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Covering Discs in Minkowski Planes

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

Horst Martini and Margarita Spirova

Abstract. We investigate the following version of the circle covering problem in strictly convex (normed or) Minkowski planes: to cover a circle of largest possible diameter by k unit circles. In particular, we study the cases $k = 3$, $k = 4$, and $k = 7$. For $k = 3$ and $k = 4$, the diameters under consideration are described in terms of side-lengths and circumradii of certain inscribed regular triangles or quadrangles. This yields also simple explanations of geometric meanings that the corresponding homothety ratios have. It turns out that basic notions from Minkowski geometry play an essential role in our proofs, namely Minkowskian bisectors, d-segments, and the monotonicity lemma.

1 Introduction

The problem of covering the unit circle with k congruent circles of minimum diameter is called the *circle covering problem* (see [5, 6, 10]). This problem was investigated by many authors, and the smallest diameter has been found for all $k \leq 10$, see also [2, 3, 7]. In the present paper we investigate the extension of this problem to (normed or) Minkowski planes. The methods developed for our approach (*e.g.,* the use of bisectors which can have a complicated geometric structure in normed planes) demand restricting ourselves to strictly convex Minkowski planes, and we consider the circle covering problem in the other direction: to cover a circle of largest possible diameter by k unit circles. We obtain results for the cases $k \in \{3, 4, 7\}$. In the Euclidean plane, these three cases are usually considered to be trivial. As we shall see, this is not the case in strictly convex Minkowski planes. Due to the large variety of possible circle shapes in normed planes, the radii of the circles to be covered for $k = 3$ and $k = 4$ are described in terms of circumradii of inscribed regular triangles and quadrangles, respectively, which have special side-lengths (and regularity is meant in the Minkowskian sense). This yields simple geometric descriptions of the correspondingly occurring homothety ratios. The case $k = 7$ yields, in contrast to $k \in \{3; 4\}$, a situation completely analogous to that in the Euclidean plane. To prove our results, we also use typical tools and notions from Minkowski geometry (*i.e.,* the geometry of finite dimensional real Banach spaces, see [4, 15, 17, 18]), such as Minkowskian bisectors, d-segments, and the monotonicity lemma.

By a (normed or) *Minkowski plane* $(\mathbb{X}, \|\cdot\|)$ with *origin* 0 we mean a two-dimensional real linear space $\mathbb{X}$ with norm $\|\cdot\|$. As usual, the *unit disc* and the *unit circle* of

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$(\mathbb{X}, \|\cdot\|)$ are defined by

$$\mathcal{D} := \{x \in \mathbb{X} : \|x\| \le 1\} \quad \text{and} \quad \mathcal{C} := \{x \in \mathbb{X} : \|x\| = 1\},$$

respectively. The set $\{x \in \mathbb{X} : \|x - p\| \le \lambda\}$, where $\lambda \in \mathbb{R}^+$, is called the *(Minkowski) disc* with center p and radius λ and denoted by $D(p, \lambda)$. The boundary of $D(p, \lambda)$ is then a (*Minkowski*) *circle* and denoted by $C(p, \lambda)$. We denote the segment between $x, y \in \mathbb{X}$ by $[xy]$, the line through x and y by $\langle xy \rangle$, the ray with origin x and passing through y by $[xy\rangle$, the triangle with vertices $x, y, z \in \mathbb{X}$ by $\triangle xyz$, and the circular arc with the endpoints x, y by $\widehat{xy}$. As usual, we denote the convex hull of a two-dimensional set K by $\operatorname{conv} K$, its interior by $\operatorname{int} K$ and, if K is closed, the boundary of K by ∂K.

A Minkowski plane $(\mathbb{X}, \|\cdot\|)$ is called *strictly convex* if $\|x+y\| = \|x\| + \|y\|$ implies that x and y are linearly dependent or, equivalently, if the unit circle does not contain a non-denegerate line segment. Any two circles in a strictly convex Minkowski plane intersect in at most two points; see [17, §3.2]. If x_1, x_2 are two different points in a strictly convex normed plane, and $y_1, y_2 \in C(x_1, \lambda) \cap C(x_2, \lambda)$ with $y_1 \ne y_2$, then

$$x_1 + x_2 = y_1 + y_2; \tag{1.1}$$

see [1].

The *bisector* of two points p and q is defined by

$$B(p, q) := \{x \in \mathbb{X} : \|x - p\| = \|x - q\|\}.$$

It is known that bisectors in every strictly convex Minkowski plane are unbounded simple curves; see [16, §8.2] and the survey [15, §4.2]. Also we note that the definition of bisectors immediately implies that $B(p, q)$ is symmetric with respect to the midpoint of $[pq]$. Many further results on and applications of bisectors in Minkowski planes and spaces are collected in Part 4 of the survey [15].

2 Some Preliminaries

Let there be given a *convex body* B, (*i.e.,* a compact, convex set with non-empty interior) in a Minkowski plane $(\mathbb{X}, \|\cdot\|)$. A collection $\{B_i\}$ of finitely many convex bodies in $(\mathbb{X}, \|\cdot\|)$ is called a *covering* of the body B if any point of B belongs to $\bigcup_i B_i$ and for every body B_i of $\{B_i\}$ there exists a point $x \in B$ such that $x \in \bigcup_{j \ne i} B_j$. Denote by $h_k(B)$ the smallest positive ratio of k homothetical copies of B whose union covers B. The following bounds on $h_k(B)$ are known for $k \in \{3, 4, 7\}$:

$$\tfrac{2}{3} \le h_3(B) \le 1, \tag{2.1}$$

$$\tfrac{1}{2} \le h_4(B) \le \frac{\sqrt{2}}{2}, \tag{2.2}$$

$$\theta \le h_7(B) \le \tfrac{1}{2}, \tag{2.3}$$

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where $\sqrt{7}/7 \le \theta \le \frac{5}{11}$; see [12, 13]. If B is centrally symmetric, *i.e.*, it can be considered as the unit disc with respect to some norm, let $R_k(B)$ be the maximal radius of all homothets of B that can be covered by k translates of B. Then $R_k(B) = 1/h_k(B)$, and one can rewrite the inequalities (2.1), (2.2), and (2.3) in the forms

$$1 \le R_3(B) \le \frac{3}{2}, \tag{2.4}$$

$$\sqrt{2} \le R_4(B) \le 2, \tag{2.5}$$

$$2 \le R_7(B) \le \frac{1}{\theta}. \tag{2.6}$$

The first lemma which is necessary for our considerations is known as the *monotonicity lemma*. It was proved in [9]; see also [8] and [17, §3.5].

Lemma 2.1 *Let $\mathcal{C}$ be the unit circle of a strictly convex normed plane $(\mathbb{X}, \|\cdot\|)$, and p, q, r be different points belonging to $\mathcal{C}$ such that the origin 0 does not belong to the open half-plane determined by $\langle pq \rangle$ which contains r. Then $\|p-q\| > \|p-r\|$.*

Remark 2.2. For any $x, y \in D(p, \lambda)$ the monotonicity lemma implies that $\|x-y\| \le 2\lambda$.

Lemma 2.3 *Let there be given a convex body B in a Minkowski plane $(\mathbb{X}, \|\cdot\|)$, and let $\mathcal{B} = \{B_i\}_{i=1}^k$ be a covering of B. If $x \in \partial B_i \cap B$, where $i \in \{1, \dots, k\}$, then there exists a body B_j from $\mathcal{B}$ different from B_i such that $x \in B_j$.*

Proof We argue by contradiction. Suppose that for any $j = 1, \dots, k$ and $j \ne i$ we have $x \notin B_j$. Then there exists a disc $D(x, \varepsilon)$ with $(\operatorname{int} D(x, \varepsilon)) \cap B_j = \varnothing$. Let y be a point of $\operatorname{int} D(x, \varepsilon)$ such that $y \notin B_i$. Denote by D the disc centered at y whose interior lies in $\operatorname{int} D(x, \varepsilon)$ and which also satisfies $D \cap B_i = \varnothing$; see Figure 1. For any $j = 1, \dots, k$, and $j \ne i$, there exists a point $y_j \in B_j \cap B$ with

$$\|y - y_j\| = \inf\{\|y - z\| : z \in B_j \cap B\},$$

by [19, Theorem 1.9.1]. If y_0 is that point among $\{y_j\}_{j=1, j\ne i}^k$ which has the smallest distance to y, then clearly $y_0 \notin \operatorname{int} D(x, \varepsilon)$. Thus for any $z \in \bigcup_{j \ne i}(B_j \cap B)$ we get $\|y - y_0\| \le \|y - z\|$. Let y^* be a point lying in $[y y_0] \cap D$. Then $\|y - y^*\| < \|y - y_0\|$, which means that $y^* \notin \bigcup_{j\ne i}(B_j \cap B)$. In view of $y^* \in D$ we have $y^* \notin B_i$. Besides this, the convexity of B implies that the point y^* belongs to B. This contradicts the fact that $\{B_i\}_{i=1}^k$ is a covering of B. ■

It is known that for any three non-collinear points in a strictly convex normed plane there exists at most one circle containing them; see [17, § 3.2]. The next lemma shows that if these points form an equilateral triangle (in the Minkowskian sense), then such a circle always exists.

Lemma 2.4 *An equilateral triangle in a strictly convex Minkowski plane $(\mathbb{X}, \|\cdot\|)$ possesses exactly one circumcircle.*

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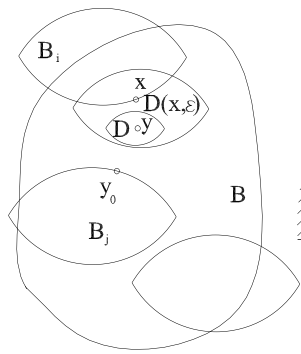

Figure 1

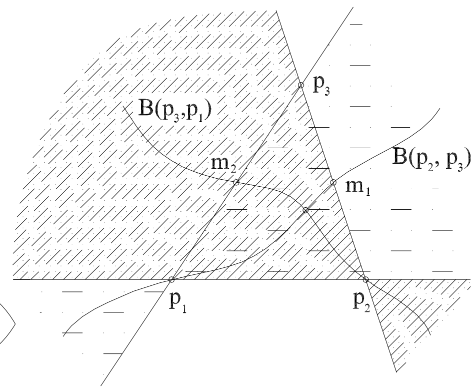

Figure 2

Proof Let there be given a triangle $\triangle p_1 p_2 p_3$ which is equilateral in $(\mathbb{X}, \|\cdot\|)$, and m_1 and m_2 be the midpoints of $[p_2 p_3]$ and $[p_3 p_1]$, respectively. Clearly, $p_1 \in B(p_2, p_3)$ and $p_2 \in B(p_3, p_1)$. By [17, Proposition 17] (see also [11]), $B(p_2, p_3)$ is contained in the double cone of p_2 and p_3 with apex p_1, and $B(p_3, p_1)$ is contained in the double cone of p_3 and p_1 with apex p_2; see Figure 2.

The curve $B(p_2, p_3)$ can intersect the segment $[p_2 p_3]$ only in m_1, so that if we denote the part of $B(p_2, p_3)$ between p_1 and m_1 by γ_1, then $\gamma_1 \subset \operatorname{conv}\{p_1, p_2, p_3\}$. Analogously, if γ_2 is the part of $B(p_3, p_1)$ between p_2 and m_2, then

$$\gamma_2 \subset \operatorname{conv}\{p_1, p_2, p_3\}.$$

If $J := [p_1 p_2] \cup [p_2 m_1] \cup \gamma_1$, then J is a Jordan curve with respect to which the point m_2 is not an interior point. Assuming that $m_2 \in \gamma \subset B(p_2, p_3)$, we have

$$\|p_2 - m_2\| = \|m_2 - p_3\| \iff \|p_2 - \tfrac{1}{2}\,(p_1 + p_3)\| = \tfrac{1}{2}\,\|p_1 - p_3\|$$
$$\iff \|p_2 - p_1 + p_2 - p_3\| = \|p_1 - p_3\|.$$

This is impossible, since d-segments are always linear segments if and only if the normed plane under consideration is strictly convex; see [4, Corollary 11.3]. (The *d-segment* $[a, b]_d$ with endpoints $a, b \in (\mathbb{X}, \|\cdot\|)$ is the set of all $x \in (\mathbb{X}, \|\cdot\|)$ satisfying $\|a - x\| + \|x - b\| = \|a - b\|$.) Hence m_2 is an exterior point with respect to J. On the other hand, there exists an $\varepsilon > 0$ such that $C(p_2, \varepsilon) \cap \gamma_1 \neq \varnothing$. If $q \in C(p_2, \varepsilon) \cap \gamma_1$, then q lies in the interior of J. This means that the part of γ_2 between q and m_2 has to intersect J. But it does not intersect $[p_1 p_2]$ or $[p_2 m_1]$ (eventually it can touch $[p_1 p_2]$ or $[p_2 m_1]$); therefore it intersects γ_1. Thus we have shown that $B(p_2, p_3)$ and $B(p_3, p_1)$ have a common point, which completes the proof. ■

The proof of Lemma 2.4 implies the following.

Lemma 2.5 *The circumcenter of any equilateral triangle in a strictly convex Minkowski plane lies in the interior of this triangle.*

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Lemma 2.6 *Let p be a point in a strictly convex Minkowski plane $(\mathbb{X}, \|\cdot\|)$ different from the origin 0. Then the bisector of the points p and $-p$ intersects the circle $C(0, \|p\|)$ in exactly two points which are opposite points of this circle.*

Proof It is clear that $B(-p, p)$ intersects $C(0, \|p\|)$ in at least two points which are opposite points of that circle, *e.g.*, in x and $-x$. We will show that, besides x and $-x$, there are no further intersection points of $B(-p, p)$ and $C(0, \|p\|)$. We have that the bisector $B(-p, p)$ is contained in the double cone V of p and $-p$ with apex x; see [11] and [17, Proposition 13]. Let us consider that part of V (denoted by V^*) which lies in the half-plane bounded by $\langle -p p\rangle$ and containing x. Thus

$$V^* = \{x+\lambda(-p-x)+\mu(p-x) : \lambda, \mu \leq 0\} \cup \{x+\lambda(-p-x)+\mu(p-x) : \lambda, \mu \in (0, 1)\}.$$

We will show that neither

$$V^- = \{x \in \mathbb{X} : x + \lambda(-p - x) + \mu(p - x) : \lambda, \mu \leq 0\},$$

nor the set

$$V^+ = \{x \in \mathbb{X} : x + \lambda(-p - x) + \mu(p - x) : \lambda, \mu \in (0, 1)\}$$

(see Figure 3 below) contains points of $C(0, \|p\|)$ which are different from x. For any $y \in V^-$ we have

$$y = x + \lambda(-p - x) + \mu(p - x) \iff (1 - \lambda - \mu)x = y + (\lambda - \mu)p,$$

where $\lambda, \mu \leq 0$. Therefore

$$(1 - \lambda - \mu)\ \|x\| = \|y + (\lambda - \mu)\ p\| < \|y\| + |\lambda - \mu|\ \|p\|$$
$$\iff \begin{cases} \|x\| \leq (1 - 2\lambda)\ \|x\| < \|y\| & \text{if } \lambda - \mu \geq 0, \\ \|x\| \leq (1 - 2\mu)\ \|x\| < \|y\| & \text{if } \lambda - \mu < 0. \end{cases}$$

Hence any $y \in V^-$ is an exterior point with respect to $C(0, \|p\|)$. On the other hand, V^+ is a triangle inscribed to the strictly convex curve $C(0, \|p\|)$. Therefore V^+ also does not contain points of $C(0, \|p\|)$ (except for p, $-p$, and x). ■

Lemma 2.7 *In a Minkowski plane $(\mathbb{X}, \|\cdot\|)$, let there be given two circles $C(x_1, \lambda_1)$ and $C(x_2, \lambda_2)$ with $\lambda_1 \neq \lambda_2$. Then the homothety*

$$\varphi\colon x \longmapsto \frac{-\lambda_2 x_1 + \lambda_1 x_2}{\lambda_1} + \frac{\lambda_2}{\lambda_1} x$$

maps $C(x_1, \lambda_1)$ into $C(x_2, \lambda_2)$. The center of φ is the point

$$s = \frac{\lambda_2}{\lambda_2 - \lambda_1} x_1 - \frac{\lambda_1}{\lambda_2 - \lambda_1} x_2.$$

The proof of this lemma is immediate.

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3 Covering a Disc by Three Translates of the Unit Disc

Now we are going to use the results from Section 2 for the case $n = 3$ of the circle covering problem in strictly convex Minkowski planes. The diameters of the circles to be covered will be given in terms of side-lengths of inscribed equilateral triangles.

Theorem 3.1 *In a strictly convex normed plane let there be given an equilateral triangle $\triangle p_1 p_2 p_3$ of side-length 2. Then the circumradius of $\triangle p_1 p_2 p_3$ is > 1, and the circumdisc of $\triangle p_1 p_2 p_3$ can be covered by three translates of the unit discs.*

Proof If $C(q, \lambda)$ is the circumcircle of $\triangle p_1 p_2 p_3$ (note that according to Lemma 2.4 this circumcircle exists), then

$$\|p_1 - q\| + \|q - p_2\| > \|p_1 - p_2\| \iff 2\lambda > 2 \iff \lambda > 1.$$

We now show that if m_i is the midpoint of $[p_j p_k]$, where $\{i, j, k\} = \{1, 2, 3\}$, then

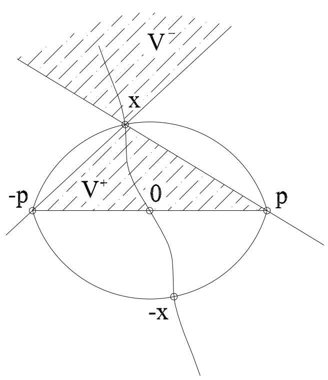

Figure 3

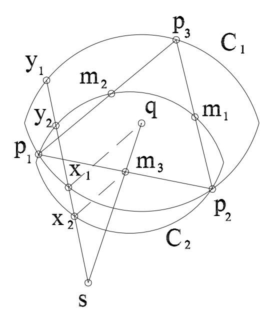

Figure 4

the discs $D(m_i, 1)$, $i = 1, 2, 3$, cover $D(q, \lambda)$. At first we check whether $m_j, m_k \in C(m_i, 1)$. Indeed,

$$\|m_i - m_j\| = \left\| \frac{p_j + p_k}{2} - \frac{p_k + p_i}{2} \right\| = \frac{1}{2}\|p_j - p_i\| = 1.$$

Thus we get that the discs $D(m_i, 1)$, $i = 1, 2, 3$, cover $\operatorname{conv}\{p_1, p_2, p_3\}$. In order to complete the proof, it remains to show that $D(m_3, 1)$ covers the arc $\widehat{p_1 p_2}$ (of course, we mean that arc which does not contain p_3), say; see Figure 4. Let us assume that $q \equiv 0$ and write $C_1 := C(0, \lambda)$ and $C_2 := C(m_3, 1)$. We consider the homothety φ mapping the circle C_1 into the circle C_2, *i.e.*, $\varphi\colon x \mapsto m_3 + 1/\lambda x$; see Lemma 2.7. Then the center of φ is $s = \frac{\lambda}{\lambda - 1} m_3$, which belongs to the opposite ray of $[m_3 0\rangle$, *i.e.*,

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s lies in the half-plane bounded by $\langle p_1 p_2 \rangle$ which does not contain p_3. We will prove that $s \notin D(m_3, 1)$ and $s \notin D(0, \lambda)$. Assume that $s \in D(m_3, 1)$. This is equivalent to

$$1 \geq \|m_3 - s\| = \|m_3 - \frac{\lambda}{\lambda - 1} m_3\| = \frac{1}{\lambda - 1}\|m_3\| = \frac{1}{2(\lambda - 1)}\|p_1 + p_2\|$$

$$\iff 2(\lambda - 1) \geq \|p_1 + p_2\| \iff 2\lambda \geq \|p_1 + p_2\| + 2.$$

The last inequality contradicts the triangle inequality referring to $\triangle p_1 p_2(-p_2)$. Assuming that $s \in D(0, \lambda)$, we get

$$\|s\| = \frac{\lambda}{\lambda - 1}\|m_1\| = \frac{\lambda}{2(\lambda - 1)}\|p_1 + p_2\| \leq \lambda \iff 2(\lambda - 1) \geq \|p_1 + p_2\|,$$

again a contradiction.

Furthermore, let x_1 belong to the arc $\widehat{p_1 p_2}$ of C_1 not containing p_3. If $\langle s x_1 \rangle \cap C_1 = \{x_1, y_1\}$, then x_1 is between s and y_1. This follows from the fact that s belongs to the same half-plane with respect to $\langle p_1 p_2 \rangle$ containing the arc $\widehat{p_1 p_2}$. If $\varphi(x_1) = x_2$, then $x_2 \in C_2$ and

$$x_2 = m_3 + \frac{1}{\lambda} x_1 \iff x_2 = \frac{\lambda - 1}{\lambda} s + \frac{1}{\lambda} x_1.$$

Since $\lambda > 1$, we state that the point x_2 is between s and x_1. Let $\langle x_1 x_2 \rangle \cap C_2 = \{x_2, y_2\}$. This means that x_2 is between s and y_2. Moreover, $y_2 = \varphi(y_1)$, equivalent to that y_2 is between s and y_1. Thus we have that the points s, x_2, x_1, y_2, y_1 are located on the line $\langle x_1 x_2 \rangle$ in this order or in the order s, x_2, y_2, x_1, y_1. But the second situation is impossible because the points x_1 and y_2 lie in different half-planes with respect to $\langle p_1 p_2 \rangle$. Therefore x_1 is between x_2 and y_2, equivalent to the fact that $D(m_3, 1)$ covers the arc $\widehat{p_1 p_3 p_2}$. ■

The next theorem gives the geometric meaning of the maximal radius $R_3(\mathcal{D})$ of all homothets of the unit disc $\mathcal{D}$ in a strictly convex normed plane that can be covered by three translates of $\mathcal{D}$.

Theorem 3.2 *If $(\mathbb{X}, \|\cdot\|)$ is a strictly convex normed plane with unit disc $\mathcal{D}$, then the quantity $R_3(\mathcal{D})$ is the maximal circumradius of equilateral triangles with side-length 2.*

Proof If λ is the maximal circumradius of all equilateral triangles of side-length 2, we will prove that the disc $D(0, \lambda + \varepsilon)$, where $\varepsilon > 0$, cannot be covered by three translates of the unit disc $\mathcal{D}$. Let $\triangle p_1 p_2 p_3$ be an equilateral triangle of side-length 2 inscribed in $D(0, \lambda)$. If φ is the homothety mapping $C(0, \lambda)$ into $C(0, \lambda + \varepsilon)$ and $\varphi(p_1, p_2, p_3) = (p_1', p_2', p_3')$, then $\triangle p_1' p_2' p_3'$ is equilateral and of side-length $2 + 2\varepsilon/\lambda$. Assume that $D(0, \lambda + \varepsilon)$ can be covered by the translates D_1, D_2, and D_3 of $\mathcal{D}$. If $p_1' \in D_1$, say, then $p_2', p_3' \notin D_1$, by Remark 2.2. If $p_2' \in D_2$, say, then $p_3' \notin D_2$. Therefore $p_3' \in D_3$. For $\{i, j, k\} = \{1, 2, 3\}$, let $\widehat{p_i' p_j'}$ be the circular arc of $C(0, \lambda + \varepsilon)$ between the points p_i' and p_j' which does not contain p_k'. Then Lemma 2.1 and Lemma 2.5 imply that any point of $\widehat{p_i' p_j'}$ does not belong to D_k. Let $\partial D_1 \cap C(0, \lambda + \varepsilon) = \{q_1, q_3\}$

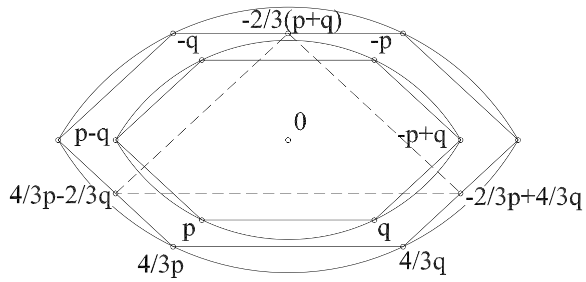

Figure 5.

and $q_1 \in \widehat{p_1'p_2'}$, $q_3 \in \widehat{p_1'p_3'}$. Note that if $q_1 \equiv q_3$, then $q_1 \equiv q_3 \equiv p_1$, which means that $D(0, \lambda + \varepsilon)$ cannot be covered by D_1, D_2, and D_3. By Lemma 2.3 we have that $q_1 \in D_2$ and $q_3 \in D_3$. Thus we obtain $\triangle q_1q_2q_3$ inscribed in $C(0, \lambda + \varepsilon)$ such that $q_1 \in \widehat{p_1'p_2'}$, $q_2 \in \widehat{p_2'p_3'}$, $q_3 \in \widehat{p_3'p_1'}$, $q_1 \neq p_1$, $q_1 \neq p_2$, $q_2 \neq p_2$, $q_2 \neq p_3$, $q_3 \neq p_3$, $q_3 \neq p_1$, and the sides of $\triangle q_1q_2q_3$ are of length ≤ 2. Moreover, it is easy to see that the interior of $\operatorname{conv}\{q_1, q_2, q_3\}$ contains the origin 0. Let $\triangle q_1q_2q_3$ be positively oriented, say. Construct an equilateral triangle $\triangle u_1u_2u_3$ of side-length 2, which is positively oriented and such that $\langle u_1u_2\rangle$ is parallel to $\langle q_1q_2\rangle$. According to [17, Proposition 33], for a given segment $[u_1u_2]$ there exists exactly one such triangle. Let $C(0, \mu)$ be a translate of the circumcircle of $\triangle u_1u_2u_3$. Then $\mu \leq \lambda$. If v_1, v_2, v_3 are the images of u_1, u_2, u_3 with respect to this translation, let v_1', v_2', v_3' be the images of v_1, v_2, v_3 with respect to the homothety mapping $C(0, \mu)$ into $C(0, \lambda + \varepsilon)$. Thus we obtain that $\triangle v_1'v_2'v_3'$ is an equilateral triangle inscribed in $C(0, \lambda + \varepsilon)$ and of side-length $2(\lambda + \varepsilon)/\mu > 2$, where the side $[v_1'v_2']$ is parallel to $[q_1q_2]$. Therefore the side $[v_1'v_2']$ lies in the open half-plane with respect to $\langle q_1q_2\rangle$ that contains the origin 0. The third vertex v_3' of $\triangle v_1'v_2'v_3'$ belongs either to the arc $\widehat{q_1q_3}$ or to the arc $\widehat{q_2q_3}$. But both these cases contradict the monotonicity lemma. ■

The next proposition gives an upper bound on $R_3(\mathcal{D})$, where $\mathcal{D}$ is the unit disc in a strictly convex normed plane. This upper bound strengthens the second inequality in (2.4) for the case that B is centrally symmetric and strictly convex.

Proposition 3.3 *In a normed plane $(\mathbb{X}, \|\cdot\|)$ with unit disc $\mathcal{D}$ we have $R_3(\mathcal{D}) \leq 4/3$ if $\mathcal{D}$ is strictly convex, and $R_3(\mathcal{D}) = 4/3$ if $\partial\mathcal{D}$ is an affine regular hexagon.*

Proof Let $\pm p, \pm q, \pm(p + q)$ be the vertices of a hexagon that is regular in the norm (*i.e.,* an affine regular hexagon with sides of the same Minkowskian length) and inscribed in the unit circle $\mathcal{C} = \partial\mathcal{D}$. Note that this is possible; see, [17, §4]. The triangle with vertices $\frac{4}{3}p - \frac{2}{3}q$, $-\frac{2}{3}p + \frac{4}{3}q$, and $-\frac{2}{3}(p + q)$ is equilateral with side-length 2 and inscribed in $\frac{4}{3}\mathcal{C}$; see Figure 5. Therefore, if $\mathcal{C}$ is strictly convex, we have $R_3(\mathcal{D}) \leq \frac{4}{3}$, and $R_3(\mathcal{D}) = \frac{4}{3}$ holds if $\mathcal{C}$ is an affine regular hexagon. ■

4 Covering a Disc by Four Translates of the Unit Disc

In case of four covering circles we will see that, somehow analogous to the considerations in the former section, regular quadrangles inscribed to a circle will play an essential role.

Lemma 4.1 *In a strictly convex Minkowski plane let there be given two points x_1 and x_3 that are opposite points of the unit circle $\mathcal{C}$; see Figure 6. If $B(x_1, x_3) \cap \mathcal{C} = \{x_2, x_4\}$, then the intersection points of $C(x_i, 1)$ and $C(x_{i+1}, 1)$ (with $i = 1, 2, 3, 4$ and $x_5 \equiv x_1$), which are different from 0, lie on the same circle C with radius between 1 and 2.*

Proof Note that according to Lemma 2.6 the intersection of $B(x_1, x_3)$ and $\mathcal{C}$ consists of exactly two points, which are opposite in $\mathcal{C}$. The monotonicity lemma implies $\|x_i - x_{i+1}\| < 2$, $i = 1, \dots, 4$. Hence $C(x_i, 1)$ and $C(x_{i+1}, 1)$ have exactly two points in common. Clearly, the origin 0 is one of them, and we denote by p_i the other intersection point. Thus, by (1.1) we obtain $x_i + x_{i+1} = p_i$. Since x_1 and x_3 are opposite points, as are x_2 and x_4, it follows that $\|p_i\| = \|x_{i+1} - x_{i+2}\|$ (note that $x_6 \equiv x_2$). But we have $x_2, x_4 \in B(x_1, x_3)$, *i.e.*,

$$\|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_4\| = \|x_4 - x_1\| = \lambda,$$

which is equivalent to $p_i \in C(0, \lambda)$. Moreover, with respect to $\triangle x_1 x_2 x_3$ we have

$$2 = \|x_1 - x_3\| < \|x_1 - x_2\| + \|x_2 - x_3\| \Longleftrightarrow 1 < \lambda.$$

On the other hand, applying the fact that for any convex quadrangle (and thus also for $x_1 x_2 x_3 x_4$) the sum of lengths of its diagonals is at least the sum of lengths of two opposite sides (see [17, Proposition 7]), we obtain

$$\|x_1 - x_3\| + \|x_2 - x_4\| > \|x_1 - x_2\| + \|x_4 - x_3\| \Longleftrightarrow 2 > \lambda. \qquad \blacksquare$$

Remark 4.2. The construction of the points x_1, x_2, x_3, x_4 in Lemma 4.1 shows that in every circle of a strictly convex Minkowski plane a regular quadrangle can be inscribed.

Remark 4.3. It is easy to check that the points p_1, p_2, p_3, and p_4 (see the proof of Lemma 4.1) form a parallelogram all of whose sides are of Minkowskian length 2 and whose two diagonals have the same length. The proof of Lemma 4.1 also implies that for any given direction such a parallelogram with two sides parallel to this direction can be constructed.

Theorem 4.4 *If, in a strictly convex normed plane, $C(x_i, 1)$, $i = 1, 2, 3, 4$, and C are determined as in Lemma 4.1, then $C(x_i, 1)$ with $i = 1, 2, 3, 4$ is a covering of C.*

Proof In view of the constructions of $C(x_i, 1)$, $i = 1, 2, 3, 4$, it is enough to prove that, *e.g.*, $D(x_1, 1)$ covers the circular arc $\widehat{p_1 p_4}$ (meaning that arc which does not contain the points p_2 and p_3). This can be verified in a way quite similar to the proof of Theorem 3.1, using the homothety that maps $C(0, \lambda)$ into $C(x_1, 1)$. $\blacksquare$

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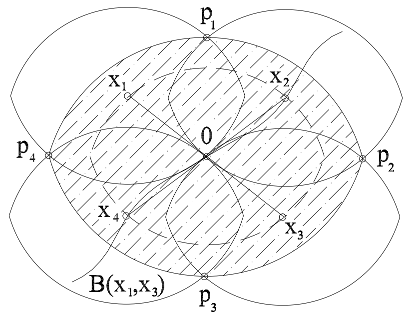

Figure 6.

Based on Remark 4.3 the next theorem can be proved in the same way as Theorem 3.2.

Theorem 4.5 *In a strictly convex normed plane with unit disc $\mathcal{D}$, the quantity $R_4(\mathcal{D})$ is the maximal circumradius of all parallelograms whose four sides are of Minkowskian length* 2*, and whose two diagonals have the same length.*

5 Covering a Disc by Seven Translates of the Unit Disc

Levi [14] proved that every plane convex body B can be covered by 7 homothetical copies of ratio $\frac{1}{2}$; see also [13]. In this section we state that, as in the Euclidean situation (for which we refer to [10]), this is the best such covering if B is a centrally symmetric, strictly convex body.

For the proof of our final theorem we need the next lemma.

Lemma 5.1 *In a strictly convex normed plane, a hexagon of side-length $< \mu$ cannot be inscribed in a circle of radius μ.*

Proof Assume that there exists a hexagon $p_1 p_2 \cdots p_6$ inscribed in the circle $C(p, \mu)$ such that $\|p_i - p_{i+1}\| < \mu$ with $i = 1, \dots, 6$ and $p_7 = p_1$. In $C(p, \mu)$ we can inscribe a regular hexagon $q_1 q_2 \cdots q_6$ of side-lengths μ such that $q_1 \equiv p_1$. Denote by $\widehat{p_i p_{i+1}}$ the circular arc which does not contain the remaining vertices of $p_1 p_2 \cdots p_6$. Then the monotonicity lemma implies $q_2 \in \widehat{p_2 p_3}$ and $q_6 \in \widehat{p_5 p_6}$ such that $q_2 \neq p_2$ and $q_6 \neq p_6$. Clearly, it is impossible that $q_3 \in \widehat{p_2 p_3}$ and $q_5 \in \widehat{p_5 p_6}$. Therefore the points q_3, q_4, and q_5 belong to $\widehat{p_5 p_4} \cup \widehat{p_4 p_3}$. This means that at least two of them lie on the same arc, which is impossible. ■

Theorem 5.2 *A disc of radius > 2 cannot be covered by seven unit discs.*

Proof Assume that the disc $D(0, \lambda)$, where $\lambda > 2$, is covered by seven unit discs $D_i = D(x_i, 1)$. Then at most six of them can have a common point with $C(0, \lambda) =$

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$\partial D(0,\lambda)$. Indeed, let $0 \in D_1$, say. Assume that $D_1 \cap C(0,\lambda) \neq \varnothing$ and $y_1 \in D_1 \cap C(0,\lambda)$. Since $y_1 \in C(0,\lambda)$, we get $\|y_1\| = \lambda > 2$. But $y_1 \in D_1$ implies $\|y_1\| \leq 2$, a contradiction. Thus, by Lemma 2.3 we get that an n-gon with $n \leq 6$ can be inscribed in $C(0,\lambda)$. The case $n = 6$ is contradictory to Lemma 5.1. Hence $C(0,2)$ cannot be covered by six unit discs. This means that $C(0,2)$ also cannot be covered by $n < 6$ unit discs. ■

From Theorem 5.2 and (2.6) we immediately get the following statement.

Corollary 5.3 *In a strictly convex normed plane with unit disc $\mathcal{D}$ the maximal radius $R_7(\mathcal{D})$ of all homothets of $\mathcal{D}$ that can be covered by 7 translates of $\mathcal{D}$ is 2.*

References

[1] E. Asplund and B. Grünbaum, *On the geometry of Minkowski planes.* Enseignement Math. **6**(1960), 299–306.

[2] K. Bezdek, *Über einige Kreisüberdeckungen.* Beiträge Algebra Geom. **14**(1983), 7–13.

[3] ———, *Über einige optimale Konfigurationen von Kreisen.* Ann. Univ. Sci. Bedapest Eötvös Sect. Math. **27**(1984), 143–151.

[4] V. Boltyanski, H. Martini, and P. S. Soltan, *Excursions into Combinatorial Geometry.* Springer-Verlag, Berlin, 1996.

[5] K. Böröczky Jr., *Finite Packing and Covering.* Cambridge Tracts in Mathematics 154, Cambridge University Press, 2004.

[6] G. Fejes Tóth, *Packing and covering.* In: Handbook of Discrete and Computational Geometry. CRC Press, Ser. Discrete Math. Appl., Boca Raton, FL, 1997, Ch. 2.

[7] ———, *Thinnest covering of a circle by eight, nine or ten circles.* In: Combinatorial and Computational Geometry. Math. Sci. Res. Inst. Publ. 52, Cambridge University Press, 2005, pp. 361–376.

[8] B. Grünbaum, *On a conjecture of H. Hadwiger.* Pacific J. Math. **11**(1961), 215–219.

[9] H. Hadwiger, *Über Treffanzahlen bei translationsgleichen Eikörpern.* Arch. Math. **8**(1957), 212–213.

[10] A. Heppes, *Covering a planar domain with sets of small diameters.* Period. Math. Hungar. **53**(2006), no. 1-2, 157–168.

[11] J. R. Holub, *Rotundity, orthogonality, and characterizations of inner product spaces.* Bull. Amer. Math. Soc. **81**(1975), 1087–1089.

[12] M. Lassak, *Covering a plane convex body by four homothetical copies with the smallest positive ratio.* Geom. Dedicata **21**(1986), no. 2, 157-167.

[13] ———, *Covering plane convex bodies with smaller homothetical copies.* In: Intuitive Geometry. Colloq. Math. Soc. János Bolyai 48. North-Holland, Amsterdam, 1987, pp. 331–337.

[14] F. W. Levi, *Ein geometrisches Überdeckungsproblem.* Arch. Math. **5**(1954), 476–478.

[15] H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces–a survey. II.* Expo. Math. **22**(2004), no. 2, 93–144.

[16] ———, *Antinorms and Radon curves.* Aequationes Math. **72**(2006), no. 1-2, 110–138.

[17] H. Martini, K. J. Swanepoel, and G. Weiss, *The geometry of Minkowski spaces–a survey. I.* Expo. Math. **19**(2001), no. 2, 97–142.

[18] A. C. Thompson, *Minkowski Geometry.* Encyclopedia of Mathematics and its Applications 63, Cambridge University Press, 1996.

[19] R. Webster, *Convexity.* Oxford University Press, New York, 1994.

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