DISCRETE GROUPS WHOSE MULTIPLIER
REPRESENTATIONS ARE TYPE *I*

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Abstract

We determine necessary and sufficient conditions for the multiplier representations of a discrete group to be type *I*. This result extends the corresponding result for ordinary representation given by Kaniuth in [4].

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1. Introduction

Multiplier representations arise in applying the Mackey analysis of ordinary representations of a group G with a regularly embedded normal subgroup N. Specifically, to each factor representation of G which restricts to an orbit through λ in $N^\smallfrown$, there corresponds a unique σ-representation of the little group H_λ/N, with the same commutant, where H_λ is the stabilizer of λ in $N^\smallfrown$ and the multiplier σ is the Mackey obstruction to extending λ to H_λ as an ordinary representation ([1], page 62). Moreover, if G is discrete, then the induced representation $\lambda\!\uparrow_N^G$ has the same commutant as the regular σ-representation of H_λ/N ([5], Theorem 7).

Suppose, then, that $\omega\colon G \times G \to T$ is a normalized multiplier on the discrete group G and let $V(G)$ denote the von Neumann algebra generated by the regular ω-representation of G. We say that (G, ω) is type *I* if all ω-representations of G are type *I*. The main result to be proved in this paper is the following.

THEOREM 1.1. *Let G be a discrete group and ω a normalized multiplier on G, then the following are equivalent.*

(i) *$V(G)$ is type I,*

(ii) *(G, ω) is type I,*

(iii) *G has an abelian subgroup A of finite index $[G : A]$ in G such that $\omega(x, y) = \omega(y, x)$ for all $x, y \in A$.*

An interesting consequence of this theorem is that the central extension G^{ω} of G (where G is discrete and G^{ω} is given any locally compact group topology) is type *I* if (G, ω) is type *I*. It should be noted that this is not a simple consequence of the fact that the ω-representations of G give rise to (ordinary) representations of the central extension G^{ω}, since not all representations of G^{ω} are obtained this way. However, looking at the central extension is a successful approach and we use it in conjunction with Robertson's characterization of Moore groups [7].

In addition to Theorem 1.1, we shall give an explicit description of the maximal type *I* central projection in $V(G)$ in terms of the properties of G, and an application to induced representations.

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2. Notation and some elementary facts

Throughout this paper we shall be interested in pairs (G, ω) where G is a discrete group and ω a normalized multiplier on G. For the definitions of multiplier and multiplier representations see Kleppner [5]. A multiplier is normalized if $\omega(x, y)^{-1} = \omega(y^{-1}, x^{-1})$ for all $x, y \in G$. Denote by ρ the left regular ω^{-1}-representation of G on $L^2(G)$ given by $(\rho(x)f)(y) = \omega(x^{-1}, y)f(x^{-1}y)$, $x, y \in G, f \in L^2(G)$, and by $V(G) = V(G, \omega)$ the von Neumann algebra generated by $\rho(G)$. If $A(G) = A(G, \omega)$ denotes the twisted group algebra of (G, ω)– the algebra of complex finitely supported measures on G with multiplication $(ab)_x = \sum_{y \in G} a_y b_{y^{-1}x} \omega(y^{-1}, x)$–then each ω^{-1}-representation π of G extends naturally to an algebra representation $a \to \sum_{g \in G} a_g \pi(g)$ of $A(G)$ which, when no confusion may arise, we shall denote by the same letter. With this notation, we see that $V(G)$ is the weak closure of $\rho(A(G))$ in $B(L^2(G))$, the algebra of bounded linear operators on $L^2(G)$.

If φ_x denotes the characteristic function at the point x in G, then for $a \in B(L^2(G))$, $x, y \in G$, we let $a_{x,y} = (a(\varphi_y), \varphi_x)$ and $a_x = a_{x,e}$, where $(\cdot, \cdot)$

denotes the canonical scalar product on $L^2(G)$. Most of the following elementary facts may be found in Kleppner [5]. Let $a \in B(L^2(G))$, then

2.1. $a \in V(G)$ if, and only if $a_{x,y} = \omega(x, y^{-1})a_{xy^{-1}}$ for all $x, y \in G$.

2.2. $a \in CV(G)$, the centre of $V(G)$ if, and only if $\omega(x, y)a_{y^{-1}xy} = a_x\omega(y, y^{-1}xy)$ for all $x, y \in G$.

2.3. For each $a \in V(G)$, the map $g \to a_g$ is in $L^2(G)$ and the map $a \to a_e$ is a finite faithful normal trace on $V(G)^+$ –the set of positive hermitian elements in $V(G)$.

2.4. If $a \in V(G)$, then $a = \Sigma_{g \in G} a_g\rho(g)$ in the sense of weak operator convergence. This expression is unique, that is if $a = \Sigma_{g \in G} a'_g\rho(g)$, then $a'_g = a_g$ all $g \in G$. We also have

$$(ab)_x = \sum_{y \in G} a_y b_{y^{-1}x}\omega(y^{-1}, x).$$

2.5. Let H be a subgroup of G, then there exists a normal *-isomorphism of $V(H) = V(H, \omega|_{H \times H})$ onto the von Neumann algebra $\{a \in V(G): a_g = 0$ if $g \notin H\}$: $a \to \Sigma_{h \in H} a_h\rho(h)$, ρ being the regular ω^{-1}-representation of G.

In view of 2.5, since we are interested only in the type structure of $V(G)$, we shall henceforth always regard $V(H)$ as a sub von Neumann algebra of $V(G)$ whenever H is a subgroup of G.

3. The ω-finite class group of G

Let $C(x) = C_G(x)$ denote the centralizer of x in G and $C_\omega(x) = C_{\omega,G}(x)$ the ω-centralizer of x in G: $C_\omega(x) = \{g \in C(x): \omega(g, x) = \omega(x, g)\}$.

LEMMA 3.1. (i) $C_\omega(x) = \{g \in G: \rho(x)\rho(g) = \rho(g)\rho(x)\}$,
(ii) $C_\omega(x)$ is a subgroup of G.

PROOF. If $g \in C_\omega(x)$, then

$$\rho(g)\rho(x)\omega(g, x) = \rho(gx) = \rho(xg) = \rho(x)\rho(g)\omega(x, g)$$

implies $\rho(g)\rho(x) = \rho(x)\rho(g)$. Conversely, suppose $g \in G$ satisfies $\rho(x)\rho(g) = \rho(g)\rho(x)$, then

$$(\rho(gx)\varphi_e)(y)/\omega(g, x) = (\rho(xg)\varphi_e)(y)/\omega(x, g)$$

all $y \in G$, so by letting $y = gx$ and simplifying,

$$\omega(g, x)^{-1} = \omega(g^{-1}x^{-1}, gx)\varphi_e(g^{-1}x^{-1}y)/\omega(x, g),$$

hence $\varphi_e(g^{-1}x^{-1}y)$ cannot be zero, but the only way this can occur is if $xg = y = gx$. Part (i) now follows. The proof of (ii) follows easily from (i).

Denote by $\Delta = \Delta_G$ the ω-finite class group of (G, ω) defined by $\Delta = \{x \in G: [G : C_\omega(x)] < \infty\}$. To see that Δ is indeed a subgroup, we need only note that $C_\omega(x^{-1}y) \supseteq C_\omega(x) \cap C_\omega(y)$ and

$$[G : C_\omega(x) \cap C_\omega(y)] \leqslant [G : C_\omega(x)][G : C_\omega(y)].$$

The group Δ has the property that all its conjugacy classes are finite, such groups are called *FC* (finite class) groups.

The following theorem points out the significance of Δ.

THEOREM 3.2. *Let G be a discrete group, ω a normalized multiplier and Δ the ω-finite class group of G. If $V(G)$ has a non-zero type I part, then*

(i) $[G : \Delta] < \infty$, *and*

(ii) $|\Delta'| < \infty$ (Δ' *denotes the commutator subgroup of Δ).*

The idea of the proof of (i) comes from Smith ([9], Theorem 9.4) and the proof of (ii) is similar to that of Smith ([10], Theorem 1). We need the following lemma.

LEMMA 3.3. *Let $CV(G)$ denote the centre of $V(G)$, then $CV(G) \subseteq V(\Delta)$.*

PROOF. Let $a \in CV(G)$ and $x \in G$ such that $a_x \neq 0$, then by 2.5 it suffices to show that $x \in \Delta$. From 2.2, $\omega(x, y)a_{y^{-1}xy} = a_x\omega(y, y^{-1}xy)$, thus $C(x) = C_\omega(x)$, but by 2.3, $[G : C(x)] < \infty$, hence $x \in \Delta$.

PROOF OF THEOREM 3.2(i). Let e_n be a non zero central projection in $V(G)$ such that $e_n V(G)$ is type I_n. From 2.3 we see that n must be finite. Since $e_n V(G)$ is a full matrix algebra over its centre $Ce_n V(G)$, it is of dimemsion at most n^2 over $Ce_n V(G) \subseteq e_n CV(G) \subseteq CV(G) \subseteq V(\Delta)$. Hence if $g_1, \ldots, g_{n^2+1}$ are $n^2 + 1$ elements of G belonging to distinct right cosets of Δ, then there exist elements $C_1, \ldots, C_{n^2+1} \in V(\Delta)$ such that $\sum_{i=1}^{n^2+1} C_i(e_n\rho(g_i)) = \sum_{i=1}^{n^2+1}(C_i e_n)\rho(g_i) = 0$ with not all $(C_i e_n)\rho(g_i) = 0$, but $C_i e_n \in V(\Delta)$, so this cannot happen because the sum $V(\Delta)\rho(g_1) \oplus \cdots \oplus V(\Delta)\rho(g_{n^2+1})$ is a direct sum of vector spaces. This shows that $[G : \Delta] \leqslant n^2 < \infty$.

LEMMA 3.4. *Suppose G is FC, H a subgroup of G and e a projection in $CV(G)$ such that both $eV(G)$ and $eV(H)$ are type I_n. If $C = \{x \in G: \rho(h)\rho(x) = \rho(x)\rho(h)$ all $h \in H\} = \bigcap_{h \in H} C_{\omega,G}(h)$ is the ω-centralizer of H in G, then $e(\rho(\sigma)\rho(\tau) - \rho(\tau)\rho(\sigma)) = 0$ for every $\sigma \in C_\omega$, $\tau \in G$. Consequently $[C_\omega, G]$ (the subgroup generated by elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$, $\sigma \in C_\omega$, $\tau \in G$) is finite.*

This result is proved by making appropriate and obvious modifications to Smith [10], Lemma 2.

PROOF OF THEOREM 3.2(ii). It is well known that a von Neumann algebra contains only type I_k summands for $k \leqslant n$ if, and only if it satisfies the standard polynomial in $2n$ variables (for details see for example [11], page 213). Suppose $V(G)$ has a non-zero type I part, then there exists a central projection $e_n \neq 0$ such that $e_n V(G)$ is type I_n. By the above remark $e_n V(G)$, and hence $e_n V(\Delta)$ satisfy the symmetric polynomial of degree $2n$; consequently $e_n V(\Delta)$ is type I.

Now let e_n be a non-zero central projection in $V(\Delta)$ such that $e_n V(\Delta)$ is type I_n. As in the proof of Smith [10],Theorem 1, we can find a finitely generated normal subgroup H of Δ such that $e_n V(H)$ has a non-zero type I_n summand $eV(H)$ for some $e \in Ce_n V(H)$. We wish to show that $e \in CV(\Delta)$. Because H is normal in Δ, for any $\sigma \in \Delta$, the automorphism $V(H) \to V(H)$: $a \to \rho(\sigma^{-1})a\rho(\sigma)$ leaves the type I_n summand $eV(H)$ fixed, that is $\rho(\sigma^{-1})e\rho(\sigma)V(H) = eV(H)$ and so by the uniqueness of e, $\rho(\sigma^{-1})e\rho(\sigma) = e$. It follows that $e \in CV(\Delta)$. Since $e \leqslant e_n$, $eV(\Delta)$ is clearly also type I_n, and thus by Lemma 3.4, $|(C_\omega)'| < \infty$ where $C_\omega = C_{\omega,\Delta}(H)$. Since H is finitely generated, $[\Delta : C_\omega] < \infty$ and thus the result $\Delta' < \infty$ follows by [6], Lemma 4.1.

4. The type I part of $V(G)$

For any discrete group G, we denote by G_0 the von Neumann kernel of G, that is the intersection of the kernels of all finite dimensional (ordinary) representations of G. If H is a subgroup of finite index in G, then $G_0 = H_0$ (see [3], Satz 7.2.2).

LEMMA 4.1. *Let G be a discrete group and H a subgroup of finite index in G such that $|H'| < \infty$, then there exists a subgroup K of G such that $[G : K] < \infty$ and $K' = G_0 = \cap \{L' : [G : L] < \infty\}$.*

PROOF. If $[G : L] < \infty$ then, since the characters of L/L' separate points, we have $G_0 = L_0 \subseteq L'$, hence $G_0 \subseteq \cap \{L' : [G : L] < \infty\}$. Since $|H'| < \infty$ and H/H_0 is maximally almost periodic, by Robertson's characterization of Moore groups ([7], Theorem 1), H/H_0 is type I, so by Kaniuth [4], Satz 2, H has a subgroup K containing H_0, of finite index in H such that K/H_0 is abelian. Since $[G : K] < \infty$ and $K' \subseteq H_0 = G_0$, the result follows.

Now let G be a discrete group and ω a normalized multiplier for G. We denote by G^ω the discrete group extension of T by G which is the set $T \times G$ with multiplication $(s, x)(t, y) = (st\omega(x, y), xy)$, $(s, x), (t, y) \in G^\omega$. Note that G^ω is discrete, it is not, as is usual, endowed with the Weil topology. Whenever H is a subgroup of G, we identify H^ω in the obvious way with a subgroup of G^ω. The following facts are worth noting: (i) $C_{G^\omega}(t, x) = (C_\omega(x))^\omega$ for all $(t, x) \in G^\omega$, (ii) Δ^ω is the finite class group of G^ω, (iii) $[G : H] = [G^\omega : H^\omega]$ for all subgroups H of G.

The next lemma is the key step to our proofs of the subsequent results.

LEMMA 4.2. *If there exists a subgroup H of G such that $[G : H] < \infty$, $|H'| < \infty$ and $\omega^n|_{H \times H}$ is trivial for some n, then there exists a subgroup K of G such that $[G : K] < \infty$, $G_0 = K'$ and $(G^\omega)_0 = (K^\omega)$.*

PROOF. Because of Lemma 4.1 we can assume without loss of generality that $H' = G_0$. For some map $\gamma: H \to T$, $\omega^n(x, y) = \gamma(x)\gamma(y)/\gamma(xy)$, all $x, y \in H$. An easy calculation shows that

$$(H^\omega)' = \left\{ (\omega(x, y)\omega(x^{-1}, y^{-1})\omega(xy, x^{-1}y^{-1}), xyx^{-1}y^{-1}): x, y \in H \right\}.$$

Since $[\omega(x, y)\omega(x^{-1}, y^{-1})\omega(xy, x^{-1}y^{-1})]^n = \gamma(xyx^{-1}y^{-1})^{-1}$, $(H^\omega)' < \infty$. By Lemma 4.1, G^ω has a subgroup M such that $[G^\omega : M] < \infty$ and $M' = (G^\omega)_0$. Let L be the image of the projection $M \to G$: $(\lambda, x) \to x$. L is a subgroup of G with the property $M \subseteq L^\omega$, hence $[G^\omega : L^\omega] < \infty$; furthermore $(L^\omega)' = M'$ and thus $K = L \cap H$ has the desired properties.

THEOREM 4.3. *Let G be a discrete group with normalized multiplier ω and let e (resp. e_n, $n = 1, 2, \ldots, n \neq \infty$) be the maximal type I (resp. type I_n) central projections in $V(G)$. By 2.3, $V(G)$ is a finite von Neumann algebra hence $\sum_{n < \infty} e_n = e$. The following are equivalent.*

(a) $e \neq 0$,

(b) *there exists a subgroup H of G such that $[G : H] < \infty$, $|H'| < \infty$ and $\omega|_{H \times H}$ is trivial.*

Suppose $e \neq 0$, then there exists a 1 dimensional ω-representation γ of G_0 such that

(i) $\pi|_{G_0} = \dim(\pi) \cdot \gamma$, *for all finite dimensional ω-representations π of G, where $\dim(\pi)$ denotes the dimension of π, and*

(ii) $e = \frac{1}{|G_0|}\sum_{g \in G_0} \gamma(g)\rho(g)$.

PROOF. Suppose $e \neq 0$. Since $e_n \neq 0$ for some integer n, an irreducible algebra representation τ of $e_n V(G)$ will give rise to a finite dimensional ω-representation

π: $g \to \tau(e_n\rho(g^{-1}))^*$ of G, where A^* denotes the adjoint of A as an operator on the dual space of the Hilbert space on which A acts.

After taking determinants, $\omega^n(x, y)$ det $\pi(xy) =$ det $\pi(x)$ det $\pi(y)$, we see that ω^n is trivial, so by Theorem 3.2 and Lemma 4.2, there exists a group K such that $[G : K] < \infty$, $G_0 = K' < \infty$ and $(G^\omega)_0 = (K^\omega)'$. Let π be an irreducible finite dimensional ω-representation of K, then π^0: $(\lambda, x) \to \lambda\pi(x)$ is a finite dimensional representation of K^ω, so

$$\pi^0((1, x)(1, y)(1, x)^{-1}(1, y)^{-1}) = \pi(x)\pi(y)\pi(x)^{-1}\pi(y)^{-1} = I \quad \text{all } x, y \in K,$$

but π is irreducible so it is one dimensional, consequently $\omega|_{K \times K}$ is trivial. This proves (a) implies (b).

Conversely, suppose (b) is true. Since $\omega|_{H \times H}$ is trivial, H has a 1 dimensional ω-representation, hence by inducing we see that G possesses a finite dimensional ω-representation. Let π be such a representation. Lemma 4.2 is applicable, so as in the preceding paragraph, $\pi(x)\pi(y) = \pi(y)\pi(x)$ all $x, y \in K$, and $G_0 = K'$, where K is the subgroup we obtain from Lemma 4.2, consequently

$$\pi(xyx^{-1}y^{-1}) = \omega(y, x)\omega(y^{-1}, x^{-1})\omega(yx, y^{-1}x^{-1}).I.$$

Since the left hand side of this expression is independent of the way we express $xyx^{-1}y^{-1}$ as a commutator, and since the right hand side depends only on the dimension of π, the function

$$\gamma: G_0 \to T: xyx^{-1}y^{-1} \to \omega(y, x)\omega(y^{-1}, x^{-1})\omega(yx, y^{-1}x^{-1}), \qquad x, y \in K,$$

extends to a well defined ω-representation of G_0 that satisfies (i).

To complete the proof of this theorem, we must show that our current assumptions lead to the statements (a) and (ii). Let $f = |G_0|^{-1}\Sigma_{g \in G_0} \gamma(g)\rho(g)$. By 2.2, f is central in $V(G)$ and since $g \to \gamma(g)\rho(g)$ is an ordinary representation of G_0, $f^2 = f$. Now, using the relation $\pi(x)\pi(y) = \pi(y)\pi(x)$ all $x, y \in K$, and the fact that $G_0 = K'$, one can show that $fV(K)$ is abelian. If we write $fV(G)$ as the module direct sum $fV(K)\rho(g_1) \oplus \cdots \oplus fV(K)\rho(g_k)$ for some set of coset representatives $g_1, \ldots, g_k$ modulo K, then (as in Smith [10], page 404) $fV(G)$ is a matrix algebra over the abelian algebra $fV(K)$ and is thus type I ([11], page 213). This proves (a).

By looking at the irreducible algebra representations of $e_n V(G)$, whenever this is not zero, we see that $eV(G)$ has enough finite dimensional representations to separate the points of $eV(G)$. But if π is a non-zero finite dimensional representation of $eV(G)$, $g \to \pi(\rho(g^{-1}))^*$ is a finite dimensional ω-representation of G, hence by (i), $\pi(\rho(g)\gamma(g)) = I$ all $g \in G_0$, hence $\pi(f) = |G_0|^{-1}\Sigma_{g \in G_0} \pi(\rho(g)\gamma(g)) = I$, from which we obtain $\pi(e - f) = \pi(e) - \pi(f) = 0$, and since π was arbitrary, we conclude that $e = f$. This completes the proof of Theorem 4.3.

Note that for ω trivial this theorem reduces to results due to Formanek [2] and Schlichting [8]. As a corollary, we can now prove Theorem 1.1.

PROOF OF THEOREM 1.1. Suppose (i) is true, then by the proof of Theorem 4.3, there exists a group K such that $[G : K] < \infty$, $K' = G_0$ and $\omega|_{K \times K}$ trivial. Since the maximal type I central projection of $V(G)$ is the identity, by Theorem 4.3, $I = |G_0|^{-1}\Sigma_{g \in G_0} \gamma(g)\rho(g)$, hence by uniqueness (see 2.4), $G_0 = \{e\}$ and thus K is abelian. From $\omega|_{K \times K}$ trivial, we can now conclude that $\omega(x, y) = \omega(y, x)$ all $x, y \in K$. This proves (i) implies (iii).

Suppose (iii) is true. Let H be a subgroup as prescribed in (iii) and π an ω^{-1}-representation of G. If $g_1, \ldots, g_n$ is a set of coset representatives modulo H, then
$$A(G) = A(H)g_1 \oplus \cdots \oplus A(H)g_n$$
where the direct sum is a module direct sum (for a definition of $A(G)$ see Section 2). Hence $A(G)$ is a matrix algebra over the abelian algebra $A(H)$ (see Smith [10], page 404) and thus satisfies a polynomial identity. Since the von Neumann algebra generated by π is the weak closure of $\pi(A(G))$, it too satisfies a polynomial identity and thus (see [11], page 213) the von Neumann algebra generated by π is type I. This proves (iii) implies (ii).

The implication (ii) implies (i) is trivial.

By doing some extra calculations, one can prove that the abelian subgroup in Theorem 1.1 may be taken to be $Z_\omega(\Delta) = \{g \in Z(\Delta): \omega(g, x) = \omega(x, g)$ all $x \in \Delta\}$, where $Z(\Delta)$ denotes the centre of Δ.

Keeping this in mind, we have the following examples.

EXAMPLE 4.4. Let G be abelian and discrete with normalized multiplier ω.

If $Z_\omega(G) = \{g \in G: \omega(x, g) = \omega(g, x)$ all $x \in G\}$ has finite index in G, then clearly (G, ω) is type I. Conversely, if (G, ω) is type I, then for each $x \in G$, $C_\omega(x)$ contains the group $Z_\omega(\Delta)$ which according to Theorem 1.1, has finite index in G, thus $\Delta = G$ and $[G : Z_\omega(G)] < \infty$.

This criterion is well known and applies to pairs such as $G = Z \times Z$, $\omega((m, n), (m', n')) = e^{2\pi i\alpha(mn' - m'n)}$ for which (G, ω) is type I if, and only if α is rational.

EXAMPLE 4.5. Let p be a prime and let G be the group with infinitely many generators $b, a_1, a_2, \ldots$ and the defining relations
$$b^p = a_1^p = a_2^p = \ldots = a_n^p = \ldots = 1,$$
$$a_i b = ba_i, \quad i = 1, 2, \ldots,$$
$$a_{i+k}a_i = ba_i a_{i+k}, \quad i, k = 1, 2, \ldots.$$

It is clear that the commutator subgroup of G coincides with the centre of G and is equal to the finite cyclic group $\langle b \rangle$. This implies that there is a bound on the size of the conjugacy classes of G.

Let ω be the multiplier $\omega(b^{s_0}a_1^{s_1}a_2^{s_2} \ldots a_n^{s_n}, b^{s_0'}a_1^{s_1'}a_2^{s_2'} \ldots a_n^{s_n'}) = e^{2\pi i/p(s_1 s_2' - s_1' s_2)}$. The elements of the form $b^{s_0}a_3^{s_3}a_4^{s_4} \ldots a_n^{s_n}$ form a normal subgroup H of G that is contained in Δ, hence $[G : \Delta] < \infty$; also $|\Delta'| \leqslant |G'| = p$, so by Theorem 4.3, $V(G, \omega)$ has a nonzero maximal type I part. However, because $Z(\Delta) \cap H \subseteq Z(H) = \langle b \rangle$, the group $Z(\Delta)$ and consequently $Z_\omega(\Delta)$ has finite index in G, thus by Theorem 1.1, $V(G, \omega)$ is not type I.

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5. Induced characters

The aim of this section is to prove the following theorem.

THEOREM 5.1. *Let G be a discrete group and N a normal subgroup of G. If λ is a character of N, that is a 1 dimensional representation of N, denote by $H_\lambda = \{ g \in G: gng^{-1}n^{-1} \in \ker \lambda$ all $n \in N \}$ the stabilizer of λ, then the von Neumann algebra V_λ generated by the induced representation $\lambda \uparrow_N^G$ is type I if, and only if H_λ contains a subgroup A such that $[H_\lambda : A] < \infty$ and $A' \subseteq \ker \lambda$.*

LEMMA 5.2. (Kleppner [5], Theorem 7). *Assume the notation of the above theorem. If ω is a normalized multiplier of H_λ/N obtained from an extension of λ to H_λ ([5], Theorem 5), then there exists a normal *-isomorphism between V_λ and $V(H_\lambda/N) = V(H_\lambda/N, \omega)$.*

PROOF OF THEOREM 5.1. Let $P \subseteq H_\lambda$ be a set of coset representatives modulo N (including the identity of H_λ). For each $x \in H_\lambda$ denote by α_x the unique element in P such that $xN = \alpha_x N$. Let ψ be the function $G \to N: x \to \alpha_x^{-1}x$. The relations $\psi(n) = n$, $\psi(xn) = \psi(x)n$ and $\psi(nx) = \psi(x)x^{-1}nx$, for $x \in H$, $n \in N$ are evident. It is clear that $\tilde{\lambda}: x \to \lambda(\psi(x))$ is an extension of λ to H_λ and that the multiplier $\sigma(x, y) = \lambda(\psi(xy))\lambda(\psi(x)\psi(y))^{-1}$ associated with this extension can be factored through H_λ/N to yield a multiplier ω on H_λ/N. We may assume without loss of generality that ω is normalized. Using Theorem 1.1 and Lemma 5.2, we deduce that V_λ is type I if, and only if there exists a subgroup A in H_λ such that both (i) $[H_\lambda : A] < \infty$ and (ii) $A' \subseteq N$, $\sigma(x, y) = \sigma(y, x)$ all $x, y \in A$, hold. It remains to show that (ii) is equivalent to $A' \subseteq \ker \lambda$. Suppose (ii) is true. Let $x, y \in A$. Since $x^{-1}y^{-1}xy \in N$, $\psi(xy) = \psi(yxx^{-1}y^{-1}xy) = \psi(yx)x^{-1}y^{-1}xy$, but from $\sigma(x, y) = \sigma(y, x)$ follows $\lambda(\psi(yx)) = \lambda(\psi(xy))$ so that $\lambda(x^{-1}y^{-1}xy) = 1$. Conversely if $A' \subseteq \ker \lambda \subseteq N$, then for $x, y \in A$, $\lambda(xy) = \lambda(\psi(yxx^{-1}y^{-1}xy)) = \lambda(\psi(yx))\lambda(x^{-1}y^{-1}xy) = \lambda(\psi(yx))$ and thus $\sigma(x, y) = \sigma(y, x)$. This completes the proof.

6. Concluding remarks

Given a discrete group G with a normalized multiplier ω, Theorem 3.2 serves the purpose of obtaining a subgroup of finite index in G whose commutator subgroup is finite, provided $V(G, \omega)$ has a non-zero type I part. Such a subgroup can also be obtained by combining Kleppner ([5], page 564) with Taylor's version ([11], Theorem 3) of Kanuith's result ([12], Satz 2).

As a consequence of Theorem 1.1, one finds that for locally compact groups G with normalized Borel multiplier ω, Theorem 1.1 can be generalized along the lines of Moore ([13], Theorem 1) and Taylor ([11], Theorem 2).

References

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