it is a branch point of the path-component containing it.

(iii) Let X be an arbitrary topological space and let x be a branch point of X. A *created path-component* of $X - \{x\}$ is a path-component of $X - \{x\}$ which is not a path-component of X.

(iv) A point x of X is called a *local branch point* of X if there exists a neighbourhood N of x such that x is a branch point of N.

LEMMA 6.1. *Suppose that X is a T_1 space and $h_t: X \to X$ is an isotopy such that $h_0 = 1_X$. Let x_0 be a local branch point of X; then $h_t(x_0) = x_0$ for all t.*

Proof. Let $H(x, t) = h_t(x)$. Let N be a neighbourhood of which x_0 is a branch point. We define a path

$$\sigma: I \to X: t \to h_t(x_0).$$

Let t_0 be the greatest lower bound of $\sigma^{-1}(X - \{x_0\})$. The case $t_0 = 1$ being trivial, assume that $t_0 < 1$.

Since H is continuous and $[0, t_0]$ compact, there exists a neighbourhood $U \times Q$ of $\{x_0\} \times [0, t_0]$ in $N \times I$ such that: $H(U \times Q) \subset N$. Let $\{C_\alpha\}$ be the created path-components of $X - \{x_0\}$. Note that $C_\alpha \cap U \neq \emptyset$ for any α, and any neighbourhood U about x_0.

There exists a created path-component C_1 and an open interval $(t_0, t_1) \subseteq Q$ such that $H(x_0, t) \in C_1$ for all $t \in (t_0, t_1)$. Then, for any two other created path-components C_2 and C_3, it is possible to choose points x_2 and x_3 such that $x_i \in P_i$, a created path-component of $U \cap C_i$, and such that $H(x_i, \frac{1}{2}(t_0 + t_1)) \in C_1$, $i = 2, 3$. Define paths $g_i(t) = H(x_i, t)$ and let r_i be the smallest t such that $g_i(t) = x_0$. At $t = r_3$, $H(x_3, r_3) = x_0$, so that $H(P_2, r_3) \subseteq C_1$. Hence, $H(x_2, r_3) \in C_1$, and hence $r_2 < r_3$. But, similarly, $r_2 > r_3$. This contradiction proves the lemma.

LEMMA 6.2. *With the same notation as in the preceding lemmas, there exist $P_\alpha \subseteq C_\alpha$ such that $P_\alpha \cup \{x_0\}$ is path-connected and $h_t(P_\alpha) \subseteq C_\alpha$ for all $t \in I$ and for all α.*

Proof. By the preceding lemma, since $H: X \times I \to X$ is continuous, $H(U \times I) \subseteq N$ for some neighbourhood U of x_0. Define P_α to be a created path-component of $U \cap C_\alpha$. Let $x \in P_\alpha$. Then $\sigma(t) = h_t(x)$ defines a path which must remain entirely in C_α since its only possible exit passes over x_0 and $h_t(x_0) = x_0$ for all t. Therefore, $h_t(P_\alpha) \subseteq C_\alpha$.

THEOREM 6.3. *The cardinality of local branch points is an isotopy invariant for T_1-spaces.*

Proof. Let $f: X \to Y$ be an isotopy equivalence and $g: Y \to X$ be an isotopy inverse. Let $x_0 \in X$ be a local branch point; then we will show that $f(x_0)$ is a local branch point of Y using the notation of the lemmas.

By the preceding lemmas and the definition of isotopy equivalence, $g(f(x_0)) = x_0$ and $g \circ f(P_\alpha) \subseteq C_\alpha$.

Now $N \cap g(Y)$ is a neighbourhood of x_0 relative to $g(Y)$, and x_0 is a branch point of $N \cap g(Y)$ with the path-components C_α^* of $g(Y) \cap C_\alpha$ (for which $C_\alpha^* \supseteq g \circ f(P_\alpha)$, where P_α is a created path-component of $N \cap U$) serving

as created path-components. Thus, $g^{-1}(x_0)$ must be a local branch point of Y since $g^{-1}:g(Y) \to Y$ is a homeomorphism. But $g^{-1}(x_0) = f(x_0)$.

Thus, for every local branch point x_0 of X, $f(x_0)$ is a local branch point of Y and similarly for every local branch point y_0 of Y, $g(y_0)$ is a local branch point of X.

We can sharpen the concept of branch point by the following definition.

Definition. Let n be any cardinal number. An *n-branch point* x of X is a branch point of X such that there are n-created path-components of $X - \{x\}$.

THEOREM 6.4. *The cardinality of n-branch points is an isotopy invariant for T_1-spaces.*

Proof. Suppose that $f:X \to Y$ is an isotopy equivalence and suppose that x_0 is an n-branch point of X with $\{C_\alpha | \alpha \in A\}$ the created path-components. By Lemma 6.2 with $N = U = X$ we have $g \circ f(C_\alpha) \subseteq C_\alpha$, where g is an isotopy inverse of f.

We know that $f(x_0)$ is a local branch point of Y and it is easy to see that $f(x_0)$ is, in fact, a branch point of Y. Let $\{D_\beta | \beta \in B\}$ be the created path-components of $Y - \{f(x_0)\}$.

Now, for any $\alpha \in A$, there is a $\beta \in B$ such that $f(C_\alpha) \subseteq D_\beta$. Also, for any $\beta \in B$, there is a $\gamma \in A$ such that $g(D_\beta) \subseteq C_\gamma$. Since $C_\gamma \supseteq g(D_\beta) \supseteq g(f(C_\alpha)) \subseteq C_\alpha$, $\alpha = \gamma$. This shows that the cardinality of A is not greater than the cardinality of B.

By symmetry, the latter is not greater than the former. Hence they are equal.

In extending the concept of n-branch point to local n-branch point, one must guard against spaces such as five different circles tangent at one point.

As an example of the application of this theorem, consider the family of spaces $B(n, l)$ of I^n, $n > 0$, with l rays emanating from $(1, 1, \ldots, 1)$ with the inherited topology of R^n. Now, $B(n, l)$ is homeomorphic to $B(n', l')$ if and only if $n = n'$ and $l = l'$. On the other extreme, $B(n, l)$ is contractible, so the family lies in just one homotopy equivalence class. However, by use of the previous theorem and the invariance of dimension, we see that $B(n, l)$ is isotopically equivalent to $B(n', l')$ if and only if $n = n'$ and $l = l'$ when $l \geqslant 2$. Since $B(n, 0)$ is isotopically equivalent to $B(n, 1)$, we have an example of a family of spaces whose isotopy type is different from both the homotopy type and the homeomorphism type.

We end this paper by showing that perfection itself is an isotopy property.

A space X is called *perfect* if every point is a limit point of X.

PROPOSITION. 6.5 *The property of being perfect is an isotopy property for T_1-spaces.*

Proof. Let X be isotopically equivalent to Y. Suppose Y is a perfect T_1-space and X is not. X must be T_1 since T_1 is an isotopy property.

Let $x_0 \in X$ be a point which is not a limit point; that is, there exists a neighbourhood N of x_0 such that $N \cap (X - \{x\}) = \emptyset$. Hence, x_0 is an open set of X. Since X is T_1, x_0 is closed. Therefore, any continuous mapping of the unit interval into X which contains x_0 in its image must be the constant map.

Now, let $f:X \to Y$ be an isotopy equivalence and let $g:Y \to X$ be an isotopy inverse. Now g must map the point $f(x_0) \in Y$ to x_0, i.e., $g(f(x_0)) = x_0$ by the following argument. We know there exists a map $H:X \times I \to X$ such that $H(x, 0) = g \circ f(x)$ and $H(x, 1) = x$. Define $\sigma:I \to X$ such that $\sigma(t) = H(x_0, t)$. σ is continuous and $\sigma(1) = H(x_0, 1) = x_0$. Now σ must be a constant map, so $\sigma(0) = x_0 = g \circ f(x_0)$.

Since $x_0 \in g(Y)$ is open in X, $g^{-1}(x_0)$ is open in Y. This means that Y is not perfect.

This contradiction proves the proposition.

PROPOSITION 6.6. *The cardinality of non-limit points is an isotopy invariant for T_1-spaces.*

Proof. It was shown in the proof of the preceding proposition that if $X \cong Y$ with $f:X \to Y$ an isotopy equivalence, the image of any non-limit point of X is a non-limit point of Y. Hence, for every non-limit point of X there is a non-limit point of Y and, symmetrically, for every non-limit point of Y there corresponds a non-limit point of X. Hence, X and Y have the same number of non-limit points.

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