The Bulletin of Symbolic Logic
Volume 27, Number 1, March 2021

STRONG COLORINGS OVER PARTITIONS

WILLIAM CHEN-MERTENS, MENACHEM KOJMAN, AND JURIS STEPRĀNS

Abstract. A strong coloring on a cardinal κ is a function $f : [\kappa]^2 \to \kappa$ such that for every $A \subseteq \kappa$ of full size κ, every color $\gamma < \kappa$ is attained by $f \restriction [A]^2$. The symbol

$$\kappa \nrightarrow [\kappa]^2_\kappa$$

asserts the existence of a strong coloring on κ.

We introduce the symbol

$$\kappa \nrightarrow_p [\kappa]^2_\kappa$$

which asserts the existence of a coloring $f : [\kappa]^2 \to \kappa$ which is *strong over a partition* $p : [\kappa]^2 \to \theta$. A coloring f is strong over p if for every $A \in [\kappa]^\kappa$ there is $i < \theta$ so that for every color $\gamma < \kappa$ is attained by $f \restriction ([A]^2 \cap p^{-1}(i))$.

We prove that whenever $\kappa \nrightarrow [\kappa]^2_\kappa$ holds, also $\kappa \nrightarrow_p [\kappa]^2_\kappa$ holds for an arbitrary finite partition p. Similarly, arbitrary finite p-s can be added to stronger symbols which hold in any model of ZFC. If $\kappa^\theta = \kappa$, then $\kappa \nrightarrow_p [\kappa]^2_\kappa$ and stronger symbols, like $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \chi)_p$ or $\mathrm{Pr}_0(\kappa, \kappa, \kappa, \aleph_0)_p$, also hold for an arbitrary partition p to θ parts.

The symbols

$$\aleph_1 \nrightarrow_p [\aleph_1]^2_{\aleph_1}, \quad \aleph_1 \nrightarrow_p [\aleph_1 \circledast \aleph_1]^2_{\aleph_1}, \quad \aleph_0 \circledast \aleph_1 \nrightarrow_p [1 \circledast \aleph_1]^2_{\aleph_1},$$
$$\mathrm{Pr}_1(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_p, \quad \text{and} \quad \mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_p$$

hold for an arbitrary countable partition p under the Continuum Hypothesis and are independent over ZFC $+\neg$CH.

§1. Introduction. The theory of strong colorings branched off Ramsey theory in 1933 when Sierpinski constructed a coloring on $[\mathbb{R}]^2$ that contradicted the uncountable generalization of Ramsey's theorem. For many years, pair-colorings which keep their range even after they are restricted to all unordered pairs from an arbitrary, sufficiently large set were called "bad"; now they are called "strong."

Definition 1. Let $\lambda \leq \kappa$ be cardinals. A *strong λ-coloring on κ* is a function $f : [\kappa]^2 \to \lambda$ such that $\lambda = \mathrm{ran}(f \restriction [A]^2)$ for every $A \in [\kappa]^\kappa$.

Received February 19, 2020.

2020 *Mathematics Subject Classification*. Primary 03E02, 03E17, 03E35, Secondary 03E50.

Key words and phrases. strong coloring, Ramsey theory, Generalized Continuum Hypothesis, forcing, Martin axiom.

1079-8986/21/2701-0003
DOI :10.1017/bsl.2021.5

By Ramsey's theorem there are no strong λ-colorings on ω for $\lambda > 1$. Sierpinski constructed a strong two-coloring on the continuum and on $\aleph_1$.

Assertions of existence of strong colorings with various cardinal parameters are conveniently phrased with partition-calculus symbols. The (negative) square-brackets symbol

$$\kappa \nrightarrow [\kappa]^2_\lambda,$$

asserts the existence of a strong λ-coloring on κ. Recall that the symbol for Ramsey's theorem for pairs,

$$\omega \rightarrow (\omega)^2_n \tag{1}$$

reads "for every $f : [\omega]^2 \rightarrow n$ there is an infinite subset $A \subseteq \omega$ such that $f \restriction [A]^2$ is constant (omits all colors but one)." The square brackets in place of the rounded ones stand for "omits at least one color"; with the negation on the arrow, the symbol $\kappa \nrightarrow [\kappa]^2_\lambda$ means, then, "not for all colorings $f : [\kappa]^2 \rightarrow \lambda$ at least one color can be omitted on $[A]^2$ for some $A \subseteq \kappa$ of cardinality $|A| = \kappa$." That is, there exists a strong λ-coloring on κ.

When 2 is replaced with some $d > 0$ the symbol states the existence of an analogous coloring of unordered d-tuples. As Ramsey's theorem holds for all finite $d > 0$, strong d-dimensional colorings can also exist only on uncountable cardinals. In what follows we shall address almost exclusively the case $d = 2$.

Definition 2. Given a coloring $f : [\kappa]^2 \rightarrow \lambda$, a set $X \subseteq [\kappa]^2$ is *f-strong* if $\operatorname{ran}(f \restriction X) = \operatorname{ran}(f)$.

The collection of f-strong subsets of $[\kappa]^2$ is clearly upwards closed and not necessarily closed under intersections.

Different square-bracket symbols require that different families of sets are f-strong with respect to the coloring f whose existence each symbol asserts. The symbol above asserts the existence of f such that every *κ-square*, that is, every $[A]^2$ for some $A \in [\kappa]^\kappa$, is f-strong. A (λ, κ)-rectangle in $[\kappa]^2$ is a set of the form $A \circledast B = \{\{\alpha, \beta\} : \alpha < \beta < \kappa,\ \alpha \in A \text{ and } \beta \in B\}$. Every κ-square contains a (μ_1, μ_2)-rectangle if $\mu_1 \leq \mu_2 \leq \kappa$; the symbol

$$\kappa \nrightarrow [\mu_1 \circledast \mu_2]^2_\lambda,$$

which asserts the existence of $f : [\kappa]^2 \rightarrow \lambda$ such that every (μ_1, μ_2)-rectangle $A \circledast B \subseteq [\kappa]^2$ is f-strong, is, then, stronger than $\kappa \nrightarrow [\kappa]^2_\lambda$.

The next two strong-coloring symbols go beyond specifying which sets ought to be f-strong. They require the existence of certain patterns in the preimage of each color.

Definition 3.

(1) A coloring $f : [\kappa]^2 \rightarrow \lambda$ witnesses the symbol

$$\operatorname{Pr}_1(\kappa, \mu, \lambda, \chi)$$

if for every $\xi < \chi$ and a pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^{<\xi}$ of cardinality $|\mathcal{A}| = \mu$ and color $\gamma < \lambda$ there are $a, b \in \mathcal{A}$ with $\max a <$

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

min b such that $f(\alpha, \beta) = \gamma$ for all $\alpha \in a$ and $\beta \in b$. The quantified ξ above is needed only in the case that $\chi \geq \mathrm{cf}(\kappa)$, which received attention very recently. When $\chi < \mathrm{cf}(\kappa)$ we omit ξ from the definition and require only that $A \subseteq [\kappa]^{<\chi}$ instead.

(2) A coloring $f : [\kappa]^2 \to \lambda$ witnesses the symbol

$$\mathrm{Pr}_0(\kappa, \mu, \lambda, \chi)$$

if for every $\xi < \chi$, a pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\xi$ of cardinality $|\mathcal{A}| = \mu$ and a matrix $\{\gamma_{i,j} : i, j < \xi\} \subseteq \lambda$ there are $a, b \in \mathcal{A}$ with $\max a < \min b$ such that $f(\alpha(i), \beta(j)) = \gamma_{i,j}$ for all $i, j < \xi$, where $a(i), b(j)$ are the i^{th} and j^{th} elements of a and of b, respectively, in increasing order.

For $\chi > 2$ and $\mu \geq \aleph_0$, $\mathrm{Pr}_1(\kappa, \mu, \lambda, \chi)$ implies $\kappa \nrightarrow [\mu]^2_\lambda$ (see 8 below). If $\chi < \mathrm{cf}(\mu)$ then $\mathrm{Pr}_0(\kappa, \mu, \lambda, \chi)$ implies $\mathrm{Pr}_1(\kappa, \mu, \lambda, \chi)$.

Let us conclude the introduction with the remark that some authors use the term "strong coloring" only for colorings which witness Pr_1 or a stronger symbol.

§2. A brief history of strong colorings.

Strong κ-colorings on various cardinals κ were constructed by Erdős, Hajnál, Milner and Rado in the 1950s and 1960s from instances of the GCH. For every cardinal κ they were able to construct from $2^\kappa = \kappa^+$ colorings $f : [\kappa^+]^2 \to \kappa^+$ which witnessed

$$\kappa^+ \nrightarrow [\kappa \circledast \kappa^+]^2_{\kappa^+},$$

and even colorings which witnesses the stronger

$$\kappa^+ \nrightarrow [{}^{\kappa \circledast \kappa^+}\!/_{1 \circledast \kappa^+}]^2_{\kappa^+}$$

whose meaning is that inside every (κ, κ^+)-rectangle $A \circledast B \subseteq \kappa^+$ there is a $(1, \kappa^+)$-rectangle $\{\alpha\} \circledast B \subseteq A \circledast B$ such that $\mathrm{ran}(f \restriction (\{\alpha\} \circledast B)) = \kappa^+$ (see Section 49 in [5]). A coloring $f[\kappa^+]^2 \to \kappa^+$ witnesses this symbol if and only if for every $B \in [\kappa^+]^{\kappa^+}$, for all but fewer than κ ordinals $\alpha < \kappa^+$ the full range κ^+ is attained by f on the set $\{\alpha\} \circledast B = \{\{\alpha, \beta\} : \alpha < \beta \in B\}$.

Galvin [15], who was motivated by the problem of productivity of chain conditions and by earlier work of Laver, used $2^\kappa = \kappa^+$ to obtain a new class of two-colorings, which in modern notation witness $\mathrm{Pr}_1(\kappa^+, \kappa^+, 2, \aleph_0)$, and used these colorings for constructing counter examples to the productivity of the κ^+-chain condition. A straightforward modification of Galvin's proof actually gives $\mathrm{Pr}_1(\kappa^+, \kappa^+, \kappa^+, \aleph_0)$ on all successor cardinals from $2^\kappa = \kappa^+$.

A remarkable breakthrough in the theory of strong colorings was the invention of the method of ordinal-walks by Todorčević [41] (or, as it was originally called, minimal walks). Todorčević applied his method to construct strong colorings on all successors of regulars in ZFC with no additional axioms. With the same method Todorčević [43] got in ZFC the square bracket symbol for triples $\omega_2 \nrightarrow [\omega_1]^3_\omega$ and proved that $\omega_2 \nrightarrow [\omega_1]^2_{\omega_1}$ is equivalent to the negation of the $(\aleph_2, \aleph_1)$ Chang conjecture. The rectangular

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

symbol $\kappa^+ \nrightarrow [\kappa^+ \circledast \kappa^+]^2_{\kappa^+}$ has been obtained since in ZFC on all succesors of uncountable regular cardinals κ by Shelah via further developments of ordinal-walks. Moore [20] developed ordinal-walks further and provided the missing $\kappa^+ = \aleph_1$ case. Rinot and Todorčević [28] present a unified proof of the rectangle version for all successors of regulars with a completely arithmetic oscillation function.

Shelah, following Galvin [15], phrased the strong coloring relations $\mathrm{Pr}_1(\kappa, \mu, \lambda, \chi)$ and $\mathrm{Pr}_0(\kappa, \mu, \lambda, \chi)$ (and a few more!) and proved $\mathrm{Pr}_1(\kappa^{++}, \kappa^{++}, \kappa^{++}, \kappa)$ for every regular κ in ZFC [34]. Shelah also proved a criterion for stepping up from Pr_1 to Pr_0: if $\mathrm{Pr}_1(\kappa, \kappa, \lambda, \chi)$ holds, $\lambda = \lambda^{<\chi}$ and there is some "interpolant" cardinal ρ such that $\rho^{<\chi} \leq \lambda$, $2^\rho \geq \kappa$ and $\mathrm{cf}(\kappa) > \rho^{<\chi}$, then $\mathrm{Pr}_0(\kappa, \kappa, \lambda, \chi)$ holds (Lemma 4.5(3), p. 170 of [38]). In particular, chosing $\rho = \lambda$ as the interpolant, $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \aleph_0) \Rightarrow \mathrm{Pr}_0(\lambda^+, \lambda^+, \lambda^+, \aleph_0)$ for every cardinal λ; so for all regular cardinals κ, $\mathrm{Pr}_0(\kappa^{++}, \kappa^{++}, \kappa^{++}, \aleph_0)$ holds in ZFC ($\mathrm{Pr}_1(\aleph_1, \aleph_1, 3, \aleph_0)$ cannot hold in ZFC because under MA the product of two ccc spaces is ccc). See the survey in [25] for more background on strong colorings and non-productivity of chain conditions.

On successors of singulars, Todorčević [41] proved that the pcf assumption $\mathrm{pp}(\mu) = \mu^+$ for a singular μ implies $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$. Shelah proved $\mathrm{Pr}_1(\mu^+, \mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ for every singular μ (4.1 p. 67 of [38]). Eisworth [9] proved $\mathrm{Pr}_1(\mu^+ \mu^+, \mu^+, \mathrm{cf}(\mu))$ from $\mathrm{pp}(\mu) = \mu^+$. Then Rinot, building on Eisworth's [9, 10], proved that for every singular μ, $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds iff $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ holds. In particular, via Shelah's criterion, $\mathrm{Pr}_0(\mu^+, \mu^+, \mu^+, \aleph_0) \iff \mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ for all singular μ [23]. Quite recently, Peng and Wu proved in [21] that $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, n)$ holds for all $n < \omega$ outright in ZFC.

The most recent progress on strong colorings is made in a series of papers by Rinot and his collaborators. The result in [24], shown to be optimal in Theorem 3.4 in [19], establishes the property $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \chi)$ for regular $\lambda > \chi^+$ from a non-reflecting stationary subset of λ composed of ordinals of cofinality $\geq \chi$ (using a new oscillation function called $\mathrm{P}\ell_6$). In [25], Rinot gets the same result from $\square(\lambda)$, thus establishing that if $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ and the λ-chain condition is productive, then λ is weakly compact in L. Then Rinot and Zhang prove in [29] that for every regular cardinal κ, $2^\kappa = \kappa^+$ implies $\mathrm{Pr}_1(\kappa^+, \kappa^+, \kappa^+, \kappa)$ and for every inaccessible λ such that $\square(\lambda)$ and $\diamondsuit^*(\lambda)$ both hold, $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \lambda)$ holds as well (this is the case in which our remark about ξ at the end of (1) of Definition 3 is relevant). In the other direction it is proved in [29] that $\mathrm{Pr}_1(\kappa^+, \kappa^+, 2, \kappa)$ fails for every singular cardinal κ and that $\mathrm{Pr}_1(\kappa^+, \kappa^+, 2, \mathrm{cf}(\kappa)^+)$ fails for a singular limit κ of strongly compact cardinals.

Ramsey's theorem prohibits the existence of strong colorings with more than one color on countable sets for which all infinite subsets are strong, but in topological partition theory, strong colorings may exist also on countable spaces. Baumgartner [3], following some unpublished work by Galvin, constructed a coloring $c : [\mathbb{Q}]^2 \to \omega$ which attains all colors on

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

every homeomorphic copy of $\mathbb{Q}$. Todorčević [44] obtained the rectangular version of Baumgartner's result and very recently, Raghavan and Todorčević [22] proved that if a Woodin cardinal exists then for every natural number $k > 2$, for every coloring $c : [\mathbb{R}]^2 \to k$ there is homeomorphic copy of $\mathbb{Q}$ in $\mathbb{R}$ on which at most two colors occur, confirming thus a conjecture of Galvin from the 1970s. They also proved that any regular topological space of cardinality $\aleph_n$ admits a coloring of $(n+2)$-tuples which attains all ω colors on every subspace which is homeomorphic to $\mathbb{Q}$.

§3. Strong-coloring symbols over partitions. We introduce now the main new notion of symbols with an additional parameter p, where p is a partition of unordered pairs. Suppose $p : [\kappa]^2 \to \theta$ is a partition of unordered pairs from κ. A preliminary definition of the square brackets symbol $\kappa \nrightarrow_p [\kappa]^2_\kappa$ with parameter p has been mentioned in the abstract: there exists a coloring $f : [\kappa]^2 \to \kappa$ such that for every $A \in [\kappa]^\kappa$ there is some p-cell $i < \theta$ such that for all $\gamma < \kappa$ there is $\{\alpha, \beta\} \in [A]^2$ such that $p(\alpha, \beta) = i$ and $f(\alpha, \beta) = \gamma$.

However, for Pr_1 or for Pr_0 it is not possible to require a prescribed pattern on $a \circledast b$ in both f and p when a, b belong to an arbitrary $\mathcal{A}$, as all such $a \circledast b$ might meet more than one p-cell. What we do, then, is replace this definition by a different one. The new definition is equivalent to the initial definition in all square-bracket symbols by Fact 5 below, and works for Pr_1 and Pr_0.

Definition 4. Suppose $f : [\kappa]^d \to \lambda$ is a coloring and $p : [\kappa]^d \to \theta$ is a partition for a cardinal κ and natural $d > 0$. Then:

(1) For a function $\zeta : \theta \to \lambda$ and $\overline{\alpha} \in [\kappa]^d$ we say that f hits ζ over p at $\overline{\alpha}$, if $f(\overline{\alpha}) = \zeta(p(\overline{\alpha}))$.
(2) A set $X \subseteq [\kappa]^d$ is (f, p) *-strong* if for every $\zeta \in \lambda^\theta$ there is $\overline{\alpha} \in X$ such that f hits ζ over p at $\overline{\alpha}$.

Thus, the initial definition of an (f, p)-strong $X \subseteq [\kappa]^d$— that $(X \cap p^{-1}(i))$ is f-strong for some *fixed* p-cell i—is replaced in (2) above with the requirement that every *assignment of colors to p-cells* $\zeta : \theta \to \lambda$ is hit by some $\overline{d} \in X$. The advantage of the new definition is that an assignment ζ can be hit in any p-cell, so defining Pr_1 and Pr_0 over a partition will now make sense.

Topologically, a set $X \subseteq [\kappa]^d$ is (f, p)-strong iff the collection $\{u_{\langle p(\overline{\alpha}), f(\overline{\alpha})\rangle} : \overline{\alpha} \in X\}$ is an open cover of the space λ^θ of all θ-sequences over λ with the product topology, where $u_{\langle i,\gamma\rangle}$ is the basic open set $\{\zeta \in \lambda^\theta : \zeta(i) = \gamma\}$.

The definitions of the main symbols over partitions which we shall work with are in Definition 7 below; an impatient reader can proceed there directly. We precede this definition with two useful facts about (f, p)-strong sets.

If $X \subseteq [\kappa]^d$ is (f, p)-strong then for every $\gamma < \lambda$ there is $\overline{\alpha} \in X$ such that $f(\overline{\alpha}) = \gamma$ since if ζ is the constant sequence with value γ and $\overline{\alpha} \in X$ is such that $f(\overline{\alpha}) = \zeta(p(\overline{\alpha}))$ then $f(\overline{\alpha}) = \gamma$. This also follows from the next fact:

Fact 5. *A set $X \subseteq [\kappa]^d$ is (f, p)-strong if and only if there is some $i < \theta$ such that $\lambda = \mathrm{ran}(f \restriction (X \cap p^{-1}(i)))$.*

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

Proof. Suppose first that that $i < \theta$ is fixed so that $\lambda = \operatorname{ran}(f \restriction (X \cap p^{-1}(i)))$. Let $\zeta \in \lambda^\theta$ be arbitrary and let $\gamma = \zeta(i)$. Fix some $\overline{\alpha} \in X$ such that $f(\overline{\alpha}) = \gamma$ and $p(\overline{\alpha}) = i$. Now $f(\overline{\alpha}) = \zeta(p(\overline{\alpha}))$ as required.

For the other direction suppose to the contrary that for every $i < \lambda$ there is some $\zeta(i) \in (\lambda \setminus \operatorname{ran}(f \restriction X))$. Since X is (f, p)-strong, find $\overline{\alpha} \in X$ such that f hits ζ over p at $\overline{\alpha}$. Let $i = p(\overline{\alpha})$. Now $f(\overline{\alpha}) = \zeta(i) \notin \operatorname{ran}(f \restriction X)$—a contradiction. ⊣

Suppose that $h : [\kappa]^d \to \lambda^{<\mu}$ is some function into sequences of length $< \mu$. For every partition $p : [\kappa]^d \to \theta$ for some $\theta < \mu$, let $h_p : [\kappa]^d \to \lambda \cup \{*\}$ be defined by

$$h_p(\overline{\alpha}) = \begin{cases} h(\overline{\alpha})(p(\overline{\alpha})) & \text{if } p(\overline{\alpha}) \in \operatorname{dom}(h(\overline{\alpha})), \\ * & \text{otherwise.} \end{cases}$$

Then for every $\overline{\alpha} \in [\kappa]^d$, if $h_p(\overline{\alpha}) \neq *$ then h_p hits $h(\overline{\alpha})$ over p at $\overline{\alpha}$. In particular, every $X \subseteq [\kappa]^d$ which is h-strong is also (h_p, p)-strong for every partition p of $[\kappa]^d$ to $\theta < \mu$ cells. A simple book-keeping argument can waive the dependence of h_p on p for a set of $\leq \lambda^{<\mu}$ partitions:

Lemma 6. *Suppose $h : [\kappa]^d \to \lambda^{<\mu}$ is given and $\overline{p} = \langle p_\delta : \delta < \lambda^{<\mu} \rangle$ is a sequence such that $p_\delta : [\kappa]^d \to \theta_\delta$ and $\theta_\delta < \mu$ for all $\delta < \lambda^{<\mu}$. Then there is a single coloring $f : [\kappa]^d \to \lambda$ such that for all $X \subseteq [\kappa]^d$, if X is h-strong then X is (f, p_δ)-strong for all $\delta < \lambda^{<\mu}$.*

Proof. Suppose $h : [\kappa]^d \to \lambda^{<\mu}$ and $\overline{p} = \langle p_\delta : \delta < \lambda^{<\mu} \rangle$ are given, where $p_\delta : [\kappa]^d \to \theta_\delta$ and $\theta_\delta < \mu$ for every $\delta < \lambda^{<\mu}$.

Let $R = \bigcup \{\{\delta\} \times \lambda^{\theta_\delta} : \delta < \lambda^{<\mu}\}$. As $|R| = \lambda^{<\mu}$, we may fix a bijection $t : \lambda^{<\mu} \to R$ and let $g = t \circ h$. So $g : [\kappa]^d \to R$ and every $X \subseteq [\kappa]^d$ is g-strong iff it is h-strong.

Define $f : [\kappa]^d \to \lambda$ by

$$f(\overline{\alpha}) = \zeta(i) \text{ if } g(\overline{\alpha}) = \langle \delta, \zeta \rangle \text{ and } p_\delta(\overline{\alpha}) = i.$$

Let $X \subseteq [\kappa]^d$ be given and assume that X is h-strong. Let $\delta < \lambda^{<\mu}$ and some desirable $\zeta \in \lambda^{\theta_\delta}$ be given. As X is h-strong, it is also g-strong, so fix $\overline{\alpha} \in X$ such that $g(\overline{\alpha}) = \langle \delta, \zeta \rangle$. Now it holds by the definition of f that $f(\overline{\alpha}) = \zeta(p_\delta(\overline{\alpha}))$, that is f hits ζ over p_δ at $\overline{\alpha} \in X$. ⊣

We define now the main symbols over a partition. We state only the case for pairs. The definitions of the square-bracket symbols for $d \neq 2$ are similar.

Definition 7. Suppose $p : [\kappa]^2 \to \theta$ is a partition of all unordered pairs from a cardinal κ.

(1) The symbol

$$\kappa \nrightarrow_p [\mu]^2_\lambda$$

asserts the existence of a coloring $f : [\kappa]^2 \to \lambda$ such that for all $A \in [\kappa]^\mu$, for every $\zeta \in \lambda^\theta$ there is $\{\alpha, \beta\} \in [A]^2$ such that $f(\alpha, \beta) = \zeta(p(\alpha, \beta))$.

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

(2) The symbol

$$\kappa \nrightarrow_p [\mu_1 \circledast \mu_2]^2_\lambda$$

asserts the existence of a coloring $f : [\kappa]^2 \to \lambda$ such that for all $A \in [\kappa]^{\mu_1}$ and $B \in [\kappa]^{\mu_2}$, for every $\zeta \in \lambda^\theta$ there is $\{\alpha, \beta\} \in A \circledast B$ such that $f(\alpha, \beta) = \zeta(p(\alpha, \beta))$.

(3) The symbol

$$\mathrm{Pr}_1(\kappa, \mu, \lambda, \chi)_p$$

asserts the existence of a coloring $f : [\kappa]^2 \to \lambda$ such that for every $\xi < \chi$ and a family $\mathcal{A} \subseteq [\kappa]^{<\xi}$ of pairwise disjoint nonempty subsets of κ such that $|\mathcal{A}| = \mu$, for every $\zeta \in \lambda^\theta$ there are $a, b \in \mathcal{A}$ such that $\max a < \min b$ and $f(\alpha, \beta) = \zeta(p(\alpha, \beta))$ for all $\{\alpha, \beta\} \in a \circledast b$.

(4) The symbol

$$\mathrm{Pr}_0(\kappa, \mu, \lambda, \chi)_p$$

asserts the existence of a coloring $f : [\kappa]^2 \to \lambda$ such that for every $\xi < \chi$, a pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\xi$ of cardinality $|\mathcal{A}| = \mu$ and a matrix $\{\zeta_{i,j} : i, j < \xi\} \subseteq \lambda^\theta$ there are $a, b \in \mathcal{A}$ with $\max a < \min b$ such that $f(a(i), b(j)) = \zeta_{i,j}(p(a(i), b(j))$ for all $i, j < \xi$, where $a(i), b(j)$ are the i^{th} and j^{th} elements of a and of b, respectively, in increasing order. If $\chi < \mathrm{cf}(\mu)$ then $\mathrm{Pr}_0(\kappa, \mu, \lambda, \chi)_p$ implies $\mathrm{Pr}_1(\kappa, \mu, \lambda, \chi)_p$.

(5) Suppose $\overline{p} = \langle p_\delta : \delta < \delta(*)\rangle$ is a *sequence of partitions* $p_\delta : [\kappa]^2 \to \theta_\delta$. In each of the four symbols above, writing $\overline{p}$ instead of p means there exists a single coloring which witnesses simultaneously the relation with p_δ in place of p for each $\delta < \delta(*)$.

By Fact 5, the first two symbols are equivalently defined by requiring that for every $X \subseteq [\kappa]^2$ which is a μ-square or is a (μ_1, μ_2)-rectangle there is a single cell $i < \theta$ (which depends on X) such that $X \cap p^{-1}(i)$ is f-strong.

Fact 8. *Suppose $\kappa \geq \mu \geq \lambda$ are cardinals. Then every coloring f which witnesses $\mathrm{Pr}_1(\kappa, \mu, \lambda, 3)_p$ witnesses also $\kappa \nrightarrow_p [\mu \circledast \mu]^2_\lambda$. In particular,*

$$\mathrm{Pr}_1(\kappa, \mu, \lambda, 3)_p \;\;\Rightarrow\;\; \kappa \nrightarrow_p [\mu \circledast \mu]^2_\lambda$$

for every partition $p : [\kappa]^2 \to \theta$.

Proof. Fix $f : [\kappa]^2 \to \lambda$ which witnesses $\mathrm{Pr}_1(\kappa, \mu, \lambda, 3)_p$. Let $A \circledast B \subseteq [\kappa]^2$ be an arbitrary (μ, μ)-rectangle. Find inductively a pair-wise disjoint $\mathcal{A} = \{a_i : i < \mu\} \subseteq A \circledast B$. Given some $\zeta \in \lambda^\theta$, fix $a = \{\alpha, \beta\}$ and $b = \{\gamma, \delta\}$ from $\mathcal{A}$ such that $\alpha < \beta < \gamma < \delta$ and such that f hits ζ over p at all (four) elements $\{x, y\} \in a \circledast b$. In particular, f hits ζ over p at $\{\alpha, \delta\}$ which belongs to $A \circledast B$. ⊣

The next lemma is the main tool for adding a partition parameter to a strong-coloring symbol.

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

LEMMA 9. *Suppose $\kappa \geq \mu \geq \lambda \geq \rho$ are cardinals. Then for every sequence of partitions $\overline{p} = \langle p_\delta : \delta < \lambda^{<\rho} \rangle$ in which $p_\delta : [\kappa]^2 \to \theta_\delta$ and $\theta_\delta < \rho$ for $\delta < \lambda^{<\rho}$:*

(1) $$\kappa \nrightarrow [\mu]^2_{\lambda^{<\rho}} \quad \Rightarrow \quad \kappa \nrightarrow_{\overline{p}} [\mu]^2_\lambda.$$

(2) *For all $\mu' \leq \mu$,*
$$\kappa \nrightarrow [\mu' \circledast \mu]^2_{\lambda^{<\rho}} \quad \Rightarrow \quad \kappa \nrightarrow_{\overline{p}} [\mu' \circledast \mu]^2_\lambda.$$

(3) $$\kappa^+ \nrightarrow [{}^{\kappa \circledast \kappa^+}\!/\!{}_{1 \circledast \kappa^+}]^2_{\lambda^{<\rho}} \quad \Rightarrow \quad \kappa^+ \nrightarrow_{\overline{p}} [{}^{\kappa \circledast \kappa^+}\!/\!{}_{1 \circledast \kappa^+}]^2_\lambda.$$

(4) *For all $\chi > 0$,*
$$\mathrm{Pr}_0(\kappa, \mu, \lambda^{<\rho}, \chi) \quad \Rightarrow \quad \mathrm{Pr}_0(\kappa, \mu, \lambda, \chi)_{\overline{p}}.$$

(5) *For all $\chi > 0$,*
$$\mathrm{Pr}_1(\kappa, \mu, \lambda^{<\rho}, \chi) \quad \Rightarrow \quad \mathrm{Pr}_1(\kappa, \mu, \lambda, \chi)_{\overline{p}}.$$

PROOF. Given any of the first three symbols in the hypotheses above, fix a coloring $h : [\kappa]^2 \to \lambda^{<\rho}$ which witnesses it. Suppose $\overline{p} = \langle p_\delta : \delta < \lambda^{<\rho} \rangle$ is given, where $p_\delta : [\kappa]^2 \to \theta_\delta$ and $\theta_\delta < \rho$ for every $\delta < \lambda^{<\rho}$.

By Lemma 6 fix $f : [\kappa]^2 \to \lambda$ such that every $X \subseteq [\kappa]^2$ which is h-strong is also (f, p_δ)-strong for all $\delta < \lambda^{<\rho}$. Let $\delta < \lambda^{<\rho}$ be arbitrary. Suppose that $X \subseteq [\kappa]^2$ is some μ-square $[A]^2$ or X is some (μ', μ)-rectangle $A \circledast B$. Then X is (f, p_δ)-strong. This proves the first two implications. For the third, let $A \circledast B$ be some (κ, κ^+)-rectangle. By the hypothesis, there is some $\alpha \in A$ such that $\{\alpha\} \circledast B$ is h-strong, hence it is also (f, p_δ)-strong.

To prove the fourth implication, let, as in the proof of Lemma 6, $R = \bigcup \{ \{\delta\} \times \lambda^{\theta_\delta} : \delta < \lambda^{<\rho} \}$, let $g : [\kappa]^2 \to R$ witness $\mathrm{Pr}_0(\kappa, \mu, \lambda^{<\rho}, \chi)$ and let $f(\alpha, \beta) = \zeta(p_\delta(\alpha, \beta))$ when $g(\alpha, \beta) = \langle \delta, \zeta \rangle$. Suppose $\xi < \chi$ and $\mathcal{A} \subseteq [\kappa]^\xi$ is pair-wise disjoint and $|\mathcal{A}| = \mu$. Given any $\delta < \lambda^{<\rho}$ and $\{\zeta_{i,j} : i, j < \xi\} \subseteq \lambda^{\theta_\delta}$, use the fact g witnesses $\mathrm{Pr}_0(\kappa, \mu, \lambda^{<\rho}, \chi)$ to fix $a, b \in \mathcal{A}$ such that $\max a < \min b$ and $f(\alpha(i), \beta(j)) = \langle \delta, \zeta_{i,j} \rangle$ for all $i, j < \xi$, where $a(i)$ and $b(j)$ are the i^{th} and j^{th} members of a and of b respectively. Now $f(a(i), b(j)) = \zeta_{i,j}(p_\delta(a(i), b(j))$ as required.

The proof of the last implication is gotten from the fourth by using constant $\zeta_{i,j} = \zeta$. ⊣

§4. Valid symbols over partitions in ZFC and in ZFC with additional axioms.

QUESTION 10. *Suppose $\kappa \geq \rho$ are cardinals. Which strong-coloring symbols in κ hold over all $< \rho$ partitions?*

Clearly, every coloring which witnesses a strong-coloring symbol Φ over some partition p, witnesses the symbol gotten by deleting p from Φ. The question of existence of strong colorings over partition therefore refines the question of existence of strong colorings in the classical sense.

Let us mention two obvious constraints on obtaining strong-coloring symbols over partitions. Given *any* coloring $f : [\kappa]^2 \to \lambda$ with $\lambda \geq 2$, let us define, for $\alpha < \beta < \kappa$, $p(\alpha, \beta) = 0 \iff f(\alpha, \beta) = 0$ and $p(\alpha, \beta) = 1$ otherwise. Then f does *not* witness $\kappa \nrightarrow_p [\kappa]^2_\lambda$. Hence:

FACT 11. *No single coloring witnesses $\kappa \nrightarrow_p [\kappa]^2_\lambda$ for all two-partitions p if $\lambda > 1$.*

If $\theta \geq \mathrm{cf}(\kappa)$ then there is a partition $p : [\kappa]^2 \to \theta$ with $|p^{-1}(i)| < \kappa$ for every $i < \theta$, so $\kappa \nrightarrow_p [\kappa]^2_\kappa$ cannot hold. This narrows down the discussion of $\kappa \nrightarrow_p [\kappa]^2_\kappa$ to partitions $p : [\kappa^2] \to \theta$ with $\theta < \mathrm{cf}(\kappa)$.

4.1. Symbols which are valid in ZFC. Every infinite cardinal λ satisfies $\lambda^{<\aleph_0} = \lambda$. Therefore, by Lemma 9, every symbol with $\lambda \geq \aleph_0$ colors which holds in ZFC continues to hold in ZFC over any sequence of length λ of finite partitions.

Let us state ZFC symbols over partitions whose classical counterparts were mentioned in §2 above:

THEOREM 12. *For every regular cardinal κ and a sequence of length κ^+ of finite partitions of $[\kappa^+]^2$,*

$$\kappa^+ \nrightarrow_{\overline{p}} [\kappa^+ \circledast \kappa^+]^2_{\kappa^+}.$$

PROOF. The symbol without $\overline{p}$ holds by the results of Todorčević, Moore and Shelah. Now apply Lemma 9(1). ⊣

In particular,

COROLLARY 13. *For every finite partition $p : [\omega_1]^2 \to n$,*

$$\omega_1 \nrightarrow_p [\omega_1 \circledast \omega_1]^2_{\omega_1} \quad \textit{and} \quad \omega_1 \nrightarrow_p [\omega_1]^2_{\omega_1}.$$

THEOREM 14. *For every sequence of length ω_2 of finite partitions of $[\omega_2]^3$,*

$$\omega_2 \nrightarrow_{\overline{p}} [\omega_1]^3_\omega,$$

and $\omega_2 \nrightarrow_{\overline{p}} [\omega_1]^3_{\omega_1}$ is equivalent to the negation of the $(\aleph_2, \aleph_1)$ Chang conjecture.

PROOF. The symbol $\omega_2 \nrightarrow [\omega_1]^3_\omega$ holds by Todorevic's [43], and now apply Lemma 6 as in the proof of Lemma 9. ⊣

THEOREM 15. *For every cardinal κ and a list $\overline{p}$ of length κ^{++} of finite partitions of $[\kappa^{++}]^2$,*

$$\mathrm{Pr}_1(\kappa^{++}, \kappa^{++}, \kappa^{++}, \kappa)_{\overline{p}}.$$

PROOF. By Shelah's [34] and Lemma 9(4). ⊣

THEOREM 16. *For every cardinal κ and a list $\overline{p}$ of length κ^{++} of finite partitions of $[\kappa^{++}]^2$,*

$$\mathrm{Pr}_0(\kappa^{++}, \kappa^{++}, \kappa^{++}, \aleph_0)_{\overline{p}}.$$

PROOF. By Shelah's [34], 4.5(3) p. 170 in [38] and Lemma 9(5). ⊣

THEOREM 17. *For every singular cardinal μ and a sequence of length $\mathrm{cf}(\mu)$ of finite partitions of $[\mu^+]^2$,*

$$\mathrm{Pr}_1(\mu^+, \mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))_{\overline{p}}.$$

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

PROOF. By Shelah's 4.1 p. 67 of [38] and Lemma 9(4). ⊣

THEOREM 18. *For every singular μ and a sequence $\overline{p}$ of length μ^+ of finite partitions of* $[\mu^+]^2$,

$$\mu^+ \nrightarrow [\mu^+]^2_{\mu^+} \Rightarrow \mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))_{\overline{p}} \wedge \mathrm{Pr}_0(\mu^+, \mu^+, \mu^+, \aleph_0)_{\overline{p}}.$$

PROOF. Suppose $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$. By Rinot's [23], also $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds. The first conjuct now follows by Lemma 9(4). To get the second conjunct observes that by the first conjunct we have in particular $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \aleph_0)$. The second conjunct follows now by 4.5(3) in [38] and Lemma 9(5). ⊣

4.2. Symbols from instances of the GCH or of the SCH. If the GCH holds then every regular cardinal λ satisfies $\lambda^{<\lambda} = \lambda$. Thus,

THEOREM 19 (GCH). *In Theorems (12)–(18) above, "finite partitions" may be replaced by $< \lambda$-partitions.*

The GCH also makes Shelah's implication 4.5(3) from [38] valid in additional cases. For example,

THEOREM 20 (GCH). *For every regular cardinal κ and a sequence $\overline{p}$ of length κ^{++} of κ^+-partitions,*

$$\mathrm{Pr}_0(\kappa^{++}, \kappa^{++}, \kappa^{++}, \kappa)_{\overline{p}}.$$

PROOF. By Shelah's [34] we have $\mathrm{Pr}_1(\kappa^{++}, \kappa^{++}, \kappa^{++}, \kappa)$ in ZFC. Let $\rho = \kappa^+$. By the GCH, $\rho^{<\kappa} = \rho$ and $(\kappa^{++})^{<\kappa} = \kappa^{++}$, so ρ qualifies as an interpolant in 4.5(3) p. 170 in [38] and $\mathrm{Pr}_0(\kappa^{++}, \kappa^{++}, \kappa^{++}, \kappa)$ follows. Now use GCH again with Lemma 9(5). ⊣

THEOREM 21. *For every cardinal κ, if $2^\kappa = \kappa^+$ then for every sequence p of length κ^+ of κ-partitions of* $[\kappa]^2$,

$$\kappa^+ \nrightarrow_{\overline{p}} [{}^{\kappa \circledast \kappa^+}\!/_{1 \circledast \kappa^+}]^2_{\kappa^+}.$$

PROOF. The symbol $\kappa^+ \nrightarrow [{}^{\kappa \circledast \kappa^+}\!/_{1 \circledast \kappa^+}]^2_{\kappa^+}$ follows from $2^\kappa = \kappa^+$ by the and Erdoős–Hajnal–Milner theorem (see Section 49 in [5]). Use now Lemma 9(2). ⊣

THEOREM 22. *For every singular cardinal μ, if $\mathrm{pp}(\mu) = \mu^+$ then for every sequence $\overline{p}$ of length μ^+ of finite partitions of* $[\mu^+]^2$,

$$\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))_{\overline{p}}.$$

PROOF. By $\mathrm{pp}(\mu) = \mu^+$ and Eisworth's theorem [10], $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds. Now use Lemma 9(4). ⊣

THEOREM 23 (GCH). *For every singular cardinal μ and a sequence $\overline{p}$ of length μ^+ of μ-partitions of* $[\mu^+]^2$,

$$\mathrm{Pr}_0(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))_{\overline{p}}.$$

PROOF. By Eisworth's theorem it holds that $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$. By the GCH and Shelah's 4.5(3) in [38], also $\mathrm{Pr}_0(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds.

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

Finally, as $(\mu^+)^\mu = \mu^+$, by Lemma 9(5), for every sequence $\overline{p}$ of length μ^+ of μ-partitions of $[\mu]^+$ it holds that $\mathrm{Pr}_0(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))_{\overline{p}}$. $\dashv$

In the next theorem a different cardinal arithmetic assumption appears:

THEOREM 24. *If μ is a singular cardinal and $2^{\mathrm{cf}(\mu)} > \mu$ then for every sequence $\overline{p}$ of length μ^+ of finite partitions of $[\mu^+]^2$,*

$$\mathrm{Pr}_0(\mu^+, \mu^+, \mathrm{cf}(\mu), \aleph_0)_{\overline{p}}.$$

PROOF. By Shelah's 4.1 p. 67 the symbol $\mathrm{Pr}_1(\mu^+, \mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ holds in ZFC. Choose $\rho = \mathrm{cf}(\mu)$. So $2^\rho \geq \mu^+$, $\rho^{<\aleph_0} = \rho$ and $\mathrm{cf}(\mu^+) > \rho^{<\aleph_0}$, so ρ qualifies as an interpolant cardinal in 4.5(3) p. 170 in [38] and $\mathrm{Pr}_0(\mu^+, \mu^+, \mathrm{cf}(\mu), \aleph_0)$ follows. Now use Lemma 9(5). $\dashv$

Lastly in this section, we show that $\text{⫯}(\kappa)$, an axiom (stated in the proof below), which does not imply $2^\kappa = \kappa^+$, implies the following rectangular square-brackets symbol.

THEOREM 25. *If κ is a cardinal and $\text{⫯}(\kappa^+)$ holds then for every sequence of partitions $\overline{p} = \langle p_\gamma : \gamma < \kappa^+ \rangle$, where $p_\gamma : [\kappa^+]^2 \to \lambda_\gamma$ and $\lambda_\gamma < \mathrm{cf}(\kappa)$ for each $\gamma < \kappa^+$, it holds that*

$$\kappa^+ \not\twoheadrightarrow_{\overline{p}} [{}^{\kappa \circledast \kappa^+} / {}_{1 \circledast \kappa^+}]^2_{\kappa^+}.$$

That is, there exists a coloring $f : [\kappa^+]^2 \to \kappa^+$ such that for every (κ^+, κ^+)-rectangle $A \circledast B$ and $\gamma < \omega_1$ there is $j < \lambda_\gamma$ and $X \in [A]^\kappa$ such that

$$\kappa^+ = \mathrm{ran}\big(f \restriction [(X \circledast B) \cap p_\gamma^{-1}(j)]\big).$$

PROOF. Suppose a sequence of partitions $\overline{p} = \langle p_\gamma : \gamma < \kappa^+ \rangle$ is given as above and we shall define the required f assuming $\text{⫯}(\kappa^+)$. Fix a sequence $\langle X_i : i < \kappa^+ \rangle$ which witnesses $\text{⫯}(\kappa^+)$, that is: $X_i \subseteq \kappa^+$, $\mathrm{otp}(X_i) = \kappa$ for each $i < \kappa^+$ and for every $A \in [\kappa^+]^{\kappa^+}$ there exists some $i < \kappa^+$ such that $X_i \subseteq A$.

Let $\beta < \kappa^+$ be arbitrary. Towards defining $f(\alpha, \beta)$ for $\alpha < \beta$, let us define, for every triple $\langle \gamma, i, j \rangle$ such that $\gamma, i < \beta$ and $j < \lambda_\gamma$,

$$A^\beta_{\langle \gamma, i, j \rangle} = \{\alpha < \beta : \alpha \in X_i \wedge p_\gamma(\alpha, \beta) = j\}. \tag{2}$$

Let

$$\mathcal{A}_\beta = \left\{ A^\beta_{\langle \gamma, i, j \rangle} : \gamma, i < \beta \wedge j < \lambda_\gamma \wedge |A^\beta_{\langle \gamma, i, j \rangle}| = \kappa \right\}. \tag{3}$$

As $\mathcal{A}_\beta$ is a family of at most κ subsets of β, each of cardinality κ, we may fix a disjoint refinement $\mathcal{D}_\beta = \{D^\beta_{\langle \gamma, i, j \rangle} : A^\beta_{\langle \gamma, i, j \rangle} \in \mathcal{A}_\beta\}$, that is, each $D^\beta_{\langle \gamma, i, j \rangle} \subseteq A^\beta_{\langle \gamma, i, j \rangle}$ has cardinality κ and $\langle \gamma, i, j \rangle \neq \langle \gamma', i', j' \rangle \Rightarrow D^\beta_{\langle \gamma, i, j \rangle} \cap D^\beta_{\langle \gamma', i', j' \rangle} = \emptyset$ for any $A^\beta_{\langle \gamma, i, j \rangle}, A^\beta_{\langle \gamma', i', j' \rangle} \in \mathcal{A}_\beta$.

Let us define now $f(\alpha, \beta)$ for all α below our fixed β by cases. For each $D^\beta_{\langle \gamma, i, j \rangle} \in \mathcal{D}_\beta$ define $f \restriction (D^\beta_{\langle \gamma, i, j \rangle} \circledast \{\beta\})$ to be some function *onto* β. This is possible since $|D^\beta_{\langle \gamma, i, j \rangle}| = \kappa$ and $\beta < \kappa^+$ (so $|D^\beta_{\langle \gamma, i, j \rangle}| = |\beta|$) and because the

$D^\beta_{\langle\gamma,i,j\rangle}$ are pairwise disjoint, hence $(D^\beta_{\langle\gamma,i,j\rangle} \circledast \{\beta\}) \cap (D^\beta_{\langle\gamma',i',j'\rangle} \circledast \{\beta\}) = \emptyset$ when $\langle\gamma, i, j\rangle \neq \langle\gamma', i', j'\rangle$.

For $\alpha \in \beta \setminus \bigcup \mathcal{D}_\beta$ define $f(\alpha, \beta)$ arbitrarily (say, as 0). As β was arbitrary, we have defined $f(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$. By this definition, for all $\beta < \kappa^+$ and $D^\beta_{\langle\gamma,i,j\rangle} \in \mathcal{D}_\beta$,

$$\beta = \operatorname{ran}(f \restriction (D^\beta_{\langle\gamma,i,j\rangle} \circledast \{\beta\}). \tag{4}$$

To see that f satisfies what Theorem 25 states, let $A, B \subseteq \kappa^+$ be arbitrary with $|A| = |B| = \kappa^+$ and let $\gamma < \kappa^+$ be given. Using the properties of the $\clubsuit(\kappa^+)$-sequence, fix some $i < \kappa^+$ such that

$$X_i \subseteq A. \tag{5}$$

As $X_i \subseteq \kappa^+$ and $\operatorname{otp}(X_i) = \kappa$, $\sup(X_i) < \kappa^+$, hence $\beta_0 := \max\{\gamma, i, \sup X_i\} < \kappa^+$.

If $\beta \in B$ is any ordinal such that $\beta > \beta_0$ then $X_i \subseteq \beta$ and as $|X_i| = \kappa$ while $\lambda_\gamma < \operatorname{cf}\kappa$, there exists some $j(\beta) < \lambda_\gamma$ such that

$$|\{\alpha \in X_i : p_\gamma(\alpha, \beta) = j\}| = \kappa,$$

that is, by (2) and (3),

$$A^\beta_{\langle\gamma,i,j(\beta)\rangle} \in \mathcal{A}_\beta.$$

By the regularity of κ^+ and the assumption that $\lambda_\gamma < \operatorname{cf}\kappa < \kappa^+$, we can fix some $B' \subseteq B \setminus (\beta_0 + 1)$ and $j(*) < \lambda_\gamma$ such that $j(\beta) = j(*)$ for all $\beta \in B$.

For each $\beta \in B'$ it holds, then, that $A^\beta_{\langle\gamma,i,j(*)\rangle}$ belongs to $\mathcal{A}_\beta$, and therefore also

$$D^\beta_{\langle\gamma,i,j(*)\rangle} \in \mathcal{D}_\beta. \tag{6}$$

Now, for each $\beta \in B'$ we have by (6) and (4) that

$$\beta = \operatorname{ran}(c \restriction D^\beta_{\langle\gamma,i,j(*)\rangle} \circledast \{\beta\}),$$

and as $D^\beta_{\langle\gamma,i,j(*)\rangle} \subseteq X_i \cap p_\gamma^{-1}(j(*))$ by (2),

$$\beta = \operatorname{ran}(f \restriction [(X_i \circledast \{\beta\}) \cap p_\gamma^{-1}(j(*))]).$$

As $B' \subseteq B$ is unbounded in κ^+ it follows, after setting $X = X_i$ and $j = j(*)$, that

$$\kappa^+ = f \restriction [(X \circledast B) \cap p_\gamma^{-1}(j)]. \qquad \dashv$$

§5. Independence results on $\aleph_1$. In this section we shall show that the existence of strong colorings over *countable* partitions of $[\omega_1]^2$ is independent over ZFC and over ZFC + $2^{\aleph_0} > \aleph_1$.

THEOREM 26. *If the CH holds, then the following five symbols are valid for every sequence of partitions $\overline{p} = \langle p_\delta : \delta < \omega_1\rangle$ where $p_\delta : [\omega_1]^2 \to \omega$:*

- $\aleph_1 \nrightarrow_{\overline{p}} [\aleph_1]^2_{\aleph_1}$,
- $\aleph_1 \nrightarrow_{\overline{p}} [\aleph_1 \circledast \aleph_1]^2_{\aleph_1}$,

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

- $\aleph_1 \nrightarrow_{\overline{p}} [{}^{\aleph_0 \circledast \aleph_1}\!\!\diagup_{1 \circledast \aleph_1}]^2_{\aleph_1}$,
- $\mathrm{Pr}_1(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\overline{p}}$,
- $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\overline{p}}$.

PROOF. Assume CH, that is, $2^{\aleph_0} = \aleph_1$. Then $\mathrm{Pr}_1(\aleph_1, \aleph_1, \aleph_1, \aleph_0)$ holds by (a slight strengthening of) Galvin's theorem. By Shelah's 4.5(3) from [38], also $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)$ holds. The CH also implies that $(\aleph_1)^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_1$. By Lemma 9(5), then, for every ω_1-sequence $\overline{p}$ of countable partitions of $[\omega_1]^2$ it holds that

$$\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\overline{p}}$$

and therefore by Lemma 8 also

$$\mathrm{Pr}_1(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\overline{p}}, \quad \omega_1 \nrightarrow_{\overline{p}} [\omega_1]^2_{\omega_1} \quad \text{and} \quad \omega_1 \nrightarrow_{\overline{p}} [\omega_1 \circledast \omega_1]^2_{\omega_1}.$$

Similarly, by the CH and Theorem 21 in the previous section,

$$\aleph_1 \nrightarrow_{\overline{p}} [{}^{\aleph_0 \circledast \aleph_1}\!\!\diagup_{1 \circledast \aleph_1}]^2_{\aleph_1}. \qquad \dashv$$

We prove next that these five symbols are valid in all models of ZFC obtained by adding $\aleph_2$ Cohen reals over an arbitrary model V of ZFC, and, more generally, by forcing with a finite-support ω_2-iteration of σ-linked posets over an arbitrary model V of ZFC.

Before proving yet another combinatorial property in a Cohen extension let us recall Roitman's [30] proof that the addition of a single Cohen real introduces an S-space, Todorčević's presentation in [42], p. 26 and Rinot's blog-post [26] in which it is shown that a single Cohen real introduces $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$. For a short proof of Shelah's theorem that a single Cohen real introduces a Suslin line see [40]. Fleissner [14] proved that adding λ Cohen reals introduces two ccc spaces whose product is not λ-cc. Hajnal and Komjath [17] proved that adding one Cohen subset to a cardinal $\kappa = \kappa^{<\kappa}$ forces the statement $Q(\kappa^+)$ they defined, following [4]: for every graph $G = \langle \kappa^+, E\rangle$ with $\chi(G) = \kappa^+$ there is a coloring $f : E \to \kappa^+$ such that for every partition of κ^+ to κ parts, all colors are gotten by f on edges from a single part. It is still open if $Q(\aleph_1)$ holds in ZFC.

THEOREM 27. *If $\mathbb{C}_{\aleph_2}$ is the partial order for adding $\aleph_2$ Cohen reals then for every sequence $\overline{p} = \langle p_\delta : \delta < \omega_1\rangle$ of partitions $p_\delta : [\omega_1]^2 \to \omega$ in the forcing extension by $\mathbb{C}_{\aleph_2}$,*

$$1 \Vdash_{\mathbb{C}_{\aleph_2}} \text{“}\aleph_1 \nrightarrow_{\overline{p}} [{}^{\aleph_0 \circledast \aleph_1}\!\!\diagup_{1 \circledast \aleph_1}]^2_{\aleph_1}.\text{''}$$

PROOF. Let $\mathbb{C}_\alpha$ be the partial order of finite partial functions from $[\alpha]^2$ to ω. Let V be a model of set theory and let $G \subseteq \mathbb{C}_{\omega_2}$ be generic over V. Then $\bigcup G : [\omega_2]^2 \to \omega$.

Now suppose that $\overline{p} = \langle p_\delta : \delta < \omega_1\rangle$ is an arbitrary sequence of partitions $p_\delta : [\omega_1]^2 \to \omega$ in $V[G]$. As there is some $\alpha \in \omega_2$ such that $\overline{p} \in V[G \cap \mathbb{C}_\alpha]$, it may be assumed that $\overline{p} \in V$. Let $c = \bigcup G \restriction [\omega_1]^2$. So $c : [\omega_1]^2 \to \omega$. In V, fix a sequence $\langle e_\alpha : \omega \leq \alpha < \omega_1\rangle$, where $e_\alpha : \omega \to \alpha$ is a bijection. In the

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

generic extension, define a coloring $f : [\omega_1]^2 \to \omega_1$ by

$$f(\alpha, \beta) = e_\beta(c(\alpha, \beta)),$$

for $\beta \geq \omega$ and as 0 otherwise.

To see that f witnesses $\aleph_1 \not\to_{\overline{p}} [\aleph_1]^2_{\aleph_1}$ suppose that there is some $\delta < \omega_1$ for which f fails to witness $\aleph_1 \not\to_{\overline{p}} [\aleph_1]^2_{\aleph_1}$. This means that in V^G there are $A \in [\omega_1]^\omega$ and $B \in [\omega_1]^{\omega_1}$ such that for all $\alpha \in A$ there is some $W(\alpha) \in \omega_1^\omega$ such that for all $\beta \in B \setminus (\alpha + 1)$ it holds that $f(\alpha, \beta) \neq W(\alpha)(p_\delta(\alpha, \beta))$. Let $\dot{A}$ and $\dot{W}$ be countable names for A and W and let $\dot{B}$ be a name for B. Let $r \in G$ decide δ and force

$$r \Vdash \text{“}(\forall \alpha \in \dot{A})(\forall \beta \in \dot{B} \setminus (\alpha + 1))\,(f(\alpha, \beta)) \neq \dot{W}(\alpha)(p_\delta(\alpha, \beta)).\text{”}$$

Let $\mathfrak{M}$ be a countable elementary submodel of $H(\omega_2, \dot{A}, \dot{B}, \dot{W}, r)$.

Fix an extension $r' \in G$ of r and an ordinal $\beta \in \omega_1 \setminus \sup(\mathfrak{M} \cap \omega_1)$ such that $r' \Vdash \beta \in \dot{B}$. Let $r_0 = r' \cap \mathfrak{M}$. Inside $\mathfrak{M}$ extend r_0 to r_1 such that $r_1 \Vdash$ “$\alpha \in \dot{A}$” for an ordinal α which is not in $\bigcup \operatorname{dom}(r')$ and r_1 decides $W(\alpha)(p_\delta(\alpha, \beta))$. Thus, $\{\alpha, \beta\} \notin \operatorname{dom}(r' \cup r_1)$. Let

$$r^* = r' \cup r_1 \cup \left\{\langle \{\alpha, \beta\}, e_\beta^{-1}(W(\alpha)(p_\delta(\alpha, \beta)))\rangle\right\}.$$

Since r^* extends r and $f(\alpha, \beta) = e_\beta(c(\alpha, \beta)) = W(\alpha)(p_\delta(\alpha, \beta))$, this is a contradiction to the choice of r. ⊣

The forcing for adding a single Cohen real is obviously σ-linked. Thus, the next theorem applies to a broader class of posets than Cohen forcing. The previous theorem holds also in this generality.

THEOREM 28. *If $\mathbb{P}$ is an ω_2-length finite support iteration of σ-linked partial orders then*

$$1 \Vdash_{\mathbb{P}} \text{“}\operatorname{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\bar{p}}\text{''}$$

for any ω_1 sequence of partitions $\bar{p} = \langle p_\delta : \delta < \omega_1\rangle$ such that $p_\delta : [\omega_1]^2 \to \omega$ for all $\delta < \omega_1$.

PROOF. Let $\mathbb{P}_\alpha$ be the finite support iteration of the first α partial orders and suppose that

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}\mathbb{Q}_\alpha = \bigcup_{n \in \omega} \mathbb{Q}_{\alpha,n} \text{ and each } \mathbb{Q}_{\alpha,n} \text{ is linked,”} \tag{7}$$

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}\{\dot{q}_{\alpha,n}\}_{n \in \omega} \text{ is a maximal antichain in } \mathbb{Q}_\alpha.\text{”} \tag{8}$$

Let $B : [\omega_2]^2 \to \omega_2$ be a bijection and let $e_\xi : \omega \to \xi$ be a bijection for each infinite $\xi \in \omega_1$. Let V be a model of set theory, let $G \subseteq \mathbb{P}$ be generic over V and let G_α be the generic filter induced on $\mathbb{Q}_\alpha$ by G.

Now suppose that a sequence of partitions $\bar{p} = \langle p_\delta : \delta < \omega_1\rangle$ such that $p_\delta : [\omega_1]^2 \to \omega$ belongs to $V[G]$. As there is some $\alpha \in \omega_2$ such that $\bar{p} \in V[G \cap \mathbb{P}_\alpha]$, it may be assumed that $\bar{p} \in V$. There is no harm in assuming that B maps $[\omega_1]^2$ to ω_1 so let $c : [\omega_1]^2 \to \omega_1$ be defined by $c(\alpha, \beta) = \xi$ if and only if $q_{B(\alpha,\beta),k} \in G_{B(\alpha,\beta)}$ and $\xi = e_{B(\alpha,\beta)}(k)$.

To see that c witnesses $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\bar{p}}$ suppose that:

- $\delta \in \omega_1$.
- $k > 0$.
- $\dot{\alpha}_{\xi,i} < \omega_1$ are $\mathbb{P}$-names for $\xi < \omega_1$ and $1 \leq i \leq k$ of distinct ordinals such that for every $\xi < \omega_1$ the sequence $\langle \dot{\alpha}_{\xi,i} : 1 \leq i \leq \kappa \rangle$ is increasing with i.
- $(\dot{N}_{i,j})$ is a $\mathbb{P}$-name for a $k \times k$ matrix with entries in ${\omega_1}^{\omega}$.

We may fix $q_\xi \in G$ such that:

(1) $q_\xi \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_{\xi,i} = \alpha_{\xi,i}$" for all $1 \leq i \leq k$,
(2) $\{\alpha_{\xi,1}, \alpha_{\xi,2}, \dots, \alpha_{\xi,k}\} \subseteq d_\xi = \bigcup B^{-1}(\mathrm{dom}\,(q_\xi))$,
(3) for all $\mu \in \mathrm{dom}\,(q_\xi)$ there is $n_{\mu,\xi}$ such that $q_\xi \restriction \mu \Vdash_{\mathbb{P}_\mu}$ "$q_\xi(\mu) \in \mathbb{Q}_{\mu, n_{\mu,\xi}}$."

Let $\{\mathfrak{M}_\eta\}_{\eta \in \omega+1}$ be countable elementary submodels of

$$H(\omega_2, \{q_\xi, \{\alpha_{\xi,1}, \alpha_{\xi,2}, \dots, \alpha_{\xi,k}\}\}_{\xi\in\omega_1}, B, \dot{N}, G)$$

such that $\mathfrak{M}_j \prec \mathfrak{M}_{j+1} \prec \mathfrak{M}_\omega$ and $\omega_1 \cap \mathfrak{M}_j \in \mathfrak{M}_{j+1}$ for each $j \in \omega$. Let $\xi_\omega \in \omega_1 \setminus \mathfrak{M}_\omega$. By elementarity there are $\xi_j \in \omega_1 \cap \mathfrak{M}_j$ such that:

(1) $\mathrm{dom}\,(q_{\xi_\omega}) \cap \mathfrak{M}_j \subseteq \mathrm{dom}\,(q_{\xi_j})$.
(2) $n_{\mu,\xi_j} = n_{\mu,\xi_\omega}$ for each $\mu \in \mathrm{dom}\,(q_{\xi_\omega}) \cap \mathfrak{M}_j$.

Note that $\{\alpha_{\xi_\omega,1}, \alpha_{\xi_\omega,2}, \dots, \alpha_{\xi_\omega,k}\} \cap \mathfrak{M}_\omega = \varnothing$ and hence

$$(\forall j \in \omega)(\forall u \in k)(\forall v \in k)\ B(\alpha_{\xi_j,u}, \alpha_{\xi_\omega,v}) \notin \mathfrak{M}_\omega. \tag{9}$$

Furthermore, note that $\bigcup B^{-1}(\mathrm{dom}\,(q_{\xi_\omega}))$ is finite and so there is J such that

$$\bigcup B^{-1}(\mathrm{dom}\,(q_{\xi_\omega})) \cap \mathfrak{M}_\omega \subseteq \mathfrak{M}_J.$$

From (9) it follows that

$$B(\alpha_{\xi_J,u}, \alpha_{\xi_\omega,v}) \notin \mathrm{dom}\,(q_{\xi_J}) \cup \mathrm{dom}\,(q_{\xi_\omega}). \tag{10}$$

From condition (19) in the choice of q_ξ and condition (1) in the choice of ξ_j, it follows that there is q^* such that $q^* \leq q_{\xi_J}$ and $q^* \leq q_{\xi_\omega}$ and $\mathrm{dom}\,(q^*) = \mathrm{dom}\,(q_{\xi_J}) \cup \mathrm{dom}\,(q_{\xi_\omega})$.

Let $\mathcal{A} \in \mathfrak{M}_\omega$ be a maximal antichain such that for every conditions $r \in \mathcal{A}$,

$$r \Vdash_{\mathbb{P}} \text{“}M_{u,v} = \dot{N}_{u,v}(p_\delta(a_{\xi_J,u}, a_{\xi_\omega,v}))\text{”}$$

for some $k \times k$ matrix $(M_{i,j})$ with entries in ω_1.

By the countable chain condition, $\mathcal{A}$ is countable and hence $\mathcal{A} \subseteq \mathfrak{M}_\omega$. Let $r \in \mathcal{A}$ be such that r is compatible with q^* and let $(M_{i,j})$ be the $k \times k$ matrix which witnesses that $r \in \mathcal{A}$. Let $q^{**} \leq q^*, r$.

Note that $B(\alpha_{\xi_J,u}, \alpha_{\xi_\omega,v}) \notin \mathrm{dom}\,(q^{**})$ because $\mathrm{dom}\,(q^{**}) \setminus (\mathrm{dom}\,(q_{\xi_J}) \cup \mathrm{dom}\,(q_{\xi_\omega})) \subseteq \mathfrak{M}_\omega$ and (9) and (10) hold. Let

$$\hat{q}(\theta) = \begin{cases} q^{**}(\theta) & \text{if } \theta \notin \{B(a_{\xi_J,u}, a_{\xi_\omega,v})\}_{u,v\in k}, \\ q_{\theta, e^{-1}_{B(a_{\xi_J,u}, a_{\xi_\omega,v})}(M_{u,v})} & \text{if } \theta = B(a_{\xi_J,u}, a_{\xi_\omega,v}). \end{cases}$$

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

Then by the definition of c

$$\hat{q} \Vdash_{\mathbb{P}} \text{“}c(\alpha_{\xi_J,u}, \alpha_{\xi_\omega,v}) = M_{u,v} = \dot{N}_{u,v}((p_\delta(a_{\xi_J,u}, a_{\xi_\omega,v}))\text{”}$$

for each u and v as required. $\dashv$

COROLLARY 29. *It is consistent with $MA_{\aleph_1}$(σ-linked) that* $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\bar{p}}$ *holds for any ω_1 sequence of partitions $\bar{p} = \{p_\xi\}_{\xi \in \omega_1}$ such that $p_\xi : [\omega_1]^2 \to \omega$.*

Now we prove that the symbol

$$\omega_1 \nrightarrow_p [\omega_1]^2_{\omega_1}$$

can consistently *fail* for some $p : [\omega_1]^2 \to \omega$.

We actually prove more. The failure of the symbol above over a partition $p : [\omega_1]^2 \to \omega$, symbolically written as

$$\omega_1 \to_p [\omega_1]^2_{\omega_1},$$

means that for every coloring $f : [\omega_1]^2 \to \omega_1$ there is a set $A \in [\omega_1]^{\aleph_1}$ such that $f \restriction ([A]^2 \cap p^{-1}(i))$ omits at least one color for every $i < \omega$. Let us introduce the following symbol:

$$\omega_1 \to_p [\omega_1]^2_{\omega_1 \setminus \omega_1},$$

to say that for every coloring $f : [\omega_1]^2 \to \omega_1$ there is a set $A \in [\omega_1]^{\aleph_1}$ such that for every $i < \omega$ a set of size $\aleph_1$ of colors is omitted by $f \restriction ([A]^2 \cap p^{-1}(i))$. An even stronger failure (via breaking ω_1 to two disjoint equinumerous sets and identifying all colors in each part) is

$$\omega_1 \to_p [\omega_1]^2_2.$$

It is the consistency of the latter symbol which we prove. Note that with the rounded-brackets symbol in (1) from the introduction we may write this failure as:

$$\omega_1 \to_p (\omega_1)^2_2,$$

whose meaning is that for every coloring $f : [\omega_1]^2 \to 2$ there is $A \in [\omega_1]^{\aleph_1}$ such that for every $i < \omega$ the set $[A]^2 \cap p^{-1}(i)$ is f-monochromatic. Thus, while $\omega_1 \nrightarrow [\omega_1]^2_{\omega_1}$ holds in ZFC, it is consistent that for a suitable countable partition p the symbol $\omega_1 \nrightarrow_p [\omega_1]^2_{\omega_1}$ fails pretty badly.

THEOREM 30. *It is consistent that $2^{\aleph_0} = \aleph_2$ and there is a partition $p : [\omega_1]^2 \to \omega$ such that*

$$\omega_1 \to_p [\omega_1]^2_2.$$

COROLLARY 31. *It is consistent that $2^{\aleph_0} = \aleph_2$ and there is some $p : [\omega_1]^2 \to \omega$ such that*

$$\omega_1 \to_p [\omega_1]^2_{\omega_1 \setminus \omega_1} \quad \textit{and hence} \quad \omega_1 \to_p [\omega_1]^2_{\omega_1}.$$

PROOF OF THE THEOREM. Let $\mathbb{P}$ be the partial order of finite partial functions from $[\omega_1]^2 \to \omega$ ordered by inclusion. More precisely, each

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

condition $q \in \mathbb{P}$ has associated to it a finite subset of ω_1 which, abusing notation, will be called $\operatorname{dom}(q)$. Then q is a function $[\operatorname{dom}(q)]^2 \to \omega$.

Given any partition $p : [\omega_1]^2 \to \omega$ and a colouring $c : [\omega_1]^2 \to 2$ define the partial order $\mathbb{Q}(p, c)$ to be the set of all pairs (h, w) such that

- $w \in [\omega_1]^{<\aleph_0}$,
- $h : m \to 2$ for some $m \in \omega$ so that $m \supseteq p([w]^2)$,
- $c(\{\alpha, \beta\}) \neq h(p(\{\alpha, \beta\}))$ for each $\{\alpha, \beta\} \in [w]^2$

and order $\mathbb{Q}(p, c)$ by coordinatewise extension. Let V be a model of set theory in which $2^{\aleph_1} = \aleph_2$ and let $\{c_\xi\}_{\xi \in \omega_2}$ enumerate cofinally often the subsets of hereditary cardinality less than $\aleph_2$. If $G \subseteq \mathbb{P}$ is generic over V, in $V[G]$ define $p_G = \bigcup G$. Then define a finite support iteration $\{\mathbb{Q}_\zeta\}_{\zeta \in \omega_2}$ such that $\mathbb{Q}_1 = \mathbb{P}$ and if c_ζ is a $\mathbb{Q}_\zeta$-name such that $1 \Vdash_{\mathbb{Q}_\zeta}$ "$c_\zeta : [\omega_1]^2 \to 2$" then $\mathbb{Q}_{\zeta+1} = \mathbb{Q}_\zeta * \mathbb{Q}(p_G, c_\zeta)$.

It suffices to establish the following two claims.

Claim 32. *For each $\zeta \in \omega_2$ greater than 1 and $\eta \in \omega_1$ the set of $q \in \mathbb{Q}_{\zeta+1}$ such that*

$$q \restriction \zeta \Vdash_{\mathbb{Q}_\zeta} \text{“}q(\zeta) = (h, w) \text{ and } w \setminus \eta \neq \varnothing\text{”}$$

is dense in $\mathbb{Q}_{\zeta+1}$.

Proof. Given q it may be assumed that there are h and w such that

$$q \restriction \zeta \Vdash_{\mathbb{Q}_\zeta} \text{“}q(\zeta) = (\check{h}, \check{w}).\text{”}$$

Let $\theta \in \omega_1$ be so large that $\theta > \max(\operatorname{dom}(q(0))), \max(w), \eta$. Let $f : w \to \omega$ be any one-to-one function so that $\operatorname{ran}(f) \cap \operatorname{dom}(h) = \emptyset$ and let $f_\theta : \{\{\theta, \rho\}\}_{\rho \in w} \to \omega$ be defined by $f_\theta(\{\theta, \rho\}) = f(\rho)$. Note that since $q(0) \cup f_\theta \in \mathbb{P}$ it is possible to find $\bar{q} \leq q \restriction \zeta$ such that:

- $f_\theta \subseteq \bar{q}(0)$ and
- $\bar{q} \Vdash_{\mathbb{Q}_\zeta}$ "$c_\zeta(\{\theta, \rho\}) = \check{k}_\rho$" for some family of integers $\{k_\rho\}_{\rho \in w}$ equal to 0 or 1.

Then let $\bar{h} \supseteq h$ be any finite function such that $\bar{h}(f(\rho)) = 1 - k_\rho$ and let $\bar{w} = w \cup \{\theta\}$. Then $\bar{q} * (\bar{h}, \bar{w})$ is the desired condition. ⊣

Claim 33. *The partial order $\mathbb{Q}_{\omega_2}$ satisfies the ccc.*

Proof. By a standard argument, there is a dense subset of $\mathbb{Q}_{\omega_2}$ of conditions q such that for each $\zeta \in \operatorname{dom}(q)$ with $\zeta > 0$, there are h and w so that $q \restriction \zeta \Vdash_{\mathbb{Q}_{\omega_2}}$ "$q(\zeta) = (\check{h}, \check{w})$." We will assume that all conditions that we work with are members of this dense subset.

Let $\{q_\xi : \xi < \omega_1\}$ be conditions in $\mathbb{Q}_{\omega_2}$. By thinning out, we can assume that their domains form a Δ-system with root $\{0, \zeta_0, \zeta_1, \dots, \zeta_k\}$. We can further assume that:

- each of the sets $\{\operatorname{dom}(q_\xi(0)) : \xi < \omega_1\}$, and $\{w_{\xi,\zeta_i} : \xi < \omega_1\}$ for each $i \leq k$ form a Δ-system,

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

- the functions $q_\xi(0)$ agree on the root of the Δ-system of their domains, and
- there are h_i, $i \leq k$, so that for all $\xi < \omega$ we have $h_i = h_{\xi,\zeta_i}$,

where $q_\xi \restriction \zeta \Vdash_{\mathbb{Q}_{\omega_2}}$ "$q_\xi(\zeta) = (\check{h}_{\xi,\zeta}, \check{w}_{\xi,\zeta})$."

Let $\delta = \max\{\mathrm{dom}\,(q_0(0)), w_{0,\zeta_i} : i \leq k\}$. Pick $\gamma < \omega_1$ so that each of the values

$$\min(\mathrm{dom}\,(q_\gamma(0)) \setminus \mathrm{dom}\,(q_0(0))), \min(w_{\gamma,\zeta_i} \setminus w_{0,\zeta_i}) \text{ for } i \leq k,$$

are above δ (if defined).

Arguing as in Claim 32, we see that q_0 and q_γ are compatible conditions. ⊣

This completes the proof of the theorem. ⊣

DEFINITION 34. The symbol

$$\kappa \rightarrow_p [\kappa]^2_{\lambda,<\mu}$$

for a partition $p : [\kappa]^2 \to \theta$ means that for every coloring $f : [\kappa]^2 \to \lambda$ there is a set $A \in [\kappa]^\kappa$ such that $|\mathrm{ran}(f \restriction ([A]^2 \cap p^{-1}(i)))| < \mu$ for all $i < \theta$.

Note that for $\mu \leq \lambda$ this symbol is stronger than $\kappa \rightarrow_p [\kappa]^2_{\lambda\setminus\lambda}$. Thus the next theorem, which uses ideas from [39], gives a stronger consistency than the previous one.

THEOREM 35. *Given any regular $\kappa > \aleph_1$ it is consistent that*:

- $\textbf{non}(\mathcal{L}) = \aleph_1$,
- $\mathfrak{b} = \aleph_2 = 2^{\aleph_0}$,
- $2^{\aleph_1} = \kappa$, *and*
- *there is a* $p : [\omega_1]^2 \to \omega$ *such that* $\omega_1 \rightarrow_p [\omega_1]^2_{\omega,<\omega}$.

THEOREM 36. *Given any regular $\kappa > \aleph_1$ it is consistent that*:

- $\textbf{non}(\mathcal{M}) = \aleph_1$,
- $\mathfrak{b} = \aleph_1$,
- $\mathfrak{d} = \aleph_2 = 2^{\aleph_0}$,
- $2^{\aleph_1} = \kappa$, *and*
- *there is a* $p : [\omega_1]^2 \to \omega$ *such that* $\omega_1 \rightarrow_p [\omega_1]^2_{\omega,<\omega}$.

The proofs of both theorems are similar, using ideas from [39]; only the proof of Theorem 35 will be given in detail. Both rely on the following definition:

DEFINITION 37. Let μ be some probability measure on ω under which each singleton has positive measure, for example $\mu(\{n\}) = 2^{-n}$. A sequence of functions $\mathcal{P} = \{p_\eta\}_{\eta\in\omega_1}$ will be said to have full outer measure if:

- $p_\eta : \eta \to \omega$ and
- for each $\eta \in \omega_1$ the set $\{p_\beta \restriction \eta\}_{\beta>\eta}$ has measure one in the measure space (ω^η, μ^η).

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

The sequence $\mathcal{P}$ is defined to be nowhere meagre similarly, by requiring that for each $\eta \in \omega_1$ the set $\{p_\beta \restriction \eta\}_{\beta>\eta}$ is nowhere meagre in (ω^η, μ^η) with the usual product topology. In both cases define $p = p(\mathcal{P})$ by $p(\alpha, \beta) = p_\beta(\alpha)$ if $\alpha < \beta$.

By enumerating all functions from a countable ordinal into ω, we have:

PROPOSITION 38. *Assuming the Continuum Hypothesis there is a sequence $\mathcal{P} = \{p_\eta\}_{\eta\in\omega_1}$ such that $\{p_\beta \restriction \eta\}_{\beta>\eta} = \omega^\eta$ for each $\eta \in \omega_1$. Hence $\mathcal{P}$ has full outer measure as in* Definition 37.

While it is, of course, impossible to preserve the property that $\{p_\beta \restriction \eta\}_{\beta>\eta} = \omega^\eta$ when adding reals, the goal of the following arguments is to show that the properties of Definition 37 can be preserved in certain circumstances. The following definition is from [39] and will play a key role in this context.

DEFINITION 39. A function $\psi : \omega^{<\omega} \to [\omega_1]^{<\aleph_0}$ satisfying that $\psi(s) \cap \psi(t) = \varnothing$ unless $s = t$ will be said to have *disjoint range*. If for each $t \in \omega^{<\omega}$ there is k such that $|\psi(t^\frown j)| < k$ for all $j \in \omega$ then ψ will be called bounded with disjoint range. If G is a filter of subtrees of $\omega^{<\omega}$ and ψ has disjoint range define

$$S(G, \psi) = \bigcup_{t\in\bigcap G} \psi(t).$$

If G is a generic filter of trees over a model V define

$$\mathcal{S}_b(G) = \{S(G, \psi) \mid \psi \in V \text{ and } \psi \text{ is bounded with disjoint range}\}.$$

It is shown in [39] that Lemmas 40 and 42 hold.

LEMMA 40. *If $G \subseteq \mathbb{L}$ is generic over V then $\mathcal{S}_b(G)$ is a P-ideal in $V[G]$.*

Lemma 41 is the content of Section 3 of [1]. Recall that if $\mathcal{I}$ is an ideal then X is said to be orthogonal to $\mathcal{I}$ if $X \cap A$ is finite for each $A \in \mathcal{I}$.

LEMMA 41 (Abraham and Todorčević). *Let $\mathcal{I}$ be a P-ideal on ω_1 that is generated by a family of $\aleph_1$ countable sets and such that ω_1 is not the union of countably many sets orthogonal to $\mathcal{I}$. Then there is a proper partial order $\mathbb{P}_\mathcal{I}$, that adds no reals, even when iterated with countable support, such that there is a $\mathbb{P}_\mathcal{I}$-name $\dot{Z}$ for an uncountable subset of ω_1 such that $1 \Vdash_{\mathbb{P}_\mathcal{I}}$ "$(\forall \eta \in \omega_1)\ \dot{Z} \cap \eta \in \mathcal{I}$."*

LEMMA 42. *If $G \subseteq \mathbb{L}$ is generic over V and $\mathbb{P}_{\mathcal{S}_b(G)}$ is the partial order of* Lemma 41 *using* Lemma 40 *and $H \subseteq \mathbb{P}_{\mathcal{S}_b(G)}$ is generic over $V[G]$ then in $V[G][H]$ there is an uncountable $R \subseteq \omega_1$ such that $R \cap Y \neq \varnothing$ for each uncountable $Y \in V[G]$ and such that $[R]^{\aleph_0} \subseteq \mathcal{S}_b(G)$.*

LEMMA 43. *Let $\mathcal{P}$ be a sequence with full outer measure and suppose that $p = p(\mathcal{P})$. Suppose further that*

- *$c : [\omega_1]^2 \to \omega$,*
- *$G \subseteq \mathbb{L}$ is generic over V, and*
- *$H \subseteq \mathbb{P}_{\mathcal{S}_b(\dot{G})}$ is generic over $V[G]$.*

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

Then there is an uncountable $X \subseteq \omega_1$ in $V[G][H]$ and $L : \omega \to \omega$ such that $L(p(\alpha, \beta)) > c(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [X]^2$.

PROOF. In $V[G]$ let $L = \bigcap G$ be the Laver real. In $V[G][H]$ let R be the uncountable set given by Lemma 42. Construct by induction distinct $\rho_\xi \in R$ such that if $\eta \in \xi$ then $L(p(\rho_\xi, \rho_\eta)) > c(\rho_\xi, \rho_\eta)$. To carry out the induction assume that $R_\eta = \{\rho_\xi\}_{\xi \in \eta}$ have been chosen and satisfy the inductive hypothesis.

By the choice of R it follows that $R_\eta \in \mathcal{S}_b(G)$. Since $\mathbb{P}_{\mathcal{S}_b(\dot{G})}$ adds no new reals it follows that $R_\eta \in V[G]$ and so there is $T \in G$ and $\psi \in V$ with bounded, disjoint range such that $T \Vdash_{\mathbb{L}}$ "$\dot{R}_\eta = S(\dot{G}, \psi)$." Let μ be so large that $T \Vdash_{\mathbb{L}}$ "$\dot{R}_\eta \subseteq \mu$" and let r be the root of T. For $t \in T$ define $\mathcal{W}_t = \left\{x \in 2^\mu \;\middle|\; x \restriction \psi(t) \text{ has constant value } |t|\right\}$ and then define

$$\mathcal{W}_t^+ = \left\{x \in 2^\mu \;\middle|\; (\exists^\infty s \in \mathbf{succ}_T(t))\ x \in \mathcal{W}_s\right\}.$$

Note that $\mathcal{W}_t^+$ has measure one in 2^μ for each $t \supseteq r$. To see this note that for a random $h \in 2^\mu$ the probability that $h(\zeta) = |t| + 1$ is $2^{-(|t|+1)}$. Also, note that since ψ is bounded—see Definition 39—there is some k such that $|\psi(s)| \leq k$ for each $s \in \mathbf{succ}_T(t)$. Hence, the probability of h belonging to $\mathcal{W}_s$ is bounded below by $2^{-(|t|+1)k}$ for all $s \in \mathbf{succ}_T(t)$ and these events are independent because the $\psi(s)$ are pairwise disjoint for $s \in \mathbf{succ}_T(t)$.

Define f on $\bigcup_{j \leq |r|} \psi(r \restriction j)$ to have constant value $|r|$ and note that the domain of f is disjoint from each $\psi(s)$ where $s \supsetneq r$. Hence the probability that $f \subseteq h$ is non-zero and independent from belonging to each $\mathcal{W}^+$. Since p has full outer measure it follows that

$$\left\{\beta \in \omega_1 \;\middle|\; f \subseteq p_\beta \restriction \mu \in \bigcap_{r \subseteq t \in T} \mathcal{W}_t^+\right\}$$

is uncountable and belongs to $V[G]$. Therefore by Lemma 42 there is some $\beta \in R \setminus R_\eta$ such that $f \subseteq p_\beta \restriction \mu$ and such that for all $t \in T$ containing r there are infinitely many $s \in \mathbf{succ}_T(t)$ such that $p(\alpha, \beta) = |s|$ for all $\alpha \in \psi(s)$.

Using this and the definition of f, it is possible to start with r and successively thin out the successors of each $t \in T$ to find a tree $T^* \subseteq T$ with root r such that $p(\alpha, \beta) = |t|$ for all $t \in T^*$ and for all $\alpha \in \psi(t)$. Once again starting with r and removing only finitely many elements of $\mathbf{succ}_{T^*}(t)$ for each $t \in T^*$ it is possible to find $T^{**} \subseteq T^*$ with root r such that

$$(\forall t \in T^{**})(\forall s \in \mathbf{succ}_{T^{**}}(t))(\forall \alpha \in \psi(t))\ s(|t|) = s(p(\alpha, \beta)) > c(\alpha, \beta)$$

and this implies that

$$T^{**} \Vdash_{\mathbb{L}} \text{“}(\forall \alpha \in \dot{R}_\eta)\ \dot{L}(p(\alpha, \beta)) > c(\alpha, \beta).\text{”}$$

Since this holds for any T, genericity yields that in $V[G][H]$ there is some $\beta \in R \setminus R_\eta$ such that $L(p(\rho_\xi, \beta)) > c(\rho_\xi, \beta)$ for each $\xi \in \eta$. Define $\rho_\eta = \beta$ to continue the induction. Since limit stages are immediate, this completes the proof. ⊣

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

PROOF OF THEOREM 35. The required model is the one obtained by starting with a model of the Continuum Hypothesis in which $2^{\aleph_1} = \kappa$. Then iterate with countable support the partial order $\mathbb{L} * \mathbb{P}_{\mathcal{S}_b \dot{G})}$. In the initial model there is, by Proposition 38, a sequence with full outer measure. To see this, begin by observing that it is shown in Theorem 7.3.39 of [2] that $\mathbb{L}$ preserves $\sqsubseteq^{\textbf{Random}}$. Since $\mathbb{P}_{\mathcal{S}(\dot{G})}$ is proper and adds no new reals it is immediate that it also preserves $\sqsubseteq^{\textbf{Random}}$. It follows by Theorem 6.1.13 of [2] that the entire countable support iteration preserves outer measure sets and, hence, any sequence with full outer measure in the initial model maintains this property throughout the iteration.

To see that for every function $c : [\omega_1]^2 \to \omega$ there is an uncountable set witnessing $\aleph_1 \to_p [\aleph_1]_{\aleph_0, <\aleph_0}$ use Lemma 3.4 and Lemma 3.6 of [1] to conclude that each partial order in the ω_2 length iteration is proper and has the $\aleph_2$-pic of Definition 2.1 on page 409 of [35]. By Lemma 2.4 on page 410 of [35] it follows that the iteration has the $\aleph_2$ chain condition and, hence, that c appears at some stage. It is then routine to apply Lemma 43.

That $\mathfrak{b} = \aleph_2$ is a standard argument using that Laver forcing adds a dominating real. ⊣

REMARK 44. The proof of Theorem 36 is similar but uses Miller reals instead of Laver reals. This requires that nowhere meagreness play the role of full outer measure.

REMARK 45. Note that there is no partition p such that

$$\omega_1 \to_p [\omega_1]^2_{\omega_1, <\omega_1}$$

because a colouring $c : [\omega_1]^2 \to \omega_1$ that is a bijection will provide a counterexample.

§6. Concluding Remarks and Open Questions. It turns out, via Lemma 9, that getting strong coloring symbols over finite partitions is not harder than getting them without partitions; so one immediately gets many strong coloring symbols over partitions outright in ZFC. If the number of colors λ raised to the number of cells in a partition is not too large, Lemma 9 applies again, and consequently all GCH symbols gotten by Erdoős, Hajnal and Milner on κ^+ hold under the GCH over arbitrary κ-partitions. Even without instances of the GCH, strong colorings symbols over countable partitions are valid in Cohen-type forcing extenstions, by Theorems 27 and 28.

Yet, it is not the case that every time a strong-coloring symbol holds at a successor of a regular, it also holds over countable partitions: by Theorems 30 and 35 the ZFC symbol $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$, and hence all stronger ones, consistently fail quite badly over sufficiently generic countable partitions. Thus, strong coloring symbols over partitions are a subject of their own, in which the independence phenomenon is manifested prominently.

Many natural questions about the combinatorial and set-theoretic connections between coloring and partition arise. We hope that this subject will get attention in the near future both in the infinite combinatorics and

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

in the forcing communities. For example, by Fact 11, there is always a set of 2-partitions of $[\kappa^+]^2$ such that no coloring is strong over all of them. What is the least cardinality of such a set? In the case of $\theta = \kappa = \aleph_0$, the results in Section 5 show that this cardinal may be as small as 1 or at least as large as $\aleph_2 = \kappa^{++}$. Can this number ever be κ or, say, $\kappa^+ < 2^\kappa$?

We conclude with a short selection of open questions.

QUESTION 46. *If* $\mathrm{Pr}_1(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_p$ *holds for all countable* p, *does also* $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_p$ *hold for all countable* p?

QUESTION 47. *Suppose* $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_0, \aleph_0)_p$ *holds for some countable partition* p. *Does* $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_p$ *hold as well*?

Without partitions, both implications above hold.

QUESTION 48. *Does* $MA_{\sigma\text{-linked}}$ *or* $\mathfrak{p} = \mathfrak{c}$ *or even full* $MA_{\aleph_1}$ *imply that* $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)_{\bar{p}}$ *holds for every* ω_1 *sequence of partitions* $\bar{p} = \langle p_\delta : \delta < \omega_1 \rangle$ *such that* $p_\delta : [\omega_1]^2 \to \omega$?

QUESTION 49. *Is it consistent that there is a partition* p *such that*

$$\aleph_1 \to_p [\aleph_1]^2_{\aleph_0, <k}$$

for some integer k?

QUESTION 50. *Is*

$$\aleph_2 \not\to_p [{}^{\aleph_0 \circledast \aleph_2}\!/\!{}_{1 \circledast \aleph_2}]^2_{\aleph_2}$$

consistent for all $\aleph_0$- *or* $\aleph_1$-*partitions* p? *That is, can there be a coloring* $f : [\omega_2]^2 \to \omega_2$ *such that for every (one, or sequence of* ω_2 *many)* ω_1-*partition(s) of* $[\omega_2]^2$, *for every* $B \in [\omega_2]^{\omega_2}$, *for all but finitely many* $\alpha < \omega_2$ *there is* $i < \omega_1$ *such that for every color* $\zeta < \omega_2$ *there is* $\beta \in B$ *such that* $p(\alpha, \beta) = i$ *and* $f(\alpha, \beta) = \zeta$.

The consistency of this symbol is open even without the p. A negative answer may be easier to get with p.

Added in proof: Problems 46–49 above are solved in [18].

Acknowledgments. The first author's research for this paper was partially supported by an Israeli Science Foundation grant number 665/20. The third author's research for this paper was partially supported by NSERC of Canada.

REFERENCES

[1] U. ABRAHAM and S. TODORČEVIĆ, *Partition properties of* ω_1 *compatible with CH*. ***Fundamenta Mathematicae***, vol. 152 (1997), no. 2, pp. 165–181.

[2] T. BARTOSZYŃSKI and H. JUDAH, ***Set Theory: On the Structure of the Real Line***, A. K. Peters, Boca Raton, 1995.

[3] J. E. BAUMGARTNER, *Partition relations for countable topological spaces*. ***The Journal of Combinatorial Theory, Series A***, vol. 43 (1986), no. 2, pp. 178–195.

[4] P. Erdős, F. Galvin and A. Hajnal, *On set-systems having large chromatic number and not containing prescribed subsystems*, ***Infinite and Finite Sets*** (Colloq. Keszthely 1973; dedicated to P. Erdős on his 60th birthday,) Vol. I. Colloq. Math. Soc. J. Bolyai, Vol. 10, North Holland, Amsterdam, 1975, pp. 425–513.

[5] P. Erdős, A. Hajnal, A. Máté, and R. Rado, ***Combinatorial Set Theory Partition Relations for Cardinals***, Vol. 106, Studies in Logic and the Foundations of Mathematics, North Holland, Amsterdam, 1984.

[6] T . Eisworth, *A note on strong negative partition relations*. ***Fundamenta Mathematicae***, vol. 202 (2009), pp. 97–123.

[7] ———, *Club guessing, stationary reflection, and coloring theorems*. ***Annals of Pure and Applied Logic***, vol. 161 (2010), pp. 1216–1243.

[8] ———, *Successors of singular cardinals*, ***Handbook of Set Theory*** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 1229–1350.

[9] ———, *Getting more colors I*. ***Journal of Symbolic Logic***, vol. 78 (2013), no. 1, pp. 1–16.

[10] ———, *Getting more colors II*. ***Journal of Symbolic Logic***, vol. 78 (2013), no. 1, pp. 17–38.

[11] ———, *On idealized versions of* $Pr_1(\mu,^+, \mu^+, \mu^+, cf(\mu))$. ***Archive for Mathematical Logic***, vol. 53 (2014), pp. 809–824.

[12] T. Eisworth and S. Shelah, *Successors of singular cardinals and coloring theorems I*. ***Archive for Mathematical Logic***, vol. 44 (2005), no. 5, pp. 597–618.

[13] ———, *Successors of singular cardinals and coloring theorems II*. ***Journal of Symbolic Logic***, vol. 74 (2009), no. 4, pp. 1287–1309.

[14] W. G. Fleissner, *Some spaces related to topological inequalities proven by the Erdős-Rado theorem*. ***Proceedings of the American Mathematical Society***, vol. 71 (1978), no. 2, pp. 313–320.

[15] F. Galvin, *Chain conditions and products*. ***Fundamenta Mathematicae***, vol. 108 (1980), no. 1, pp. 33–48.

[16] O. Guzman, *On* $(1, \omega_1)$*-weakly universal functions*. ***Fundamenta Mathematicae***, vol. 247 (2019), no. 1, pp. 87–98.

[17] A. Hajnal and P. Komjath, *Some remarks on the simultaneous chromatic number*. ***Combinatorica***, vol. 23 (2003), no. 1, pp. 89–104.

[18] M. Kojman, A. Rinot and J. Steprans, *Advances on strong colorings over partitions*, Preprint February 2021. Available at http://assafrinot.com/paper/49, 2021.

[19] C. Lambie-Hanson and A. Rinot, *Knaster and friends II: The C-sequence number*. ***Journal of Mathematical Logic***, vol. 21 (2021), no. 1, p. 2150002, 54.

[20] J. T. Moore, *A solution to the L space problem*. ***Journal of the American Mathematical Society***, vol. 19 (2006), pp. 717–736.

[21] Y. Peng and L. Wu, *A Lindelöf group with non-Lindelöf square*. ***Advances in Mathematics***, vol. 325 (2018), pp. 215–242.

[22] D. Raghavan and S. Todorčević, *Proof of a conjecture of Galvin*. ***Forum of Mathematics***, vol. Pi 8 (2020), p. e15.

[23] A. Rinot, *Transforming rectangles into squares, with applications to strong colorings*. ***Advances in Mathematics***, vol. 231 (2012), no. 2, pp. 1085–1099.

[24] ———, *Complicated colorings*. ***Mathematical Research Letters***, vol. 21 (2014), no. 6, pp. 1367–1388.

[25] ———, *Chain conditions of products, and weakly compact cardinals*, this Journal, vol. 20 (2014), no. 3, pp. 293–314.

[26] ———, An S-space from a Cohen real. *Blog post* Available at http://blog.assafrinot.com/?p=2707.

[27] ———, Personal communication. 2019.

[28] A. Rinot and S. Todorčević, *Rectangular square-bracket operation for successor of regular cardinals*. ***Fundamenta Mathematicae***, vol. 220 (2013), no. 2, pp. 119–128.

[29] A. Rinot and J. Zhang, *Strongest transformations*, Preprint February 2021. Available at http://assafrinot.com/paper/45, 2021.

https://doi.org/10.1017/bsl.2021.5 Published online by Cambridge University Press

[30] J. Roitman, *Adding a random or a Cohen real: Topological consequences and the effect on Martin's axiom*. ***Fundamenta Mathematicae***, vol. 103 (1979), no. 1, pp. 47–60.

[31] S. Shelah, *Was Sierpiński right? I*. ***Israel Journal of Mathematics***, vol. 62 (1988), pp. 355–380.

[32] ———, *Strong partition relations below the power set: Consistency, was Sierpiński right, II?* Proceedings of the Conference on Set Theory and its Applications in Honor of A. Hajnal and V. T. Sos., Budapest, 1/91, 1991, pp. 637–668.

[33] ———, *Was Sierpiński right? III. Can continuum–c.c. times c.c.c. be continuum–c.c.?* ***Annals of Pure and Applied Logic***, vol. 78 (1996), pp. 259–269.

[34] ———, *Colouring and non-productivity of $\aleph_2$-cc*. ***Annals Pure and Applied Logic***, vol. 84 (1997), pp. 153–174.

[35] ———, ***Proper and Improper Forcing***, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

[36] ———, *Was Sierpiński right? IV*. ***Journal of Symbolic Logic***, vol. 65 (2000), pp. 1031–1054.

[37] ———, *Successors of singulars, cofinalities of reduced products of cardinals and productivity of chain conditions*. ***Israel Journal of Mathematics***, vol. 62 (1988), pp. 213–256.

[38] ———, ***Cardinal Arithmetic***, Vol. 29, Oxford Logic Guides, The Clarendon Press, Oxford University Press, New York, NY, 1994.

[39] S. Shelah and J. Steprāns, *Universal graphs and functions on ω_1*, preprint, 2016.

[40] J. Steprāns, *Combinatorial consequences of adding Cohen reals*. ***Set Theory of the Reals***, pp. 583–617, Israel Math. Conf. Proc., 6, Bar-Ilan Univ., Ramat Gan, 1993.

[41] S. Todorčević, *Partitioning pairs of countable ordinals*. ***Acta Mathematica***, vol. 159 (1987), no. 3–4, pp. 261–294.

[42] ———, ***Partition Problems in Topology, vol.*** 84, Contemporary Mathematics, American Mathematical Society, Providence, RI, 1989.

[43] ———, *Some partitions of three-dimensional combinatorial cubes*. ***The Journal of Combinatorial Theory, Series A***, vol. 68 (1994), no. 2, pp. 410–437.

[44] ———, *Oscillations of sets of integers*. ***Advances in Applied Mathematics***, vol. 20 (1998), no. 2, pp. 220–252.

DEPARTMENT OF MATHEMATICS & STATISTICS
YORK UNIVERSITY, 4700 KEELE STREET
TORONTO, ON M3J 1P3, CANADA
E-mail: chenwb@gmail.com
E-mail: steprans@yorku.ca

DEPARTMENT OF MATHEMATICS
BEN-GURION UNIVERSITY OF THE NEGEV
P.O.B. 653, BE'ER SHEVA 84105, ISRAEL
E-mail: kojman@woobling.org