Appendix A
Reference formulas

A.1 Dirac matrix element tables

Tables A.1 and A.2 first appeared in Lepage and Brodsky (1980). Following the original authors of these tables we use the notation

$$\vec{p}_\perp \times \vec{p}\,'_\perp \equiv p^1 p'^2 - p^2 p'^1. \tag{A.1}$$

Note that $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$. The spinors in the tables are defined in Eqs. (1.50) and (1.51).

The following formulas will be useful in relating other matrix elements to those tabulated:

$$\bar{v}_{\sigma'}(p')\gamma^\mu v_\sigma(p) = \bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p'), \tag{A.2}$$

$$\bar{v}_{\sigma'}(p')\gamma^\mu\gamma^\nu\gamma^\rho v_\sigma(p) = \bar{u}_\sigma(p)\gamma^\rho\gamma^\nu\gamma^\mu u_{\sigma'}(p'), \tag{A.3}$$

$$\bar{v}_{\sigma'}(p')v_\sigma(p) = -\bar{u}_\sigma(p)u_{\sigma'}(p'). \tag{A.4}$$

These formulas allow one to obtain matrix elements constructed only from v-spinors using Table A.1 for matrix elements constructed solely from u-spinors.

For any Dirac spinors ψ and η and a 4×4 matrix Γ, the following is true:

$$\left(\bar{\psi}\Gamma\eta\right)^* = \bar{\eta}\gamma^0\Gamma^\dagger\gamma^0\psi. \tag{A.5}$$

This allows one to construct matrix elements of the type

$$\bar{u}_\sigma(p)\Gamma v_{\sigma'}(p') = [\bar{v}_{\sigma'}(p')\gamma^0\Gamma^\dagger\gamma^0 u_\sigma(p)]^* \tag{A.6}$$

from those tabulated in Table A.2. As

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \tag{A.7}$$

in the case $\Gamma = \gamma^\mu$, Eq. (A.6) gives

$$\bar{u}_\sigma(p)\gamma^\mu v_{\sigma'}(p') = \left(\bar{v}_{\sigma'}(p')\gamma^\mu u_\sigma(p)\right)^*. \tag{A.8}$$

A.2 Some useful integrals

Below we list several integrals used throughout the book. We leave their derivation as an exercise for the reader. The Green function of the Laplace equation in two dimensions is

$$\int \frac{d^2q_\perp}{q_\perp^2} e^{i\vec{q}_\perp\cdot\vec{x}_\perp} = 2\pi \ln \frac{1}{x_\perp \Lambda}, \tag{A.9}$$

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Table A.1. *Dirac matrix elements constructed from u-spinors only. Table reprinted with permission from Lepage and Brodsky (1980). Copyright 1980 by the American Physical Society.*

Matrix element	Value
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^+\frac{u_\sigma(p)}{\sqrt{p^+}}$	$2\delta_{\sigma\sigma'}$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\frac{2}{p^+p'^+}(\vec{p}_\perp\cdot\vec{p}\,'_\perp - i\sigma\vec{p}_\perp\times\vec{p}\,'_\perp + m^2)$ $-\delta_{\sigma,-\sigma}\sigma\frac{2m}{p^+p'^+}\left[(p'^1+i\sigma p'^2)-(p^1+i\sigma p^2)\right]$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\left(\frac{p'^i_\perp - i\sigma\epsilon^{ij}p'^j_\perp}{p'^+}+\frac{p^i_\perp + i\sigma\epsilon^{ij}p^j_\perp}{p^+}\right)$ $-\delta_{\sigma,-\sigma}\sigma m\left(\frac{p'^+-p^+}{p'^+p^+}\right)(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}m\frac{p^++p'^+}{p^+p'^+}-\delta_{\sigma,-\sigma}\sigma\left(\frac{p'^1+i\sigma p'^2}{p'^+}-\frac{p^1+i\sigma p^2}{p^+}\right)$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$4\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma_\perp^i\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\,4\,\frac{p'^i_\perp - i\epsilon^{ij}\sigma p'^j_\perp}{p'^+}+\delta_{\sigma,-\sigma'}\,\sigma\frac{4m}{p'^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}\,4\,\frac{p^i_\perp + i\epsilon^{ij}\sigma p^j_\perp}{p^+}-\delta_{\sigma,-\sigma'}\,\sigma\frac{4m}{p^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{u}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\gamma^+\gamma_\perp^j\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma\sigma'}2(\delta^{ij}+i\sigma\epsilon^{ij})$

where Λ is the IR cutoff on the integration. Taking the transverse gradient of Eq. (A.9) yields

$$\int d^2q_\perp e^{i\vec{q}_\perp\cdot\vec{x}_\perp}\frac{\vec{q}_\perp}{q_\perp^2} = 2\pi i\frac{\vec{x}_\perp}{x_\perp^2}. \tag{A.10}$$

Here is a variation of Eq. (A.9), for a massive Green function:

$$\int \frac{d^2q_\perp}{q_\perp^2+m^2}e^{i\vec{q}_\perp\cdot\vec{x}_\perp} = 2\pi K_0(mx_\perp). \tag{A.11}$$

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Table A.2. *Dirac matrix elements constructed from u- and v-spinors. Table reprinted with permission from Lepage and Brodsky (1980). Copyright 1980 by the American Physical Society.*

Matrix element	Value
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^+\frac{u_\sigma(p)}{\sqrt{p^+}}$	$2\delta_{\sigma,-\sigma'}$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\frac{2}{p^+p'^+}(\vec{p}_\perp\cdot\vec{p}'_\perp - i\sigma\vec{p}_\perp\times\vec{p}'_\perp - m^2)$ $-\delta_{\sigma\sigma'}\sigma\frac{2m}{p^+p'^+}\left[(p'^1+i\sigma p'^2)+(p^1+i\sigma p^2)\right]$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\left(\frac{p'^i_\perp - i\sigma\epsilon^{ij}p'^j_\perp}{p'^+}+\frac{p^i_\perp + i\sigma\epsilon^{ij}p^j_\perp}{p^+}\right)$ $-\delta_{\sigma\sigma'}\sigma m\left(\frac{p'^+ + p^+}{p'^+p^+}\right)(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma}m\frac{p'^+ - p'^+}{p^+p'^+} - \delta_{\sigma\sigma'}\sigma\left(\frac{p'^1+i\sigma p'^2}{p'^+}-\frac{p^1+i\sigma p^2}{p^+}\right)$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$4\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma^-\gamma^+\gamma_\perp^i\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\,4\frac{p'^i_\perp - i\epsilon^{ij}\sigma p'^j_\perp}{p'^+}+\delta_{\sigma\sigma'}\sigma\frac{4m}{p'^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\gamma^+\gamma^-\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\,4\frac{p^i_\perp + i\epsilon^{ij}\sigma p^j_\perp}{p^+}-\delta_{\sigma\sigma'}\sigma\frac{4m}{p^+}(\delta^{i1}+i\sigma\delta^{i2})$
$\frac{\bar{v}_{\sigma'}(p')}{\sqrt{p'^+}}\gamma_\perp^i\gamma^+\gamma_\perp^j\frac{u_\sigma(p)}{\sqrt{p^+}}$	$\delta_{\sigma,-\sigma'}\,2(\delta^{ij}+i\sigma\epsilon^{ij})$

Equations (A.10) and (A.9) can be used to show that

$$\int d^2y_\perp\frac{\vec{y}_\perp\cdot(\vec{y}_\perp+\vec{x}_\perp)}{y_\perp^2(\vec{y}_\perp+\vec{x}_\perp)^2} = 2\pi\ln\frac{1}{x_\perp\Lambda}. \tag{A.12}$$

Several angular integrals are useful too:

$$\int_0^{2\pi}\frac{d\varphi_q}{(\vec{q}_\perp-\vec{l}_\perp)^2} = \frac{2\pi}{|l_\perp^2-q_\perp^2|}, \tag{A.13}$$

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where φ_q is the angle between $\vec{q}_\perp$ and $\vec{l}_\perp$;

$$\int_0^{2\pi} d\varphi_q \frac{\vec{q}_\perp - \vec{l}_\perp}{(\vec{q}_\perp - \vec{l}_\perp)^2} = -2\pi\theta(l_\perp - q_\perp)\frac{\vec{l}_\perp}{l_\perp^2}; \tag{A.14}$$

$$\int_0^{2\pi} \frac{d\varphi_q}{q_\perp^2 + (\vec{q}_\perp - \vec{l}_\perp)^2} = \frac{2\pi}{\sqrt{4q_\perp^4 + l_\perp^4}}; \tag{A.15}$$

$$\int_0^{2\pi} d\varphi \, e^{iz\sin\phi - in\varphi} = 2\pi J_n(z), \tag{A.16}$$

for integer $n \geq 0$;

$$\int_0^{\infty} dk\, k^{\lambda-1} J_\nu(kx) = 2^{\lambda-1}x^{-\lambda}\frac{\Gamma\left(\frac{1}{2}(\nu+\lambda)\right)}{\Gamma\left(\frac{1}{2}(2+\nu-\lambda)\right)}. \tag{A.17}$$

The integral (A.17) converges for real $x > 0$ and for $-\mathrm{Re}\ \nu < \mathrm{Re}\ \lambda < 3/2$, but it can be analytically continued outside this region of λ. A useful special case is $\nu = 0$:

$$\int_0^{\infty} dk\, k^{\lambda-1} J_0(kx) = 2^{\lambda-1}x^{-\lambda}\frac{\Gamma\left(\frac{1}{2}\lambda\right)}{\Gamma\left(1-\frac{1}{2}\lambda\right)}. \tag{A.18}$$

One can also show that

$$\int_0^{\infty} dk\, k^{\lambda-1}[1 - J_0(kx)] = -2^{\lambda-1}x^{-\lambda}\frac{\Gamma\left(\frac{1}{2}\lambda\right)}{\Gamma\left(1-\frac{1}{2}\lambda\right)}. \tag{A.19}$$

This integral converges for real x and for $-2 < \mathrm{Re}\ \lambda < 0$: it can also be analytically continued outside this region of λ.

A.3 Another useful integral

Let us find here the integral

$$I_{dip} = \int d^2x_2 \frac{x_{10}^2}{x_{20}^2 x_{21}^2}, \tag{A.20}$$

which is very useful in the Mueller's dipole model. We first note that one can write

$$d^2x_2 = 2\pi x_{02}x_{12}\int_0^{\infty} dkk\, J_0(kx_{10})\, J_0(kx_{20})\, J_0(kx_{21}), \tag{A.21}$$

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where the right-hand side is non-zero only if there exists a triangle with sides x_{10}, x_{20}, and x_{21}. Equation (A.20) becomes

$$I_{dip} = 2\pi x_{10}^2 \int_0^\infty dk\, k\, J_0(k x_{10}) \int_\rho^\infty \frac{dx_{20}}{x_{20}} J_0(k x_{02}) \int_\rho^\infty \frac{dx_{21}}{x_{21}} J_0(k x_{12}), \tag{A.22}$$

where we have inserted a UV regulator ρ into the x_{20}- and x_{21}- integrals. Writing

$$\int_\rho^\infty \frac{dx}{x} J_0(kx) = \lim_{\epsilon \to 0} \left\{ \int_0^\infty dx\, x^{\epsilon - 1} J_0(kx) - \int_0^\rho dx\, x^{\epsilon - 1} J_0(kx) \right\}, \tag{A.23}$$

we use Eq. (A.18) to perform the first integral on the right while putting $J_0(kx) = 1$ in the second integral before integrating over x, thus neglecting higher powers of the UV regulator ρ. Expanding the result in ϵ and taking the limit $\epsilon \to 0$ yields

$$\int_\rho^\infty \frac{dx}{x} J_0(kx) = \ln \frac{2}{k\rho} - \gamma_E + O(\rho). \tag{A.24}$$

Substituting Eq. (A.24) into Eq. (A.22) we obtain

$$I_{dip} = 2\pi x_{10}^2 \int_0^\infty dk\, k\, J_0(k x_{10}) \left(\ln \frac{2}{k\rho} - \gamma_E \right)^2. \tag{A.25}$$

Using (for $x_{10} > 0$)

$$\int_0^\infty dk\, k\, J_0(k x_{10}) = 0, \tag{A.26}$$

$$\int_0^\infty dk\, k\, J_0(k x_{10}) \ln k = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int_0^\infty dk k^{1+\epsilon} J_0(k x_{10}) = -\frac{1}{x_{10}^2}, \tag{A.27}$$

and

$$\int_0^\infty dk\, k\, J_0(k x_{10}) \ln^2 k = \lim_{\epsilon \to 0} \frac{\partial^2}{\partial \epsilon^2} \int_0^\infty dk\, k^{1+\epsilon} J_0(k x_{10}) = \frac{2}{x_{10}^2} \left(\ln \frac{x_{10}}{2} + \gamma_E \right), \tag{A.28}$$

all of which follow from Eq. (A.18), we can rewrite Eq. (A.25) as

$$I_{dip} = 4\pi \ln \frac{x_{01}}{\rho}, \tag{A.29}$$

which is used in arriving at the last line of Eq. (4.64) in the main text.