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ON THE LATTICE OF $\Pi_{\mathfrak{I}}$-SUBNORMAL SUBGROUPS OF A FINITE GROUP

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Abstract

Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes $\mathbb{P}$. Let $\sigma_0 \in \Pi \subseteq \sigma$ and let $\mathfrak{I}$ be a class of finite σ_0-groups which is closed under extensions, epimorphic images and subgroups. We say that a finite group G is $\Pi_{\mathfrak{I}}$-*primary* provided G is either an $\mathfrak{I}$-group or a σ_i-group for some $\sigma_i \in \Pi \setminus \{\sigma_0\}$ and we say that a subgroup A of an arbitrary group G^* is $\Pi_{\mathfrak{I}}$-*subnormal* in G^* if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G^*$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary for all $i = 1, \ldots, t$. We prove that the set $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ of all $\Pi_{\mathfrak{I}}$-subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G and we describe the conditions under which the lattice $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ is modular.

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1. Introduction

Throughout this paper, G and G^* always denote a finite group and an arbitrary group, respectively. If $N \trianglelefteq G^*$, we denote by $\mathcal{L}(G^*/N)$ the lattice of all subgroups H of G^* with $N \leq H \leq G^*$.

A classical theorem of Wielandt states that the set $\mathcal{L}_{sn}(G)$ of all subnormal subgroups of G is a sublattice of the lattice $\mathcal{L}(G)$ of all subgroups of G. A generalisation of the lattice $\mathcal{L}_{sn}(G)$ was found by Kegel [9].

Let $\mathfrak{F}$ be a class of groups. A subgroup A of G is called $\mathfrak{F}$-*subnormal in G in the sense of Kegel* [9] or K-$\mathfrak{F}$-*subnormal* in G [3, Definition 6.1.4] if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i} \in \mathfrak{F}$ for all $i = 1, \ldots, t$.

In [9], Kegel proved that if the class $\mathfrak{F}$ is closed under extensions, epimorphic images and subgroups, then the set $\mathcal{L}_{\mathfrak{F}sn}(G)$ of all K-$\mathfrak{F}$-subnormal subgroups of G is a sublattice of the lattice $\mathcal{L}(G)$. For every set π of primes, we may choose the class $\mathfrak{F}$ of all π-groups. In this way we obtain infinitely many functors $\mathcal{L}_{\mathfrak{F}sn}$ assigning to every

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group G a sublattice of $\mathcal{L}(G)$ containing $\mathcal{L}_{sn}(G)$. This result has been generalised using the theory of formations (see [2, 14] and [3, Ch. 6]).

Here, we seek to generalise Kegel's result without applying the theory of formations. Following Shemetkov [11], σ denotes some partition of the set of all primes $\mathbb{P}$. Thus, $\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We denote by Π a subset of the set σ and set $\Pi' = \sigma \setminus \Pi$. We assume that $0 \in I$ and $\sigma_0 \in \Pi$.

Let $\mathfrak{I}$ denote a class of finite σ_0-groups which is closed under extensions, epimorphic images and subgroups. We will say that G is $\Pi_{\mathfrak{I}}$-*primary* provided G is either an $\mathfrak{I}$-group or a σ_i-group for some $\sigma_i \in \Pi \setminus \{\sigma_0\}$. We will omit the symbol $\mathfrak{I}$ in the notation and definitions when $\mathfrak{I}$ is the class of all σ_0-groups. Therefore, for example, we say that G is Π-*primary* provided G is a σ_i-group for some $\sigma_i \in \Pi$.

Definition 1.1. We say that a subgroup A of G^* is $\Pi_{\mathfrak{I}}$-*subnormal* in G^* if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G^*$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary for all $i = 1, \ldots, t$.

Note that a subgroup A of G is K-$\mathfrak{I}$-subnormal in G if and only if it is $\Pi_{\mathfrak{I}}$-subnormal in G, where $\Pi = \{\sigma_0\}$, and A is subnormal in G if and only if it is $\Pi_{\mathfrak{I}}$-subnormal in G, where $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$.

Example 1.2. Consider $C_{29} \rtimes C_7$, a nonabelian group of order 203, and let P be a simple $\mathbb{F}_{11}(C_{29} \rtimes C_7)$-module which is faithful for $C_{29} \rtimes C_7$. Construct the group $G = (P \rtimes (C_{29} \rtimes C_7)) \times A_5$, where A_5 is the alternating group of degree 5. Let $\sigma = \{\sigma_0, \sigma_1, \sigma_2\}$, where $\sigma_0 = \{2, 3, 5\}$, $\sigma_1 = \{11, 29\}$ and $\sigma_2 = \{2, 3, 5, 11, 29\}'$. Let $\Pi = \{\sigma_1, \sigma_2\}$. Then a subgroup H of G of order 4 is σ-subnormal in G but it is neither Π-subnormal in G nor $\sigma_{\mathfrak{I}}$-subnormal in G, where $\mathfrak{I}$ is the class of all soluble σ_0-groups (see Lemma 2.1(1) below). The subgroup C_{29} is $\Pi_{\mathfrak{I}}$-subnormal in G but it is clearly not subnormal in G.

It is not difficult to show that the intersection of any two $\Pi_{\mathfrak{I}}$-subnormal subgroups of G is also $\Pi_{\mathfrak{I}}$-subnormal in G (see Lemma 2.1(3) below). It is well known that any partially ordered set with the greatest element 1 in which each nonempty subset has a greatest lower bound is a lattice. Hence, the set $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ of all $\Pi_{\mathfrak{I}}$-subnormal subgroups of G is a lattice.

Modifying the concept of σ-nilpotency in [6], we say that G is $\Pi_{\mathfrak{I}}$-nilpotent if $G = A_1 \times \cdots \times A_t \times A$ for some $\Pi_{\mathfrak{I}}$-primary groups $A_1, \ldots, A_t$ and a nilpotent group A. Note that G is nilpotent if and only if it is $\Pi_{\mathfrak{I}}$-nilpotent, where $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$.

Our main goal is to prove the following theorem.

Theorem 1.3. *The lattice $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ is modular if and only if the following two conditions hold:*

(a) *if $T, S \in \mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$, where T is a normal subgroup of S and either S/T is $\Pi_{\mathfrak{I}}$-primary or $|S/T| = p^3$ (p a prime), then $\mathcal{L}(S/T)$ is modular;*

(b) *$\langle A, B\rangle/(A \cap B)_{\langle A,B\rangle}$ is $\Pi_{\mathfrak{I}}$-nilpotent for each $A, B \in \mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ such that A and B cover $A \cap B$ (in $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$) and $A \cap B$ is not normal in both A and B.*

Twice applying Theorem 1.3, first with $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$ and then with $\Pi = \{\sigma_0\}$ and with $\mathfrak{I}$ the class of all identity groups, gives the following corollary.

Corollary 1.4 (Cf. [15] and [10, Theorem 9.2.3]). *The following statements are equivalent.*

(i) $\mathcal{L}_{sn}(G)$ *is modular.*
(ii) *If* $T \leq S$ *are subnormal subgroups of* G*, where* T *is normal in* S *and* S/T *is a* p*-group,* p *a prime, then* $\mathcal{L}(S/T)$ *is modular.*
(iii) *If* $T \leq S$ *are subnormal subgroups of* G*, where* T *is normal in* S *and* $|S/T| = p^3$ *(*p *a prime), then* $\mathcal{L}(S/T)$ *is modular.*

We say that G is $\mathfrak{I}$*-nilpotent* if $G = A \times B$, where $A \in \mathfrak{I}$ and B are Hall subgroups of G and B is nilpotent. Note, in passing, that every subgroup of G is K-$\mathfrak{I}$-subnormal in G if and only if G is $\mathfrak{I}$-nilpotent [9]. Now we can characterise groups with modular lattice $\mathcal{L}_{\mathfrak{I}sn}(G)$.

Corollary 1.5. *The lattice* $\mathcal{L}_{\mathfrak{I}sn}(G)$ *is modular if and only if the following two conditions hold:*

(a) *if* $T, S \in \mathcal{L}_{\mathfrak{I}sn}(G)$*, where* T *is a normal subgroup of* S *and either* $S/T \in \mathfrak{I}$ *or* $|S/T| = p^3$ *(*p *a prime), then* $\mathcal{L}(S/T)$ *is modular;*
(b) $\langle A, B\rangle/(A \cap B)_{\langle A,B\rangle}$ *is* $\mathfrak{I}$*-nilpotent for each* $A, B \in \mathcal{L}_{\mathfrak{I}sn}(G)$ *such that* A *and* B *cover* $A \cap B$ *(in* $\mathcal{L}_{\mathfrak{I}sn}(G)$*) and* $A \cap B$ *is not normal in both* A *and* B*.*

The proof of Theorem 1.3 is based on many properties of $\Pi_{\mathfrak{I}}$-subnormal subgroups, which we study in Section 2. In particular, we give the proof of the following two results, which are the key steps in the proof of Theorem 1.3.

Proposition 1.6. *Let* A *be a* $\Pi_{\mathfrak{I}}$*-subnormal subgroup of* G*. If* A *is* $\Pi_{\mathfrak{I}}$*-nilpotent, then the normal closure* A^G *of* A *in* G *is also* $\Pi_{\mathfrak{I}}$*-nilpotent. Moreover, if* A *is a* σ_i*-group for some* $\sigma_i \in \Pi$*, then* A^G *is a* σ_i*-group; if* A *is a* Π'*-group, then* A^G *is also a* Π'*-group.*

Theorem 1.7. *The lattice* $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ *is a sublattice of the lattice* $\mathcal{L}(G)$ *of all subgroups of* G*.*

These two results may be of independent interest since they generalise known results. First, in the case $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$, Proposition 1.6 and Theorem 1.7 yield the following well-known result (see, for example, [4, Ch. A, Theorem 8.8]).

Corollary 1.8. *If* $A_1, \ldots, A_t$ *are nilpotent subnormal subgroups of* G*, then* $\langle A_1, \ldots, A_t\rangle$ *is also a nilpotent subnormal subgroup of* G*.*

In the case $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$, or when $\Pi = \{\sigma_0\}$, Theorem 1.7 yields the result of Wielandt mentioned above.

Corollary 1.9. *The set of all subnormal subgroups of* G *forms a sublattice of the lattice of all subgroups of* G*.*

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Another special case of Theorem 1.7 was also proved in [13].

COROLLARY 1.10. *The set of all σ-subnormal subgroups of G forms a sublattice of the lattice of all subgroups of G.*

2. $\Pi_{\mathfrak{I}}$-subnormal subgroups

Let n be an integer. We write $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$. We say that n is a Π-*number* if $\sigma(n) \subseteq \Pi$ and G is a Π-*group* if $|G|$ is a Π-number.

We use $(G^*)^{\mathfrak{I}}$ and $O^{\sigma_i}(G^*)$ respectively to denote the intersection of all normal subgroups N of G with $G^*/N \in \mathfrak{I}$ and with the property that G^*/N is a finite σ_i-group. We say that G^* is $\Pi_{\mathfrak{I}}$-*perfect* if $G^* = (G^*)^{\mathfrak{I}}$ and $O^{\sigma_i}(G^*) = G^*$ for all $\sigma_i \in \Pi$ with $i \neq 0$.

LEMMA 2.1. *Let A, K and N be subgroups of G^*. Suppose that A is $\Pi_{\mathfrak{I}}$-subnormal in G^* and N is normal in G^*.*

(1) *$A \cap K$ is $\Pi_{\mathfrak{I}}$-subnormal in K.*
(2) *If K is a $\Pi_{\mathfrak{I}}$-subnormal subgroup of A, then K is $\Pi_{\mathfrak{I}}$-subnormal in G^*.*
(3) *If K is $\Pi_{\mathfrak{I}}$-subnormal in G^*, then $A \cap K$ is $\Pi_{\mathfrak{I}}$-subnormal in G^*.*
(4) *If $N \leq K$ and K/N is $\Pi_{\mathfrak{I}}$-subnormal in G^*/N, then K is $\Pi_{\mathfrak{I}}$-subnormal in G^*.*
(5) *AN/N is $\Pi_{\mathfrak{I}}$-subnormal in G^*/N.*
(6) *If $K \leq A$ and A is $\Pi_{\mathfrak{I}}$-primary, then K is $\Pi_{\mathfrak{I}}$-subnormal in G^*.*
(7) *If A is $\Pi_{\mathfrak{I}}$-perfect, then A is subnormal in G^*.*
(8) *If $|G^* : A|$ is a Π'-number, then A is subnormal in G^*.*

PROOF. See the proof of [1, Lemma 2.2]. □

Recall that $O^{\Pi}(G)$ denotes the subgroup of G generated by all its Π'-subgroups [13]. A subgroup H of G is called a *Hall Π-subgroup* of G if $|H|$ is a Π-number and $|G : H|$ is a Π'-number, and a *σ-Hall subgroup* of G if H is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$ [12, 13].

LEMMA 2.2. *Let $\Pi_1 \subseteq \Pi$ and A be a Π-subnormal subgroup of G.*

(1) *If $H \neq 1$ is a Hall Π_1-subgroup of G and A is not a Π_1'-group, then $A \cap H \neq 1$ is a Hall Π_1-subgroup of A.*
(2) *If A is a Hall Π_1-subgroup of G, then A is normal in G.*
(3) *If $|G : A|$ is a Π_1-number, then $O^{\Pi_1}(A) = O^{\Pi_1}(G)$.*
(4) *If N is a normal Π_1-subgroup of G, then $N \leq N_G(O^{\Pi_1}(A))$.*

PROOF. Assume that the lemma is false and let G be a counterexample of minimal order. By hypothesis, there is a subgroup chain $A = A_0 < A_1 < \cdots < A_r = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. Without loss of generality, we may assume that $M < G$.

(1) First we show that $M \cap H \neq 1$ is a Hall Π_1-subgroup of M. If either $H \leq M$ or M is normal in G, it is evident. Assume that $K = M_G \neq M$ and $H \nleq M$. Then $|G : K|$ is a σ_i-number for some $\sigma_i \in \Pi$. Moreover, $\sigma_i \in \sigma(H) \subseteq \Pi_1$ since otherwise

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we have $H \le K \le M$. Therefore, if $K = 1$, then G is a σ_i-group and so $H = G$. But then $M = M \cap H$ is a Hall Π_1-subgroup of M. Now assume that $K \ne 1$. Then $HK/K \ne 1$ is a Hall Π_1-subgroup of G/K since $H \nleq M$, and M/K is a Π-subnormal subgroup of G/K such that M/K is not a Π_1'-group since $K < M$ and $|G/K|$ is a σ_i-number, where $\sigma_i \in \sigma(H) \subseteq \Pi_1$. Therefore, the choice of G implies that $(HK/K) \cap (M/K) = (H \cap M)K/K \ne 1$ is a Hall Π_1-subgroup of M/K. Hence, $|M : K(H \cap M)|$ is a Π'-number. On the other hand, $H \cap K$ is a Hall Π_1-subgroup of K since K is normal in G. Therefore, $|M : H \cap M| = |M : K(H \cap M)|\,|K : K \cap H|$ is a Π_1'-number. Hence, $M \cap H \ne 1$ is a Hall Π_1-subgroup of M. Since A is Π-subnormal in M and $|M| < |G|$, the choice of G implies that $H \cap A = (H \cap M) \cap A$ is a Hall Π_1-subgroup of A.

(2) If $A = 1$, it is clear. Now assume that $A \ne 1$. Then, for any $x \in G$, $A \cap A^x \ne 1$ is a Hall Π_1-subgroup of A by Assertion (1). Therefore, $A = A^x$ for all $x \in G$, giving (2).

(3) It is clear that $|M : A|$ and $|G : M|$ are Π_1-numbers. Moreover, A is Π-subnormal in M by Lemma 2.1(1). The choice of G implies that $O^{\Pi_1}(A) = O^{\Pi_1}(M)$. Since $|G : M|$ is a Π_1-number, G/M_G is a Π_1-number. Therefore, every Π_1'-subgroup of G is contained in M_G, so $O^{\Pi_1}(G) = O^{\Pi_1}(M) = O^{\Pi_1}(A)$.

(4) It is clear that $|AN : A|$ is a Π_1-number. On the other hand, A is Π-subnormal in AN by Lemma 2.1(1). Hence, $N \le N_{AN}(O^{\Pi_1}(AN)) = N_{AN}(O^{\Pi_1}(A))$ by Assertion (3).

The lemma is proved. □

Lemma 2.3. *Let A be a $\Pi_{\mathfrak{I}}$-subnormal subgroup of G.*

(1) *If R is a minimal normal subgroup of G and not $\Pi_{\mathfrak{I}}$-primary, then $R \le N_G(A)$.*
(2) *If $N \le H \le G$, where N is a normal subgroup of G such that $G/N \in \mathfrak{I}$, then $H^{\mathfrak{I}} = G^{\mathfrak{I}}$.*
(3) *If $G = AB$ for some subgroup B of G contained in $\mathfrak{I}$, then $A^{\mathfrak{I}} = G^{\mathfrak{I}}$.*
(4) *If a normal subgroup R of G belongs to $\mathfrak{I}$, then $R \le N_G(A^{\mathfrak{I}})$.*

Proof. Assume that the lemma is false and let G be a counterexample of minimal order. By hypothesis, there is a subgroup chain $A = A_0 < A_1 < \cdots < A_r = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. Without loss of generality, we may assume that $M < G$.

(1) First assume that R is abelian. If R is a Π'-group, then $|RA : A|$ is a Π'-number and so A is subnormal in RA by Lemma 2.1(1),(8). Hence, $R \le N_G(A)$ by [4, Ch. A, 14.3]. Now suppose that R is a p-group for some $p \in \sigma_i \in \Pi$. Then $i = 0$ and $R \notin \mathfrak{I}$ since otherwise R is $\Pi_{\mathfrak{I}}$-primary, contrary to the hypothesis. Since $\mathfrak{I}$ is closed under extensions and subgroups, every group $S \in \mathfrak{I}$ is a p'-group. But A is $\Pi_{\mathfrak{I}}$-subnormal in RA. Hence, A is subnormal in RA since $|AR : A|$ is a power of p (from the proof of Lemma 2.1(8)). Hence, $R \le N_G(A)$. Finally, suppose that R is nonabelian. Consider the group AR. Clearly, R is the product of some minimal normal subgroup of AR. If $AR < G$, then the choice of G implies that $R \le N_G(A)$. Now assume that $G = AR$. Then $R \nleq M$ since $M < G$. If M is not normal in G, then G/M_G is $\Pi_{\mathfrak{I}}$-primary and so $RM_G/M_G \simeq R/R \cap M_G \simeq R$ is $\Pi_{\mathfrak{I}}$-primary, contrary to the hypothesis. Hence, M is normal in G. Then $R = (R \cap M) \times R_0$, where $R \cap M$ and R_0 are normal

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in G and $R_0 \cap M = 1$. Hence, $R \cap M \leq \mathrm{Soc}(M)$ and $M = M \cap AR = A(R \cap M)$. The choice of G implies that $R \cap M \leq N_M(A) \leq N_G(A)$. On the other hand, $R_0 \cap M = 1$, so $R_0 \leq C_G(M) \leq C_G(A)$. Thus, $R \leq N_G(A)$. Hence, we have (1).

(2) Clearly, $G^{\Im} \leq N$ and also $N/G^{\Im} \in \Im$. Hence, $N^{\Im} \leq G^{\Im}$. Since $N^{\Im}$ is a characteristic subgroup of N, it is normal in G. Thus, $G/N^{\Im} \in \Im$ since the class $\Im$ is closed under extensions. It follows that $G^{\Im} \leq N^{\Im}$ and so $N^{\Im} = G^{\Im}$. Since N is normal in H and $H/N \in \Im$, we have also that $N^{\Im} = H^{\Im}$. Hence, $H^{\Im} = G^{\Im}$.

(3) It is clear that $|M : A|$ and $|G : M|$ are $\pi(B)$-numbers, so $|M : A|$ and $|G : M|$ are σ_0-numbers. Moreover, as A is $\Pi_{\Im}$-subnormal in M and $M = A(M \cap B)$, the choice of G implies that $A^{\Im} = M^{\Im}$. Note that $G/M_G \in \Im$. Indeed, if M is normal in G, this follows from the isomorphism $BM/M \simeq B/B \cap M$ since in this case $M_G = M$. Assume that M is not normal in G. Then either G/M_G is a σ_i-group for some $i \neq 0$ or $G/M_G \in \Im$. But the former case is impossible since $|G : M|$ is a σ_0-number. Hence, $G/M_G \in \Im$. It follows from (2) that $A^{\Im} = M^{\Im} = G^{\Im}$.

(4) By Lemma 2.1(1), A is $\Pi_{\Im}$-subnormal in AR, so $A^{\Im} = (AR)^{\Im}$ by (3). It follows that $R \leq N_{AR}(A^{\Im}) \leq N_G(A^{\Im})$.

The lemma is proved. □

Lemma 2.4. *If H is a normal subgroup of G and $\pi = \pi(H/H \cap \Phi(G))$, then H has a Hall π-subgroup E and E is normal in G. Moreover, if $H/H \cap \Phi(G) \in \Im$, then $E \in \Im$.*

Proof. See the proof of [13, Lemma 2.5]. □

Proof of Proposition 1.6. We use $G_{\Im}$ to denote the product of all normal subgroups of G belonging to $\Im$. Since the class $\Im$ is closed under extensions and subgroups, every subgroup of $G_{\Im}$ belongs to $\Im$.

If $A = 1$ or $A = G$, then $A = A^G$ is $\Pi_{\Im}$-nilpotent by hypothesis. Now assume that $1 \neq A \neq G$. By hypothesis, $A = B_1 \times \cdots \times B_t \times B$ for some $\Pi_{\Im}$-primary groups $B_1, \ldots, B_t$ and a nilpotent group B. Then $A^G = (B_1)^G \cdots (B_t)^G B^G$. Without loss of generality, we can assume that $B_1 \in \Im$, B_i is a σ_i-group for all $i = 2, \ldots, t$ and B is a Π'-subgroup of G. Therefore, in order to prove that A^G is $\Pi_{\Im}$-nilpotent, it is enough to prove the following three claims.

Claim 1. $B_1^G \in \Im$. It is enough to show that $B_1 \leq G_{\Im}$. Assume that this false and let G be a counterexample of minimal order. Let $D = G_{\Im}$. Then $B_1 \neq 1$ and $G \neq D$. Let R be a minimal normal subgroup of G. Clearly, B_1 is $\Pi_{\Im}$-subnormal in G and B_1 is $\Pi_{\Im}$-nilpotent. The choice of G and Lemma 2.1(5) imply that $B_1R/R \leq O/R = (G/R)_{\Im}$ since B_1R/R is a σ_0-group. Therefore, $R \nleq D$, so $D = 1$ and $B_1 \cap R < R$. Suppose that $L = B_1 \cap R \neq 1$. Then L is $\Pi_{\Im}$-subnormal in G by Lemma 2.1(3) and so L is $\Pi_{\Im}$-subnormal in R by Lemma 2.1(1). If $R < G$, then the choice of G implies that $L \leq R_{\Im} \leq D$. But then $R \leq D$, which is a contradiction. Hence, $R = G$ is a simple group, which is impossible since $B_1 \neq 1$ and $G \neq D$. Therefore, $R \cap B_1 = 1$.

If $O < G$, then the choice of G implies that $B_1 \leq O_{\Im} \leq G_{\Im}$, contrary to our assumption on B_1. Hence, $G/R = O/R \in \Im$. It follows from [8, Ch. I, Hilfssatz 9.6] that R is the unique minimal normal subgroup of G. If $R \leq \Phi(G)$, then G has a normal Hall subgroup $E_0 \in \mathfrak{I}$ by Lemma 2.4. It follows that $E_0 \leq D = 1$, which is a contradiction. Thus, $R \nleq \Phi(G)$, which implies that $C_G(R) \leq R$.

Now we show that $R \leq N_G(B_1)$. First assume that R is $\Pi_{\mathfrak{I}}$-primary. Then R is a σ_i-group for some $i \neq 0$ and so $O^{\sigma_i}(B_1) = B_1$. Therefore, $R \leq N_G(B_1)$ by Lemma 2.2(4). On the other hand, if R is not $\Pi_{\mathfrak{I}}$-primary, then $R \leq N_{RB_1}(B_1)$ by Lemma 2.3(1). Therefore, $B_1R = B_1 \times R$ and so $B_1 \leq C_G(R) \leq R$. This contradiction completes the proof of Claim 1.

Claim 2. If $i > 1$, then B_i^G is a σ_i-group. This claim can be proved similarly to Claim 1 using Lemma 2.2 instead of Lemma 2.3.

Claim 3. B^G is a nilpotent Π'-group. Assume that the claim is false and let G be a counterexample of minimal order. Then $B \neq G$. It is clear that B is $\Pi_{\mathfrak{I}}$-subnormal in G. Hence, there is a subgroup chain $B = A_0 < A_1 < \cdots < A_r = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{I}}$-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. Without loss of generality, we can assume that $M < G$. Then B^M is a nilpotent Π'-group by the choice of G. If M is normal in G, then B^M is subnormal in G. Using Claim 2 with $\Pi = \sigma = \{\{2\}, \{3\}, \ldots\}$, we conclude that $B^G = (B^M)^G$ is a nilpotent Π'-group. Finally, if M is not normal in G, then M/M_G is a Π-group and so $B^M \leq M_G$. Hence, as above, we conclude that $B^G = (B^M)^G$ is a nilpotent Π'-group. □

Proposition 2.5. *Let $\mathfrak{N}_{\Pi_{\mathfrak{I}}}$ be the class of all $\Pi_{\mathfrak{I}}$-nilpotent groups.*

(1) *The class $\mathfrak{N}_{\Pi_{\mathfrak{I}}}$ is closed under products of normal subgroups, homomorphic images and subgroups. Moreover, if $G/\Phi(G)$ is $\Pi_{\mathfrak{I}}$-nilpotent, then G is $\Pi_{\mathfrak{I}}$-nilpotent.*

(2) *$G \in \mathfrak{N}_{\Pi_{\mathfrak{I}}}$ if and only if every subgroup of G is $\Pi_{\mathfrak{I}}$-subnormal in G.*

Proof. Assertion (1) follows from Lemma 2.4. For Assertion (2), see the proof of [7, Proposition 3.4]. □

Proof of Theorem 1.7. In view of Lemma 2.1(3), we only need to show that if A and B are $\Pi_{\mathfrak{I}}$-subnormal subgroups of G, then $\langle A, B\rangle$ is $\Pi_{\mathfrak{I}}$-subnormal in G. Assume that this is false and let G be a counterexample of minimal order. Then $A \neq 1 \neq B$ and $\langle A, B\rangle \neq G$. Let R be a minimal normal subgroup of G.

Claim 1. $\langle A, B\rangle R = G$ and so $\langle A, B\rangle_G = 1$.

Suppose that $L = \langle A, B\rangle R \neq G$. Lemma 2.1(1) implies that A and B are $\Pi_{\mathfrak{I}}$-subnormal in L. Hence, the choice of G implies that $\langle A, B\rangle$ is $\Pi_{\mathfrak{I}}$-subnormal in L. On the other hand, $L/R = \langle A, B\rangle R/R = \langle AR/R, BR/R\rangle$, where AR/R and BR/R are $\Pi_{\mathfrak{I}}$-subnormal in G/R by Lemma 2.1(5), so the choice of G implies that L/R is $\Pi_{\mathfrak{I}}$-subnormal in G/R and so L is $\Pi_{\mathfrak{I}}$-subnormal in G by Lemma 2.1(4). But then $\langle A, B\rangle$ is $\Pi_{\mathfrak{I}}$-subnormal in G by Lemma 2.1(2). This contradiction proves Claim 1.

Claim 2. If S is a nonidentity characteristic subgroup of C, where $C \in \{A, B\}$, then $R \nleq N_G(S)$. In particular, $R \nleq N_G(C)$.

Indeed, if $R \leq N_G(S)$, then $S^G = S^{\langle A,B\rangle R} = S^{\langle A,B\rangle} \leq \langle A, B\rangle_G = 1$, which is a contradiction.

Claim 3. R is $\Pi_{\mathfrak{I}}$-primary. Hence, R is a σ_i-group for some $\sigma_i \in \Pi$.

This follows from Claim 2 and Lemma 2.3(1).

Claim 4. A and B are $\Pi_{\mathfrak{I}}$-primary.

Claim 3 implies that R is a $\Pi_{\mathfrak{I}}$-primary σ_i-group for some $\sigma_i \in \Pi$. First assume that $i = 0$. Then R belongs $\mathfrak{I}$, so $R \leq N_G(A^{\mathfrak{I}})$ by Lemma 2.3(2). Hence, $A/A^{\mathfrak{I}} = A/1 \simeq A \in \mathfrak{I}$ by Claim 2. It follows that A is an $\mathfrak{I}$-group. Now assume that R is a σ_i-group for $i \neq 0$. Then $R \leq N_G(O^{\sigma_i}(A))$ by Lemma 2.2(4). But $O^{\sigma_i}(A)$ is characteristic in A, so $O^{\sigma_i}(A) = 1$ by Claim 2. Hence, A is a σ_i-group. This shows that A is $\Pi_{\mathfrak{I}}$-primary. Similarly, we can see that B is $\Pi_{\mathfrak{I}}$-primary.

Final contradiction. From Claim 4, we know that A and B are $\Pi_{\mathfrak{I}}$-primary. Consequently, A and B are $\Pi_{\mathfrak{I}}$-nilpotent. Hence, A^G and B^G are $\Pi_{\mathfrak{I}}$-nilpotent by Proposition 1.6. It follows from Proposition 2.5(1) that $A^G B^G$ is $\Pi_{\mathfrak{I}}$-nilpotent. Therefore, by Proposition 2.5(2), $\langle A, B\rangle$ is $\Pi_{\mathfrak{I}}$-subnormal in G. This contradiction completes the proof. □

3. Proof of Theorem 1.3

First suppose that $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ is modular. We derive (a) and (b).

(a) If T and S are $\Pi_{\mathfrak{I}}$-subnormal subgroups of G, where T is normal in S, and either S/T is $\Pi_{\mathfrak{I}}$-primary or $|S/T| = p^3$ (p a prime), then $\mathcal{L}(S/T) = \mathcal{L}_{\Pi_{\mathfrak{I}}}(S/T)$ by Lemma 2.1(6). Hence, $\mathcal{L}(S/T)$ is modular.

(b) Now assume that A and B are $\Pi_{\mathfrak{I}}$-subnormal subgroups of G such that A and B cover $A \cap B$ (in $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$) and also $A \cap B$ is not normal in both A and B. We show that $\langle A, B\rangle/(A \cap B)_{\langle A,B\rangle}$ is $\Pi_{\mathfrak{I}}$-nilpotent. In view of Theorem 1.7, $\langle A, B\rangle$ is $\Pi_{\mathfrak{I}}$-subnormal in G. Hence, $\mathcal{L}_{\Pi_{\mathfrak{I}}}(\langle A, B\rangle)$ is a sublattice of the modular lattice $L_{\Pi_{\mathfrak{I}}}(G)$, so $\mathcal{L}_{\Pi_{\mathfrak{I}}}(\langle A, B\rangle)$ is modular. Then, in case $\langle A, B\rangle < G$, we see that $\langle A, B\rangle/(A \cap B)_{\langle A,B\rangle}$ is $\Pi_{\mathfrak{I}}$-nilpotent by induction. Finally, suppose that $\langle A, B\rangle = G$. Then, since A and B cover $A \cap B$ and the lattice $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ is modular, A and B are coatoms in the lattice $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ by [10, Theorem 2.1.10]. Therefore, either A is normal in G or G/A_G is $\Pi_{\mathfrak{I}}$-primary. In the former case $A \cap B$ is normal in B, contrary to our choice of A and B. Hence, G/A_G is $\Pi_{\mathfrak{I}}$-primary, so it is $\Pi_{\mathfrak{I}}$-nilpotent. Similarly, G/B_G is $\Pi_{\mathfrak{I}}$-nilpotent. Therefore, $G/A_G \cap B_G$ is $\Pi_{\mathfrak{I}}$-nilpotent by Proposition 2.5, so $G/A_G \cap B_G = G/(A \cap B)_G = \langle A, B\rangle/(A \cap B)_{\langle A,B\rangle}$ is $\Pi_{\mathfrak{I}}$-nilpotent.

The sufficiency of (a) and (b) follows from the following proposition.

Proposition 3.1. *The lattice $\mathcal{L}_{\Pi_{\mathfrak{I}}}(G)$ is modular if the following two conditions hold.*

(a) *If $T \leq S$ are $\Pi_{\mathfrak{I}}$-subnormal subgroups of G, where T is normal in S and either S/T is $\Pi_{\mathfrak{I}}$-primary or $|S/T| = p^3$ (p a prime), then $\mathcal{L}(S/T)$ is modular.*

(b) *$\langle A, B\rangle/(A \cap B)_{\langle A,B\rangle}$ is $\Pi_{\mathfrak{I}}$-nilpotent for each $A, B \in L_{\Pi_{\mathfrak{I}}}(G)$ such that A and B cover $A \cap B$, $A \cap B$ is not normal in both A and B and also $|A : A \cap B|$ and $|B : A \cap B|$ are σ-coprime, that is, $\sigma(|A : A \cap B|) \cap \sigma(|B : A \cap B|) = \emptyset$.*

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PROOF. Suppose that the proposition is false and let G be a counterexample of minimal order. Then G is neither $\Pi_{\mathfrak{J}}$-primary nor a group of order p^3 (p a prime) (otherwise, $\mathcal{L}(G) = \mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$ is modular by hypothesis).

(i) *If $A, B \in \mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$, where A covers B and B is not normal in A, then A/B_A is a nonabelian $\Pi_{\mathfrak{J}}$-primary group of order pq for some primes p and q. Hence, $|A : B|$ is a prime.*

Since B is not normal in A and A covers B (in $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$), A/B_A is $\Pi_{\mathfrak{J}}$-primary. Therefore, every subgroup of A/B_A is $\Pi_{\mathfrak{J}}$-subnormal in A/B_A by Lemma 2.1(6), so B is a maximal subgroup of A. Thus, A/B_A is a primitive group. On the other hand, by hypothesis, $\mathcal{L}(A/B_A)$ is modular. Therefore, A/B_A is a nonabelian $\Pi_{\mathfrak{J}}$-primary group of order pq with p and q distinct primes, by the Iwasawa theorem [10, Theorem 2.4.4] (see also [10, Lemma 2.4.3]).

(ii) *$L_{\Pi}(G)$ is lower semimodular.*

We need to show that if $A, B \in \mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$ are such that $\langle A, B\rangle$ covers A (in $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$), then B covers $A \cap B$ (in $\mathcal{L}_{\Pi}(G)$). Suppose that this is false. Then $G = \langle A, B\rangle$. Indeed, assume that $\langle A, B\rangle < G$. In view of Theorem 1.7, the hypothesis holds for $\langle A, B\rangle$. The choice of G implies that $\mathcal{L}_{\Pi}(\langle A, B\rangle)$ is modular. Hence, this lattice is lower semimodular by [10, Theorem 2.1.10] and so B covers $A \cap B$, which is a contradiction. Hence, $G = \langle A, B\rangle$. We show that A is not a conjugate of B. Indeed, assume that $A = B^x$. Then $A_G = B_G$. Lemma 2.1(4),(5) implies that the hypothesis holds for G/A_G. The choice of G implies that the lattice $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G/A_G)$ is modular. It is clear also that in this lattice $G/A_G = \langle A/A_G, B/A_G\rangle$ covers A/A_G. Hence, B/A_G covers $(A/A_G) \cap (B/B_G) = (A \cap B)/A_G$ and so B covers $A \cap B$ (in $\mathcal{L}_{\Pi}(G)$), which is a contradiction. Hence, $A \neq B^x$ for all $x \in G$. Now we show that A is not normal in G. Assume that $A \trianglelefteq G$. Then $AB = G$. Assume that T is a $\Pi_{\mathfrak{J}}$-subnormal subgroup of G such that $A \cap B \leq T \leq B$. Then AT is $\Pi_{\mathfrak{J}}$-subnormal in G by Theorem 1.7. Hence, from $A \leq AT \leq AB = G$, it follows that either $A = AT$ or $AT = AB$. In the former case, $T \leq A$ and so $A \cap B = T$. In the second case, $B = T(B \cap A) = T$. Hence, B covers $A \cap B$. This contradiction shows that A is not normal in G. Assume that $AB \neq G$. Then $AB/A_G \neq G/A_G$. Since $G = \langle A, B\rangle$ and G covers A, it follows that $B \nleq A$. Hence, G/A_G is a nonabelian group of order pq for some primes p and q and $|G : A|$ is a prime by (i). Since A is not a conjugate of B, A/A_G is not a conjugate of A_GB/A_G in G/A_G, so $G/A_G = (A_GB/A_G)(A/A_G) = AB/A_G$, which is a contradiction. Thus, $AB = G$. Then, by (i) again, $|G : A| = |B : A \cap B|$ is a prime. This implies that B covers $A \cap B$, which is a contradiction. Hence, we have proved (ii).

(iii) *$\mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$ is upper semimodular.*

In view of [5, page 173], it is enough to show that if $A, B \in \mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$ are such that A and B cover $A \cap B$ (in $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$), then $\langle A, B\rangle$ covers A (in $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G)$). Suppose that this is false.

Claim 1. $G = \langle A, B\rangle$. Assume that $\langle A, B\rangle < G$. Then the choice of G implies that $\mathcal{L}_{\Pi_{\mathfrak{J}}}(\langle A, B\rangle)$ is modular. Hence, this lattice is upper semimodular by [10, Theorem 2.1.10] and so $\langle A, B\rangle$ covers A, which is a contradiction. Hence, we have Claim 1.

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Claim 2. $AB \neq BA$. Indeed, if $AB = BA$, then $G = AB$ by Claim 1. Hence, for every subgroup $T \in L_{\Pi_\mathfrak{I}}(G)$ satisfying $A \leq T \leq G$, we have $T = A(T \cap B)$. On the other hand, since B covers $A \cap B$, from $A \cap B = T \cap A \cap B \leq T \cap B \leq B$ it follows that either $A \cap B = T \cap B$ or $T \cap B = B$. In the former case, $T = A(T \cap B) = A(A \cap B) = A$. In the second case, $T = G$. Hence, $G = \langle A, B\rangle = AB$ covers A, which is a contradiction. Hence, we have Claim 2.

Claim 3. $(A \cap B)_G = 1$. Assume that $(A \cap B)_G \neq 1$. Then, by Lemma 2.1(4),(5), the hypothesis holds for $G/(A \cap B)_G$. The choice of G implies that $\mathcal{L}_{\Pi_\mathfrak{I}}(G/(A \cap B)_G)$ is modular. Since B covers $A \cap B$ (in $\mathcal{L}_{\Pi_\mathfrak{I}}(G)$), $B/(A \cap B)_G$ covers $(A \cap B)/(A \cap B)_G$ (in $\mathcal{L}_{\Pi_\mathfrak{I}}(G/(A \cap B)_G)$). Hence, $\langle A, B\rangle/(A \cap B)_G$ covers $A/(A \cap B)_G$ (in $\mathcal{L}_{\Pi_\mathfrak{I}}(G/(A \cap B)_G)$), which implies that $\langle A, B\rangle$ covers A (in $\mathcal{L}_{\Pi_\mathfrak{I}}(G)$). This contradiction proves Claim 3.

Claim 4. A and B are not both $\Pi_\mathfrak{I}$-primary. Assume, for example, that A is $\Pi_\mathfrak{I}$-primary. First suppose that $A \in \mathfrak{I}$. Then $A^G \in \mathfrak{I}$ by Proposition 1.6. By Claim 1, $G = A^G B$ and so $B^\mathfrak{I} = G^\mathfrak{I}$ is normal in G by Lemma 2.3(3). Since G is not $\Pi_\mathfrak{I}$-primary, $B^\mathfrak{I} \neq 1$. Let R be a minimal normal subgroup of G contained in $B^\mathfrak{I}$. By Claim 3, $R \nleq A$, so $(A \cap B)R = B$ since B covers $A \cap B$. Hence, $AB = A(A \cap B)R = AR = BA$, contrary to Claim 2. Hence, A is a σ_i-group for some $i \neq 0$. Then A^G is a σ_i-group by Proposition 1.6. Hence, $|G : B| = |A^G B : B|$ is a σ_i-number. By Lemma 2.2(3), $O^{\sigma_i}(B) = G^{\sigma_i}(B)$ is normal in G. Since G is not $\Pi_\mathfrak{I}$-primary, $O^{\sigma_i}(B) \neq 1$ and so, as above, $AB = BA$, contrary to Claim 2. Hence, we have Claim 4.

Claim 5. $A \cap B \neq 1$. Assume that $A \cap B = 1$. Then A and B are minimal $\Pi_\mathfrak{I}$-subnormal subgroups of G, that is, A and B are atoms in $\mathcal{L}_{\Pi_\mathfrak{I}}(G)$. Hence, A and B are simple groups. By Claim 4, A and B are $\Pi_\mathfrak{I}$-perfect, so A and B are subnormal in G by Lemma 2.1(7). If one of these subgroups, say A, is nonabelian, then $R = A^G$ is a minimal normal subgroup of G and $R \leq N_G(B)$ by Lemma 2.3(1) since A is not $\Pi_\mathfrak{I}$-primary by Claim 4. But then $AB = BA$, contrary to Claim 2. Hence, $|A| = p$ and $|B| = q$ for some primes p and q. Proposition 1.6 (using the case when $\Pi = \sigma = \{\{2\}, \{3\}, \dots, \}$) implies that $A \leq O_p(G)$ and $B \leq O_q(G)$. Therefore, by Claim 2, $p = q$, so $G = O_p(G)$. But then $\mathcal{L}_{\Pi_\mathfrak{I}}(G) = L(G)$ is modular by [10, Lemma 2.3.3] and Condition (a), which is a contradiction. Hence, we have Claim 5.

Claim 6. Neither $A \leq N_G(A \cap B)$ nor $B \leq N_G(A \cap B)$. Assume, for example, that $B \leq N_G(A \cap B)$. If also $A \leq N_G(A \cap B)$, then $A \cap B$ is normal in G by Claim 1, which is impossible in view of Claims 3 and 5. Therefore, $A \nleq N_G(A \cap B)$. Then $A/(A \cap B)_A$ is a $\Pi_\mathfrak{I}$-primary σ_i-group for some $\sigma_i \in \Pi$ by (i). First suppose that $A/(A \cap B)_A \in \mathfrak{I}$. Then $(A \cap B)^\mathfrak{I} = A^\mathfrak{I}$ is normal in A by Lemma 2.3(2). On the other hand, since $B \leq N_G(A \cap B)$ and $(A \cap B)^\mathfrak{I}$ is characteristic in $A \cap B$, we have $B \leq N_G((A \cap B)^\mathfrak{I})$. Hence, $(A \cap B)^\mathfrak{I}$ is normal in G, which in view of Claim 3 implies that $(A \cap B)^\mathfrak{I} = 1$. But then A is $\Pi_\mathfrak{I}$-primary, contrary to Claim 4. Therefore, $A/(A \cap B)_A$ is a σ_i-group for some $\sigma_i \in \Pi \setminus \{\sigma_0\}$. But in this case we see similarly that A is a σ_i-group. Thus, A is $\Pi_\mathfrak{I}$-primary, contrary to Claim 4. This contradiction completes the proof of Claim 6.

Final contradiction for (iii). From (i) and Claim 6, $A/(A \cap B)_A$ and $B/(A \cap B)_B$ are $\Pi_{\mathfrak{J}}$-primary. First suppose that these two groups are σ_i-groups for the same $\sigma_i \in \Pi$. Assume that $i \neq 0$. Then $O^{\sigma_i}(A \cap B) = O^{\sigma_i}(A) = O^{\sigma_i}(B)$ by Lemma 2.2(3), so $O^{\sigma_i}(A \cap B)$ is normal in G by Claim 1. Hence, from Claim 3, $O^{\sigma_i}(A \cap B) = 1$. It follows that A and B are σ_i-groups. But then A and B are $\Pi_{\mathfrak{J}}$-primary, which contradicts Claim 4. Therefore, $i = 0$, that is, $A/(A \cap B)_A \in \mathfrak{J}$ and $B/(A \cap B)_B \in \mathfrak{J}$. Then $A^{\mathfrak{J}} = (A \cap B)^{\mathfrak{J}} = B^{\mathfrak{J}}$ by Lemma 2.3(2). From Claims 1 and 3, $(A \cap B)^{\mathfrak{J}} = 1$, which also implies that A and B are $\Pi_{\mathfrak{J}}$-primary. This contradiction shows that $A/(A \cap B)_A$ is a σ_i-group and $B/(A \cap B)_B$ is a σ_j-group for distinct σ_i and σ_j in σ. By Claim 3 and the hypothesis, $G/(A \cap B)_G = \langle A, B\rangle/(A \cap B)_G = G/1$ is $\Pi_{\mathfrak{J}}$-nilpotent by Condition (b). Thus, $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G) = \mathcal{L}(G)$ by Proposition 2.5(ii) and $G = A_1 \times \cdots \times A_t \times A_{t+1} \times \cdots \times A_n$ is the direct product of some $\Pi_{\mathfrak{J}}$-primary groups $A_1, \ldots, A_t$ and primary (that is, of prime power order) groups $A_{t+1}, \ldots, A_n$. Note that $\mathcal{L}(A_1), \ldots, \mathcal{L}(A_t)$ are modular by Condition (a) and $\mathcal{L}(A_{t+1}), \ldots, \mathcal{L}(A_n)$ are also modular by Condition (a) and [10, Lemma 2.3.3]. Therefore, $\mathcal{L}(A_1) \times \cdots \times \mathcal{L}(A_n)$ is modular. But $\mathcal{L}(G) \simeq \mathcal{L}(A_1) \times \cdots \times \mathcal{L}(A_n)$ by [10, Lemma 1.6.4]. Therefore, $\mathcal{L}_{\Pi_{\mathfrak{J}}}(G) = \mathcal{L}(G)$ is modular, which is a contradiction, proving (iii).

From (ii) and (iii) and [10, Theorem 2.1.10], the lattice $L_{\Pi_{\mathfrak{J}}}(G)$ is modular, contrary to the choice of G. The proposition is proved. □

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