UPSETS IN ROUND ROBIN TOURNAMENTS

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1. Introduction. Consider a round robin tournament in which each of n players is required to play precisely one game with each other player, and assume that each game ends in a win or a loss. The results of such a tournament can be conveniently recorded in a square (0, 1)-matrix $T = (t_{ij})$ of order n by setting $t_{ij} = 1$ if player i defeats player j, $t_{ij} = 0$ if player i loses to player j, and $t_{ii} = 0$. Thus T has 0's along the main diagonal, and in the off-diagonal positions T satisfies the "skew-symmetry" condition that $t_{ij} = 1$ if and only if $t_{ji} = 0$. We call such a (0, 1)-matrix T a *tournament matrix*.

Tournament matrices have received attention in **(1–4)**. In particular, Ryser **(4)** has studied the class of all n by n tournament matrices having specified row sums r_i, where

$$0 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n \leqslant n - 1. \tag{1.1}$$

The ith row sum of T represents the total number of wins for player i, and the ith column sum represents his losses. Thus, denoting the ith column sum by s_i, we have

$$r_i + s_i = n - 1. \tag{1.2}$$

The monotonicity assumption (1.1) implies that

$$n - 1 \geqslant s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n \geqslant 0. \tag{1.3}$$

With the notation chosen as in (1.1), a 1 above the main diagonal of T means that a player has defeated another who has a better (or at least no worse) win record, that is, an upset occurred. One of the results of **(4)** is an explicit formula for the minimum number $\tilde{\tau}$ of upsets that could have occurred, given the integers r_i:

$$\tilde{\tau} = \sum[r_i - (i - 1)], \tag{1.4}$$

the summation being over all i such that $r_i > i - 1$. It is clear that the sum in (1.4) is a lower bound for the number of upsets. That this bound can always be achieved is established in **(4)** by deducing the existence, in the class of all tournament matrices having row sums r_i, of a tournament matrix having the property that if row i contains a 0 in one or more of the positions $1, 2, \ldots, i - 1$, then row i contains only 0's in the remaining positions, for $i = 1, 2, \ldots, n$.

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In Section 2 we give a simple and direct construction for such a tournament matrix $\tilde{T}$, thereby providing an easier proof of (1.4). We then go on in Section 3 to show that the problem of finding a tournament matrix $\bar{T}$ which maximizes the number of upsets can also be solved by an equally simple construction.

Section 4 discusses the way in which both the minimum and maximum problems described above can be formulated as minimal cost-flow problems in suitable networks, and indicates how the duality theorem of linear inequality theory can be applied to deduce results about the structure of $\tilde{T}$ and $\bar{T}$.

2. The minimum number of upsets.

Let

$$R = (r_1, r_2, \ldots, r_n) \tag{2.1}$$

denote the given row-sum vector whose components r_i are arranged monotonically as in (1.1), and let

$$\mathfrak{T} = \mathfrak{T}(R) \tag{2.2}$$

denote the class of all tournament matrices having row-sum vector R and column-sum vector S, where

$$S = (s_1, s_2, \ldots. s_n). \tag{2.3}$$

The components of R and S, of course, satisfy (1.2). It is known that the class (2.2) is non-empty if and only if the inequalities

$$r_1 + r_2 + \ldots + r_e \geqslant \tfrac{1}{2}e(e-1) \tag{2.4}$$

hold for $e = 1, 2, \ldots, n$, with equality for $e = n$. The necessity of the conditions (2.4) is obvious, and sufficiency has been established in various ways **(1–4)**.

Let $T = (t_{ij})$ be in $\mathfrak{T}(R)$. Since for $i \neq j$, $t_{ij} = 1$ if and only if $t_{ji} = 0$, and since $t_{ii} = 0$, it follows that, for each $i = 1, 2, \ldots, n$,

$$\textstyle\sum_{j>i} t_{ij} - \sum_{j<i} t_{ji} = r_i - (i-1). \tag{2.5}$$

Conversely, if T is a tournament matrix whose elements above the main diagonal satisfy (2.5), then T is in $\mathfrak{T}(R)$. Our construction for a tournament matrix $\tilde{T}$ which minimizes upsets over all matrices in $\mathfrak{T}(R)$ will be based on (2.5), and hence we shall deal primarily with the vector

$$A = (a_1, a_2, \ldots, a_n) \tag{2.6}$$

whose components a_i are given by

$$a_i = r_i - (i-1). \tag{2.7}$$

The validity of Theorem 2.1 below, which rephrases the existence conditions (2.4) in terms of the vector A, is readily checked.

THEOREM 2.1. *Let the vector $A = (a_1, a_2, \ldots, a_n)$ have components defined by (2.7). Then*

$$a_1 \geqslant 0, \qquad a_{i-1} - a_i \leqslant 1 \quad (i = 2, 3, \ldots, n). \tag{2.8}$$

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The class $\mathfrak{T}(R)$ *is non-empty if and only if the inequalities*

$$a_1 + a_2 + \ldots + a_e \geqslant 0 \qquad (e = 1, 2, \ldots, n) \tag{2.9}$$

hold, with equality for $e = n$.

Conditions (2.8), which assert that the components of A can decrease by 1 at most, reflect the monotonicity assumption concerning the components of R.

We now describe a construction for a specific matrix $\tilde{T}$ in $\mathfrak{T}(R)$. From (2.5), with $i = n$, we see that the last column of $\tilde{T}$ contains $s_n = -a_n$ 1's. These 1's can be inserted in certain positions corresponding to positive components of the vector A in the following way. Find the first member of the last consecutive string of positive components of A. Starting with this position in column n of the matrix $\tilde{T}$ to be constructed, insert 1's consecutively downward until either $-a_n$ 1's have been inserted or this string of positive components of A has been exhausted. In the latter case, find the first member of the next-to-last consecutive string of positive components of A, and continue inserting 1's as above. When $-a_n$ 1's have been inserted in column n in this fashion, define a new vector A' having $n - 1$ components as follows:

$$a'_i = \begin{cases} a_i - 1 & \text{if column } n \text{ has a 1 in position } i, \\ a_i & \text{otherwise,} \end{cases} \tag{2.10}$$

for $i = 1, 2, \ldots, n - 1$. The last row of $\tilde{T}$ can now be filled in as the complement transpose of the last column. The entire procedure can then be repeated using A' and the undetermined portion of column $n - 1$, and so on.

The schema of Figure 2.1 illustrates the construction for

$$R = (1, 2, 3, 3, 3, 6, 6, 6, 6),$$
$$A = (1, 1, 1, 0, -1, 1, 0, -1, -2).$$

	1	2	3	4	5	6	7	8	9	9	8	7	6	5	4	3	2	1
1		0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
2	1		0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	
3	1	1		0	1	0	0	0	0	1	1	1	1	1	0	0		
4	1	1	1		0	0	0	0	0	0	0	0	0	0	0			
5	1	1	0	1		0	0	0	0	−1	−1	−1	−1	−1				
6	1	1	1	1	1		0	0	1	1	0	0	0					
7	1	1	1	1	1	1		0	0	0	0	0						
8	1	0	1	1	1	1	1		0	−1	−1							
9	0	1	1	1	1	0	1	1		−2								

FIGURE 2.1

We now verify that the construction produces a matrix $\tilde{T}$ in $\mathfrak{T}(R)$. In view of (2.5) and (2.10), this will surely be the case provided the construction can be carried out as described. To show that it can be, we may proceed inductively.

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We note first of all that (2.8) and (2.9) imply that A has at least $-a_n$ positive components, so the 1's can be inserted in column n as described. It now suffices to establish that the vector A' defined by (2.10) again satisfies (2.8) and (2.9). That A' satisfies (2.8) is obvious from the fact that we start with the first member of a string of positive components of A and work downward in reducing components of A. I am indebted to T. A. Brown for the following simple proof that the components of the reduced vector A' satisfy (2.9). Clearly

$$a'_1 + a'_2 + \ldots + a'_{n-1} = 0. \tag{2.11}$$

Thus if (2.9) were violated, there would be an integer e in the interval

$$1 \leqslant e \leqslant n-2, \tag{2.12}$$

such that

$$a'_1 + a'_2 + \ldots + a'_e < 0. \tag{2.13}$$

Hence $a_n < 0$. We may assume in (2.13) that

$$a'_e < 0, \qquad a'_{e+1} \geqslant 0, \tag{2.14}$$

for if this were not so, we could easily locate another integer e in the interval (2.12) for which (2.13) and (2.14) would hold.

Let

$$a'_1 + a'_2 + \ldots + a'_e = a_1 + a_2 + \ldots + a_e - p \tag{2.15}$$

and

$$a'_{e+1} + a'_{e+2} + \ldots + a'_{n-1} = a_{e+1} + a_{e+2} + \ldots + a_{n-1} - q \tag{2.16}$$

for non-negative integers p and q satisfying

$$p + q = -a_n. \tag{2.17}$$

By (2.11) and (2.13) we have

$$a'_{e+1} + a'_{e+2} + \ldots + a'_{n-1} > 0, \tag{2.18}$$

and hence

$$a_{e+1} + a_{e+2} + \ldots + a_{n-1} > 0. \tag{2.19}$$

It now follows from (2.8), (2.14), and (2.19) that the sequence $a_{e+1}, a_{e+2}, \ldots, a_{n-1}$ has more than $-a_{n-1}$ positive members, and hence has at least $-a_n$ positive members. Consequently, using (2.14), we see that our procedure for defining A' implies $q = -a_n$, $p = 0$. Thus $a_1 + a_2 + \ldots + a_e < 0$, contradicting the assumption that A satisfies (2.9). Hence A' satisfies (2.9).

THEOREM 2.2. *The matrix $\tilde{T}$ is in $\mathfrak{T}(R)$ and minimizes the number of* 1'*s above the main diagonal over all matrices in $\mathfrak{T}(R)$.*

It remains only to check the second assertion of Theorem 2.2. But this is easily done, since $\tilde{T}$ clearly has

$$\tilde{\tau} = \sum a_i = \sum [r_i - (i-1)]$$

1's above the main diagonal, the summation being over all i such that $a_i > 0$.

Note too that the construction for $\tilde{T}$ provides an independent proof of Theorem 2.1.

3. The maximum number of upsets. In constructing the tournament matrix $\tilde{T}$, we worked with elements above the main diagonal so as to minimize the number of 1's which could be inserted to satisfy the constraints (2.5). We now shift attention to elements below the main diagonal, our aim being to minimize the number of 1's required to satisfy the constraints.

$$\sum_{j<i} t_{ij} - \sum_{j>i} t_{ji} = r_i - (n-i) \qquad (i = 1, 2, \ldots, n), \tag{3.1}$$

which also characterize, via skew-symmetry, a matrix $T = (t_{ij})$ in $\mathfrak{T}(R)$. Since minimizing the number of 1's below the main diagonal is equivalent to maximizing the number of 1's above the diagonal, we are here concerned with the maximum possible number of upsets.

Let

$$B = (b_1, b_2, \ldots, b_n) \tag{3.2}$$

have components defined by

$$b_i = r_i - (n-i) \qquad (i = 1, 2, \ldots, n). \tag{3.3}$$

The monotonicity assumption (1.1) on components of R implies that

$$-(n-1) \leqslant b_1 < b_2 < \ldots < b_n, \tag{3.4}$$

and the existence conditions (2.4) imply that

$$b_1 + b_2 + \ldots + b_e \geqslant e(e-n) \qquad (e = 1, 2, \ldots, n), \tag{3.5}$$

with equality for $e = n$. However, we shall make no explicit use of (3.5) in verifying that the construction of this section produces a matrix in $\mathfrak{T}(R)$. The construction will not, in any event, maintain the strict monotonicity of (3.4) for the reduced vector B' defined below, although it will preserve monotonicity of components of B'.

Our argument will use interchanges **(4)**, which we shall think of as being generated by elements below the main diagonal. Suppose we have a (0, 1)-solution t_{ij}, $i > j$, of equations (3.1). Let

$$t_{i_1,j_1}, \qquad t_{i_1,j_2}, \qquad t_{i_2,j_2}, \qquad t_{i_2,j_1} \tag{3.6}$$

be alternately 0 and 1 (or 1 and 0) in this solution. Then interchanging 0's and 1's in (3.6) gives another (0, 1)-solution to (3.1). We shall call this operation an interchange involving the positions

$$(i_1, j_1), \qquad (i_1, j_2), \qquad (i_2, j_2), \qquad (i_2, j_1). \tag{3.7}$$

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Another type of four-way interchange involves the positions

$$(3.8) \qquad (i_1, j_1), \qquad (i_2, j_1), \qquad (j_2, i_1), \qquad (j_2, i_2),$$

where again the four corresponding values of t_{ij} are alternately 0 and 1 (or 1 and 0). Finally, a third type of interchange involves the three positions

$$(3.9) \qquad (i_1\ j_1), \qquad (i_2, j_1), \qquad (i_2, i_1),$$

again the three corresponding values of t_{ij} being alternately 0 and 1 (or 1 and 0). Each of these interchanges produces another (0, 1)-solution to equations (3.1). Hence performing an interchange and its skew-symmetric mate changes a matrix T in $\mathfrak{T}(R)$ to another matrix T' in $\mathfrak{T}(R)$. It is shown in (**4**) that one can pass through the class $\mathfrak{T}(R)$ by interchanges of these types. (Actually, a four-way interchange can be accomplished by a sequence of two three-way interchanges, but we find it convenient to ignore this fact.)

We now describe the construction of a particular matrix $\bar{T}$ in $\mathfrak{T}(R)$. The first column of $\bar{T}$ contains $s_1 = -b_1$ 1's. We insert these 1's in positions corresponding to the $-b_1$ largest components of B, with preference given to topmost positions in case of equal components. Then define a new vector B' having $n - 1$ components by

$$(3.10) \qquad b'_i = \begin{cases} b_i - 1 & \text{if a 1 has been inserted in position } i, \\ b_i & \text{otherwise,} \end{cases}$$

for $i = 2, 3, \ldots, n$. We may then fill in the first row of $\bar{T}$ by skew-symmetry. The procedure is then repeated using B' and the undetermined part of the second column, and so on.

The schema of Figure 3.1 illustrates the construction for

$$R = (1, 2, 3, 3, 3, 6, 6, 6, 6),$$
$$B = (-7, -5, -3, -2, -1, 3, 4, 5, 6).$$

								−7		1	0	0	0	0	0	0	0
							−5	−5	0		1	1	0	0	0	0	0
						−4	−4	−3	1	0		1	1	0	0	0	0
					−3	−3	−3	−2	1	0	0		1	1	0	0	0
				−3	−3	−3	−2	−1	1	1	0	0		0	1	0	0
			−1	0	0	1	2	3	1	1	1	0	1		1	1	0
		0	0	0	1	2	3	4	1	1	1	1	0	0		1	1
	0	0	0	1	2	3	4	5	1	1	1	1	1	0	0		1
0	0	0	1	2	3	4	5	6	1	1	1	1	1	1	1	0	0
9	8	7	6	5	4	3	2	1	1	2	3	4	5	6	7	8	9

FIGURE 3.1

Notice in the example of Figure 3.1 that the "tie-breaking" part of the construction which gives preference to topmost positions in case of equal components was used in assigning the 1's in column 5. This part of the rule maintains monotonicity of components of the new vector B', although not the strict monotonicity satisfied by components of the starting vector, of course. The fact that strict monotonicity may not be preserved means that the minor of $\bar{T}$ which corresponds to the new vector may not have monotonic row sums. For example, deleting the first and second rows and columns in Figure 3.1 leaves a minor whose row sums are not monotonic. This is to be contrasted with the construction of Section 2, where monotonicity of row sums was preserved.

We now prove that the construction produces a matrix $\bar{T}$ in $\mathfrak{T}(R)$ which maximizes upsets. To this end, let T be a tournament matrix in $\mathfrak{T}(R)$ which minimizes the number of 1's below the main diagonal. We shall show that the first column of T can be made to coincide with that of $\bar{T}$ by performing four-way interchanges on T. Let the bottommost 1 of the first column of T be in position e. Suppose that $e < n$, so that

$$(3.11) \qquad t_{e1} = 1, \qquad t_{e+1,1} = 0.$$

We have

$$(3.12) \quad \textstyle\sum_{j<e} t_{ej} - \sum_{j>e} t_{je} = b_e \leqslant b_{e+1} = \sum_{j<e+1} t_{e+1,j} - \sum_{j>e+1} t_{j,e+1}.$$

It follows from (3.11) and (3.12) that either there is an integer j such that

$$(3.13) \qquad e < j \leqslant n, \qquad t_{je} = 1, \qquad t_{j,e+1} = 0,$$

or there is an integer i such that

$$(3.14) \qquad 1 \leqslant i < e, \qquad t_{ei} = 0, \qquad t_{e+1,i} = 1.$$

Suppose (3.13) holds. Then $j \neq e + 1$, for otherwise an interchange involving the three positions

$$(e, 1), \qquad (e + 1, 1), \qquad (e + 1, e)$$

produces a matrix in $\mathfrak{T}(R)$ having fewer 1's below the diagonal than does T, contradicting our assumption about T. Hence $j > e + 1$ in (3.13). We may now perform a four-way interchange involving the positions

$$(e, 1), \qquad (e + 1, 1), \qquad (j, e), \qquad (j, e + 1).$$

This lowers the position of the bottommost 1 in the first column of T and yields a new matrix having the minimum number of 1's below the main diagonal. Similarly, if (3.14) holds, a four-way interchange involving the positions

$$(e, 1), \qquad (e + 1, 1), \qquad (e, i), \qquad (e + 1, i)$$

accomplishes the same result. Repetition of this argument shows that the 1's in the first column of T can be brought to appear consecutively at the bottom by four-way interchanges, thereby producing a new matrix T' in $\mathfrak{T}(R)$ having the minimum number of 1's below the diagonal.

It is important to note that we did not require the strict inequality $b_e < b_{e+1}$ in (3.12) for the above argument, but used only the fact that $b_e \leqslant b_{e+1}$. This means that we can repeat the argument on the second column, using the vector B', and so on. For later stages of the argument, we need also to observe that in case the 1's in a column of $\bar{T}$ do not all appear consecutively at the bottom (because of the tie-breaking rule for equal components), we can first bring all 1's to the bottom by interchanges which affect adjacent rows, and then raise an appropriate number of 1's by similar interchanges, since we shall be working with equal components of the reduced vector in the raising process.

Thus T can be transformed into $\bar{T}$ by four-way interchanges, and consequently $\bar{T}$ minimizes the number of 1's below the diagonal. This proves the following theorem:

THEOREM 3.1. *The matrix $\bar{T}$ is in $\mathfrak{T}(R)$ and maximizes the number of* 1'*s above the main diagonal over all matrices in* $\mathfrak{T}(R)$.

In contrast with the situation for the minimum number of upsets $\tilde{\tau}$, we do not have a simple formula for the maximum number of upsets $\bar{\tau}$. In terms of the discussion in Section 4, this is because the rule for constructing $\bar{T}$ involves "transshipment" in satisfying "demands" from "supplies," whereas the rule for constructing $\tilde{T}$ does not.

4. Flows, duality, and normal forms. Let $\mathfrak{G}$ be a directed graph consisting of nodes $1, 2, \ldots, n$ and directed arcs ij (from i to j). Suppose that each arc ij of $\mathfrak{G}$ has associated with it two non-negative numbers c_{ij} and a_{ij}, and that each node i of $\mathfrak{G}$ has associated with it a number a_i. We call c_{ij} the *capacity* of arc ij, a_{ij} the *unit cost of flow* in ij, and a_i the *supply* or *demand* at node i according as $a_i > 0$ or $a_i \leqslant 0$. We shall assume that

$$a_1 + a_2 + \ldots + a_n = 0. \tag{4.1}$$

A *feasible flow* $X = (x_{ij})$ in $\mathfrak{G}$ is a real vector having one component for each arc ij, which satisfies the equations and inequalities

$$\begin{aligned} \textstyle\sum_j (x_{ij} - x_{ji}) &= a_i \qquad (i = 1, 2, \ldots, n), \\ 0 \leqslant x_{ij} &\leqslant c_{ij} \qquad \text{for all arcs } ij. \end{aligned} \tag{4.2}$$

If, in addition, X minimizes the *flow cost*

$$\textstyle\sum_{ij} a_{ij} x_{ij} \tag{4.3}$$

over all feasible flows, we call X a *minimal cost flow*. We refer the reader to **(1**, chapter III**)** for a discussion of various iterative methods for constructing minimal cost flows. In particular, it is well known that if the supplies and demands a_i and the arc capacities c_{ij} are integers, then there is a minimal cost flow $X = (x_{ij})$ whose components x_{ij} are integers.

Both the minimum and maximum problems of Sections 2 and 3 can thus be viewed as minimal cost-flow problems in appropriate (acyclic) directed

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graphs, as suggested by (2.5) and (3.1). For the minimum upset problem, we may take the graph to consist of nodes $1, 2, \ldots, n$, with all arcs of the form ij, where $i < j$. Each arc ij has $c_{ij} = a_{ij} = 1$, and node i has supply (demand) $a_i = r_i - (i-1)$. Theorem 2.1 gives necessary and sufficient conditions for feasibility of the supplies and demands, the condition being that the cumulative net supply must be non-negative. Theorem 2.2 shows that the flow $\tilde{X}$ whose components are given by the elements of $\tilde{T}$ which lie above the main diagonal is a minimal cost flow. This flow involves no transshipment; that is, $\tilde{x}_{ij} > 0$ implies that node i is a supply node and node j is a demand node. Consequently the minimal flow cost is given by the equation

$$\tilde{\tau} = \sum a_i,$$

where the summation is over supplies a_i.

For the maximum upset problem, we take the graph $\mathfrak{G}$ to consist of nodes $1, 2, \ldots, n$, with supplies and demands given by $b_i = r_i - (n-i)$, and arcs ij, where $i > j$, with capacities and unit costs $c_{ij} = a_{ij} = 1$. The elements below the main diagonal of the matrix $\bar{T}$ then give a minimal cost flow $\bar{X}$. This flow does involve transshipment. For example, in Figure 3.1, node 1 receives shipments from nodes 3, 4, 5, which are themselves demand nodes.

The linear programming duality theorem can be applied to these minimal cost flow problems to deduce information about the structure of $\tilde{T}$ and $\bar{T}$. Consider the minimum upset problem. The dual of this can be formulated as follows: Find numbers π_i, where $i = 1, 2, \ldots, n$, and non-negative numbers α_{ij}, for $i < j$, which maximize the dual form

$$\sum_i a_i \pi_i - \sum_{ij} \alpha_{ij} \tag{4.4}$$

subject to the dual constraints

$$\pi_i - \pi_j - \alpha_{ij} \leqslant 1 \qquad (i < j;\, i, j = 1, 2, \ldots, n). \tag{4.5}$$

It is apparent from (4.4) and (4.5) that we may set

$$\alpha_{ij} = \max(0, \pi_i - \pi_j - 1), \tag{4.6}$$

and thus the dual problem becomes that of maximizing the (unconstrained) function

$$\sum_i a_i \pi_i - \sum_{ij} \max(0, \pi_i - \pi_j - 1). \tag{4.7}$$

The results of **(4)** and Section 2 show that an optimal solution to the dual problem is given by

$$\tilde{\pi}_i = \begin{cases} 1 & \text{if } a_i > 0, \\ 0 & \text{if } a_i \leqslant 0. \end{cases} \tag{4.8}$$

It is not an accident that the optimal π_i turn out to be integers. This is always the case for minimal cost-flow problems in which the a_{ij} are integers. That (4.8) constitutes an optimal solution to the dual problem follows from the fact

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that (4.8) inserted in (4.7) and $\tilde{X}$ inserted in (4.3) produce equality between primal and dual forms,

$$\sum_{ij} \tilde{x}_{ij} = \sum_i a_i \tilde{\pi}_i - \sum_{ij} \max(0, \tilde{\pi}_i - \tilde{\pi}_j - 1), \tag{4.9}$$

both sides of (4.9) being equal to $\tilde{r}$.

The optimal dual solution $\tilde{\Pi}$ given by (4.8) can be used to obtain, via well-known optimality properties for the pair of dual linear programmes being dealt with here, certain information about the structure of a minimal cost flow, hence of $\tilde{T}$. Since X and Π are optimal in their respective programmes if and only if

$$\begin{aligned} \pi_i - \pi_j < 1 \text{ implies } x_{ij} = 0 \qquad (i < j), \\ \pi_i - \pi_j > 1 \text{ implies } x_{ij} = 1 \qquad (i < j), \end{aligned} \tag{4.10}$$

it follows that a matrix T which minimizes upsets has the form illustrated in Figure 4.1. Conversely, any tournament matrix in $\mathfrak{T}(R)$ having this form minimizes upsets.

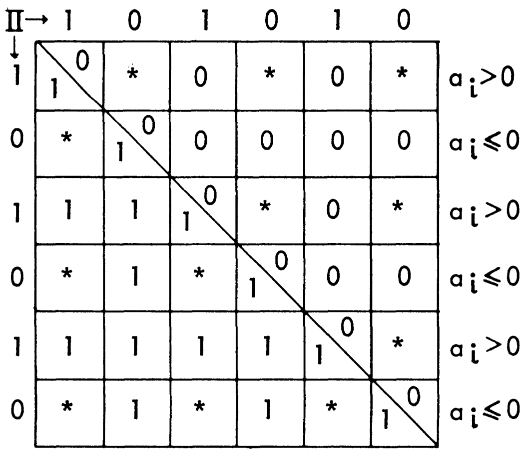

FIGURE 4.1

In Figure 4.1, 1 stands for all elements 1, 0 for all elements 0, and $*$ for undetermined portions. The partitioning is determined solely by the positivity and non-positivity of the a_i, as shown. (Rows corresponding to $a_i = 0$ could be included in either way.) Note that this form precludes transshipment; that is, it has the minimizing property deduced in **(4)**, namely that if row i has a 0 in its initial segment $(i, 1), (i, 2), \ldots, (i, i-1)$, then it has only 0's in its terminal segment.

We turn now to the maximum upset problem, expressed as a minimal cost-flow problem. The dual of this problem is that of finding π_i, where $i = 1, 2, \ldots, n$ and non-negative α_{ij}, where $i > j$, which maximize the expression

$$\sum_i b_i \pi_i - \sum_{ij} \alpha_{ij}, \tag{4.11}$$

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subject to the constraints

$$\pi_i - \pi_j - \alpha_{ij} \leqslant 1 \qquad (i > j; \quad i, j = 1, 2, \ldots, n). \tag{4.12}$$

Again we may assume (4.6), so that the dual problem asks for the (unconstrained) maximum of the function

$$g(\Pi) = \sum_i b_i \pi_i - \sum_{ij} \max(0, \pi_i - \pi_j - 1). \tag{4.13}$$

It suffices to consider vectors $\Pi = (\pi_1, \pi_2, \ldots, \pi_n)$ having integral components π_i in (4.13), as noted in the discussion of the minimum upset problem.

In view of the condition $\sum_{i=1}^{n} b_i = 0$, replacing π_i by $\pi_i + k$, $i = 1, 2, \ldots, n$, does not change (4.13), and hence we may deal with non-negative integers π_i. We shall show, in fact, that optimal π_i may be assumed to satisfy the condition

$$0 = \pi_1 \leqslant \pi_2 \leqslant \ldots \leqslant \pi_n, \tag{4.14}$$

and the inequality

$$\pi_{i+1} - \pi_i \leqslant 1 \qquad (i = 1, 2, \ldots, n-1). \tag{4.15}$$

To establish (4.14), suppose that

$$\pi_{e+1} < \pi_e \tag{4.16}$$

for some e. Let

$$\Pi = (\pi_1, \pi_2, \ldots, \pi_e, \pi_{e+1}, \ldots, \pi_n),$$
$$\Pi' = (\pi_1, \pi_2, \ldots, \pi_{e+1}, \pi_e, \ldots, \pi_n).$$

Then

$$g(\Pi') - g(\Pi) = \pi_{e+1} b_e + \pi_e b_{e+1} - \pi_e b_e - \pi_{e+1} b_{e+1} - \max(0, \pi_e - \pi_{e+1} - 1) + \max(0, \pi_{e+1} - \pi_e - 1).$$

By (4.16), this equation becomes

$$g(\Pi') - g(\Pi) = (\pi_e - \pi_{e+1})(b_{e+1} - b_e - 1) + 1.$$

Hence, since $b_e < b_{e+1}$,

$$g(\Pi') - g(\Pi) \geqslant 1.$$

Consequently, interchanging adjacent components of Π which satisfy (4.16) increases (4.13). This proves (4.14).

For the proof of (4.15), assume that

$$\pi_{e+1} \geqslant \pi_e + 2 \tag{4.17}$$

for some e, and let

$$\Pi = (\pi_1, \pi_2, \ldots, \pi_e, \pi_{e+1} \ldots, \pi_n),$$
$$\Pi' = (\pi_1, \pi_2, \ldots, \pi_e, \pi_{e+1} - 1, \ldots, \pi_n - 1),$$

the components of Π being monotonic. Then

$$(4.18)\quad g(\Pi') - g(\Pi) = -\sum_{i=e+1}^{n} b_i + \sum_{i=e+1}^{n}\sum_{j=1}^{e} [\max(0, \pi_i - \pi_j - 1) - \max(0, \pi_i - \pi_j - 2)].$$

By (4.14) and (4.17), each term in the double sum of (4.18) is at least 1, and hence

$$(4.19)\qquad g(\Pi') - g(\Pi) \geqslant -\sum_{i=e+1}^{n} b_i + e(n-e).$$

By (3.5), the right side of (4.19) is non-negative. Hence we may assume (4.15).

It follows now from (4.14), (4.15), and the optimality properties (4.10) that a tournament matrix $\bar{T}$ which maximizes upsets has the form illustrated in Figure 4.2, and conversely. For example, the matrix $\bar{T}$ of Figure 3.1 has the

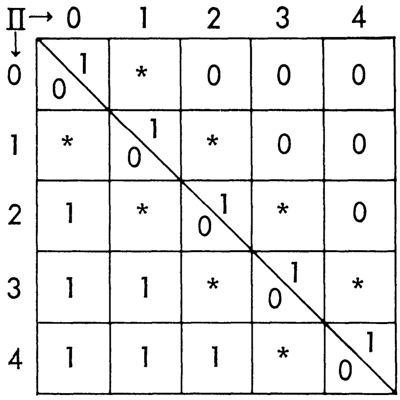

FIGURE 4.2

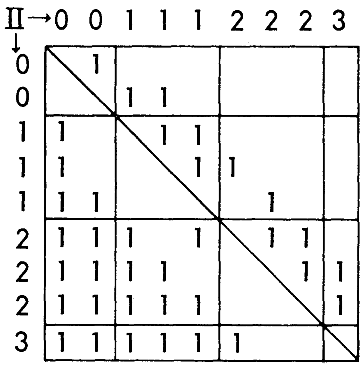

FIGURE 4.3

partitioned form shown in Figure 4.3. The partitioning is not in general unique. For instance, another optimal dual solution in this example is given by

$$\pi_1 = 0, \quad \pi_2 = 1, \quad \pi_3 = \pi_4 = \pi_5 = 2, \quad \pi_6 = \pi_7 = \pi_8 = 3, \quad \pi_9 = 4.$$

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