MATCHING TORSION AND COTORSION THEORIES

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Throughout the following note R will denote an associative ring with unit element 1. We shall denote by R-mod [resp. mod-R] the category of all unitary left [resp. right] R-modules. Morphisms in these categories will be written as acting on the side opposite scalar multiplication. All other functions will be written as acting on the left. If $\mathscr{C}$ is a category, we shall abuse notation and write " $A\in\mathscr{C}$ " when we mean " A is an object of $\mathscr{C}$ ".

We can define an equivalence relation on the class R-inj of all injective objects of R-mod by calling two modules equivalent if and only if each of them can be embedded in a direct product of copies of the other. An equivalence class on R-inj is called a *torsion theory* on R-mod. With each torsion theory τ on R-mod we associate a class $\mathscr{T}_\tau$ of τ-*torsion* modules and a class $\mathscr{F}_\tau$ of τ-*torsionfree* modules defined respectively by

$$\mathscr{T}_\tau = \{M\in R\text{-mod} \mid \mathrm{Hom}_R(M, E) = 0 \quad \text{for any} \quad E\in\tau\}$$

and

$$\mathscr{F}_\tau = \{M\in R\text{-mod} \mid M \text{ is embeddable in some element of } \tau\}.$$

This definition can easily be seen to be equivalent to the definition of a hereditary torsion theory in the sense of Dickson [**2**]. For an introduction to such theories consult [**2, 5, 8**]. If E is an element of R-inj we denote the torsion theory on R-mod containing E by $\chi(E)$.

We can partially order the set of all torsion theories on R-mod by setting $\tau \leqq \tau'$ if and only if $\mathscr{T}_\tau \subseteq \mathscr{T}_{\tau'}$. For any left R-module M with injective hull E, $\chi(E)$ is then the unique largest torsion theory relative to which M is torsionfree. Note too that there is a minimal torsion theory ξ on R-mod, namely the equivalence class of injective cogenerators of R-mod.

We can dually define an equivalence relation on the class R-proj of all projective objects of R-mod by calling two modules equivalent if and only if each of them is an epimorphic image of a direct sum of copies of the other. An equivalence class on R-proj is called a *cotorsion theory* on R-mod. Such theories are introduced and studied in [**6**]. With each cotorsion theory κ on R-mod we associate a class $\mathscr{T}^*_\kappa$ of κ-*cotorsion* modules and a class $\mathscr{F}^*_\kappa$ of κ-*cotorsionfree* modules defined respectively by

$$\mathscr{T}^*_\kappa = \{M\in R\text{-mod} \mid \mathrm{Hom}_R(P, M) = 0 \quad \text{for any} \quad P\in\kappa\}$$

and

$$\mathscr{F}^*_\kappa = \{M\in R\text{-mod} \mid M \text{ is an epimorphic image of some element of } \kappa\}.$$

If P is an element of R-proj we denote the cotorsion theory on R-mod containing P by $\eta(P)$. If $\kappa = \eta(P)$ for some finitely-generated projective left R-module P we say that κ is *finitary*.

We define torsion and cotorsion theories on mod-R in a manner similar to the above.

If U is a left R-module then the *trace* of U in R is defined to be

$$\{\Sigma u_i\alpha_i \mid u_i\in U \text{ and } \alpha_i\in\mathrm{Hom}_R(U, R)\}.$$

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This is clearly a two-sided ideal of R. If P is a projective left R-module then the trace I of P is idempotent and $IP = P$. Moreover, $I = R$ if and only if P is a generator of R-mod. The reader can easily verify that for any left R-module M, $IM = \Sigma\{P\beta_i \mid \beta \in \mathrm{Hom}_R(P, M)\}$ from which one deduces directly that two members of R-proj are equivalent if and only if their traces are equal. We can therefore define the trace of a cotorsion theory κ, denoted by I_κ, to be the trace of any one of its members. Again an easy proof which will be left to the reader shows that

$$\mathscr{T}^*_\kappa = \{M \in R\text{-mod} \mid I_\kappa M = 0\}$$

and

$$\mathscr{F}^*_\kappa = \{M \in R\text{-mod} \mid I_\kappa M = M\}.$$

If I is an idempotent two-sided ideal of R then there is a torsion theory $\delta(I)$ on R-mod satisfying $\mathscr{T}_{\delta(I)} = \{M \in R\text{-mod} \mid IM = 0\}$. Indeed, δ is a bijection between the set of idempotent two-sided ideals of R and a family of torsion theories which we will call *jansian.* (See [4], where such theories are called *TTF*-theories; these theories are interesting in their own right and have been extensively studied in the literature.) As a result, δ induces an embedding Δ of the set of all cotorsion theories on R-mod into the set of all torsion theories on R-mod defined by $\Delta : \kappa \mapsto \delta(I_\kappa)$. We then have $\mathscr{T}^*_\kappa = \mathscr{T}_{\Delta(\kappa)}$ for all cotorsion theories κ on R-mod. In particular, $\xi = \Delta(\xi^*)$, where ξ^* is the cotorsion theory of all projective generators of R-mod.

Similarly, the function δ induces an embedding Δ° of the set of all cotorsion theories on mod-R into the set of all cotorsion theories on R-mod defined by $\Delta^\circ : \lambda \mapsto \delta(I_\lambda)$, where I_λ is the trace of the cotorsion theory λ. Then $\mathscr{T}_{\Delta^\circ(\lambda)} = \{M \in R\text{-mod} \mid P \otimes_R M = 0 \text{ for any } P \in \lambda\}$.

It is natural to ask when are the functions Δ and Δ° surjections. This is far from being always true; indeed, being jansian is a strong condition for a torsion theory to satisfy. Theorem A of [3] suggests that the proper place to look is over right perfect rings and indeed we have

THEOREM. *The following conditions on a ring R are equivalent:*

(1) *R is right perfect.*
(2) *Every torsion theory on* R-mod *is of the form* $\Delta(\kappa)$ *where* κ *is a finitary cotorsion theory on* R-mod.
(3) Δ *is a bijection.*

Proof. (1)$\Rightarrow$(2): By [1, Theorem P], a right perfect ring R is *left semiartinian*; that is to say, every nonzero left R-module has a nonzero socle. (The socle of a module is the sum of its simple submodules.) Therefore the socle of any nonzero left R-module is large in it. Assume then that R is left perfect and let τ be a torsion theory on R-mod. Pick an element E of τ.

Since R is right perfect, it is in particular semiperfect and so there are only finitely many distinct isomorphism classes of simple left R-modules. In particular, we can select a set $\{M_1, \ldots, M_k\}$ of nonisomorphic simple left R-modules such that every simple submodule of E is isomorphic to one of them. Let $M = \bigoplus M_i$. Since R is semiperfect, M has a finitely-generated projective cover $\mu : P \to M$. Set $\kappa = \eta(P)$. Then we have $M = P\mu = I_\kappa P\mu = I_\kappa M$,

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and so M is κ-cotorsionfree. This implies that $I_\kappa x \neq 0$ for any $x \in M_i$ $(1 \leqq i \leqq k)$ and so $I_\kappa x \neq 0$ for any element x of the socle of E. Since the socle of E is large in E, this implies that $I_\kappa x \neq 0$ for any $x \in E$ and so E is $\Delta(\kappa)$-torsionfree. Thus $\tau = \chi(E) \geqq \Delta(\kappa)$.

Now let U be a left R-module, that does not belong to $\mathscr{T}_{\Delta(\kappa)}$. Then there exists a nonzero R-homomorphism $P \to U$ and so there is a proper submodule N of P such that P/N is isomorphic to a nonzero submodule of U. Since $\ker(\mu)$ is a small submodule of P, $N' = N + \ker(\mu)$ is also a proper submodule of P, and so P/N' is a nonzero homomorphic image of M. Since M is a completely-reducible module, it contains an isomorphic copy of each of its homomorphic images and so P/N' is isomorphic to a nonzero submodule of M. Therefore we have a nonzero homomorphism from P/N into the socle of E which, by injectivity, extends to a nonzero homomorphism $U \to E$. Therefore $U \notin \mathscr{T}_\tau$ and so $\tau \leqq \Delta(\kappa)$, proving equality.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Assume that Δ is a bijection. Then every torsion theory on R-mod is jansian. Let M be a nonzero left R-module with injective hull E. Then $\chi(E) = \Delta(\kappa)$ for some cotorsion theory κ on R-mod. If $P \in \kappa$ then by [**1**, Proposition 2.7] P has a simple homomorphic image N. Since $\mathrm{Hom}_R(P, N) \neq 0$, $N \notin \mathscr{T}_\kappa^* = \mathscr{T}_{\chi(E)}$ and so there exists a nonzero R-homomorphism $N \to E$. Since N is simple, this means that N is isomorphic to a submodule of E. Since M is large in E, this means that N is isomorphic to a submodule of M. Thus M has a nonzero socle.

We have thus shown that R is left semiartinian. This implies in particular that every torsion theory on R-mod is of the form $\chi(E)$ where E is the injective hull of a completely reducible left R-module. Since each of these is jansian, $R/J(R)$ is completely reducible by [**3**, Theorem B]. Therefore R has no infinite sets of orthogonal idempotents and so by [**1**, Theorem P], R is right perfect.

COROLLARY. *The following conditions on a ring R are equivalent:*

(1) *R is right perfect.*

(2) *Every torsion theory on* R-mod *is of the form* $\Delta^\circ(\lambda)$ *where* λ *is a finitary cotorsion theory on* mod-R.

Proof. This follows from the theorem and the fact [7] that if P is a finitely-generated projective left [resp. right] R-module then $P^* = \mathrm{Hom}_R(P, R)$ is a finitely-generated right [resp. left] R-module which has the same trace in R as P.

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