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ULTRAMETRIC AND NON-LOCALLY CONVEX ANALOGUES OF THE GENERAL CURVE LEMMA OF CONVENIENT DIFFERENTIAL CALCULUS

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Abstract. The General Curve Lemma is a tool of Infinite-Dimensional Analysis that enables refined studies of differentiability properties of maps between real locally convex spaces to be made. In this article, we generalize the General Curve Lemma in two ways. First, we remove the condition of local convexity in the real case. Second, we adapt the lemma to the case of curves in topological vector spaces over ultrametric fields.

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Introduction. The General Curve Lemma (as in [**7**, Proposition 4.2.15] or [**17**, Lemma 12.2]) is a powerful tool for the study of finite order differentiability properties of mappings between real locally convex spaces in the Convenient Differential Calculus of Frölicher, Kriegl and Michor; (see [**7**, § 4.3] and [**17**, § 12]). It allows pieces of a (suitable) given sequence of smooth curves to be combined to a single smooth curve, which runs through all of the pieces in finite time. The goal of this paper is to extend the General Curve Lemma to curves in not necessarily locally convex real topological vector spaces, and to curves in topological vector spaces over an ultrametric field.

Our studies are based on the differential calculus of smooth and C^k-maps between open subsets of topological vector spaces over a topological field developed in [**3**], which has by now been applied to a variety of questions in Differential Geometry [**2**], Lie Theory ([**8**], [**9**], [**12**]) and Dynamical Systems (see [**10**] for a survey). We recall that this approach generalizes traditional concepts. In particular, a map between open subsets of real locally convex spaces is C^k in the sense of [**3**] if and only if it is a Keller C^k_c-map (see [**3**]). Furthermore, it is known (see [**11**, Theorem 2.1]) that a map between open subsets of finite-dimensional vector spaces over a complete ultrametric field is C^k in the sense of [**3**] if and only if it is a C^k-map in the usual sense of Non-Archimedean Analysis (as in [**21**, § 84] and [**6**]). The definition of C^1-maps in [**3**] is also similar in spirit to an earlier definition used in [**18**] and [**19**].

Our General Curve Lemma in the real case (Theorem 4.1) closely resembles its classical counterpart for curves in real locally convex spaces. It subsumes the next result.

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Real Case of General Curve Lemma. *Let E be a real topological vector space and $(s_n)_{n\in\mathbb{N}}$ as well as $(r_n)_{n\in\mathbb{N}}$ be sequences of positive reals such that $\sum_{n=1}^{\infty} s_n < \infty$ and $r_n \geq s_n + \frac{2}{n^2}$ for each $n \in \mathbb{N}$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of smooth maps $\gamma_n\colon [-r_n, r_n] \to E$ which become small sufficiently fast* (*in the sense made precise in Theorem* 4.1). *Then there exists a smooth curve $\gamma\colon \mathbb{R} \to E$ and a convergent sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that $\gamma(t_n + t) = \gamma_n(t)$ for all $n \in \mathbb{N}$ and $t \in [-s_n, s_n]$.*

If $(\mathbb{K}, |.|)$ is an ultrametric field, we obtain a variant of the General Curve Lemma (Theorem 3.1) that subsumes the following result.

Ultrametric General Curve Lemma. *Let E be a topological vector space over an ultrametric field* $(\mathbb{K}, |.|)$, *and $\mathbb{O} := \{x \in \mathbb{K}\colon |x| \leq 1\}$. Let $\rho \in \mathbb{K}^\times$ with $|\rho| < 1$ and $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of maps $\gamma_n \in BC^\infty(\rho^n\mathbb{O}, E)$ that become small sufficiently fast* (*in the sense made precise in Theorem* 3.1). *Then there exists a smooth map $\gamma\colon \mathbb{K} \to E$ such that $\gamma(\rho^{n-1} + t) = \gamma_n(t)$ for all $n \in \mathbb{N}$ and $t \in \rho^n\mathbb{O}$.*

The preceding results are useful for the study of k times Hölder differentiable maps of Hölder exponent $\sigma \in]0, 1]$ ($C^{k,\sigma}$-maps, for short), as introduced in [**11**] and (for $\sigma = 1$) in [**13**]. As shown in [**11**], our General Curve Lemmas imply a characterization of $C^{k,\sigma}$-maps on metrizable spaces.

Theorem. *Let $\mathbb{K}$ be $\mathbb{R}$ or an ultrametric field. Let E and F be topological $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a map, defined on an open subset $U \subseteq E$. Let $k \in \mathbb{N}_0$ and $\sigma \in]0, 1]$. If E is metrizable, then f is $C^{k,\sigma}$ if and only if $f \circ \gamma\colon \mathbb{K}^{k+1} \to F$ is $C^{k,\sigma}$, for each C^∞-map $\gamma\colon \mathbb{K}^{k+1} \to U$.*

It would be nice to know whether smooth maps on $\mathbb{K}^{k+1}$ can be replaced by smooth maps of a single variable here, as in Boman's classical results concerning the real finite-dimensional case [**4**] and their infinite-dimensional generalizations [**17**]. The author undertook some steps in this direction jointly with S. V. Ludkovsky (cf. also Ludkovsky's preprint [**20**]). Our versions of the General Curve Lemma were created in connection with this question.

We mention that an analogue of the preceding theorem for C^k-maps can already be found in [**3**, Theorem 12.4], where it was proved with the help of variants of the Special Curve Lemma (Lemma 11.1 and Lemma 11.2 in [**3**]).

Our versions of the General Curve Lemma are more difficult to prove than the classical lemma (as reflected by the length of this text), because it does not suffice to prove merely the existence and continuity of derivatives (of all orders) for γ. Instead, to establish smoothness of γ, one has to prove existence of continuous extensions to higher difference quotient maps, which is a much more cumbersome task. To keep the effort manageable, our strategy is to manufacture, in a first step, certain smooth curves $\eta_n\colon \mathbb{K} \to E$ with pairwise disjoint supports from the given curves γ_n. In a second step, we then show that $\gamma := \sum_{n=1}^{\infty} \eta_n$ converges in $BC^\infty(\mathbb{K}, E)$. To prove convergence of this series, we introduce a notion of "absolute convergence" for series in general topological vector spaces (Definition 2.6), the topology of which need not arise from a family of continuous seminorms. In [**13**], so-called "gauges" have already been used as a substitute for continuous seminorms (cf. [**15**] for the real case). To define absolute convergence of series in general topological vector spaces, we introduce "calibrations" as a further generalization of continuous seminorms (Definition 2.1). These are sequences of gauges which are pairwise related by a certain substitute for the triangle inequality.

1. Preliminaries, notation and basic facts. In this section, we set up terminology and notation. We also compile various basic facts, for later use. These are easy to take on faith, and we recommend that the reader skip the proofs (given in Appendix A), which are not difficult.

All topological fields occurring in this article are assumed Hausdorff and non-discrete. A field $\mathbb{K}$, equipped with an absolute value $|.|\colon \mathbb{K} \to [0, \infty[$ defining a non-discrete topology on $\mathbb{K}$ is called a *valued field*. An *ultrametric field* is a valued field $(\mathbb{K}, |.|)$ whose absolute value satisfies the ultrametric inequality, $|x + y| \le \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$. If $(E, \|.\|)$ is a normed space over a valued field, $r > 0$ and $x \in E$, we define $B_r^E(x) := \{y \in E\colon \|y - x\| < r\}$ and $\overline{B}_r^E(x) := \{y \in E\colon \|y - x\| \le r\}$. Recall that if $\mathbb{K}$ is an ultrametric field, then $\overline{B}_r^{\mathbb{K}}(x)$ and $B_r^{\mathbb{K}}(x)$ are both open and closed. (This will prove to be useful for piecewise definitions of maps). Furthermore, the ultrametric inequality implies that

$$|x + y| = |x| \quad \text{for all } x, y \in \mathbb{K} \text{ such that } |y| < |x|. \tag{1}$$

All topological vector spaces over topological fields are assumed Hausdorff. As usual, $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

A differential calculus of C^k-maps between subsets of ultrametric fields was developed in [**21**]. It makes sense just as well for maps into topological vector spaces over general topological fields (cf. [**3**, § 6] for open domains), and will be used in this form here. The approach can be generalized to a differential calculus of C^k-maps between open subsets of topological vector spaces [**3**]. Compare [**18**], [**19**] for an earlier approach to infinite-dimensional calculus over ultrametric fields,[1] which however is not equivalent to ours, at least when applied to local fields of positive characteristic [**11**]. We only give the definition of C^k-maps on subsets of $\mathbb{K}$ here, following the notational conventions from [**3**] (rather than [**21**]).

Definition 1.1. Let $\mathbb{K}$ be a topological field, $U \subseteq \mathbb{K}$ be a non-empty subset without isolated points, and $\gamma\colon U \to E$ a map to a topological $\mathbb{K}$-vector space E. The map γ is said to be $C^0_{\mathbb{K}}$ if it is continuous; in this case, we set $\gamma^{<0>} := \gamma$. We call γ a $C^1_{\mathbb{K}}$*-map* if it is continuous and if there exists a continuous map $\gamma^{<1>}\colon U \times U \to E$ such that

$$\gamma^{<1>}(x_0, x_1) = \frac{\gamma(x_1) - \gamma(x_0)}{x_1 - x_0} \quad \text{for all } x_0, x_1 \in U \text{ such that } x_0 \neq x_1.$$

Recursively, having defined $C^j_{\mathbb{K}}$-maps and associated maps $\gamma^{<j>}\colon U^{j+1} \to E$ for $j = 0, \dots, k-1$ for some $k \in \mathbb{N}$, we call γ a $C^k_{\mathbb{K}}$*-map* if it is $C^{k-1}_{\mathbb{K}}$ and there is a continuous map $\gamma^{<k>}\colon U^{k+1} \to E$ such that

$$\gamma^{<k>}(x_0, x_1, \dots, x_k) = \frac{\gamma^{<k-1>}(x_k, x_1, \dots, x_{k-1}) - \gamma^{<k-1>}(x_0, x_1, \dots, x_{k-1})}{x_k - x_0}$$

for all $(x_0, \dots, x_k) \in U^{k+1}$ such that $x_0 \neq x_k$. The map γ is $C^\infty_{\mathbb{K}}$ (or *smooth*) if it is $C^k_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$. If $\mathbb{K}$ is understood, we write C^k instead of $C^k_{\mathbb{K}}$. We let $C^k(U, E)$ be the set of all C^k-maps $U \to E$. Then $C^k(U, E)$ is a vector subspace of E^U.

[1]See also the maps called C^n (in contrast to $C^{[n]}$) in [**20**].

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Here $\gamma^{<k>}$ is uniquely determined, and $\gamma^{<k>}$ is symmetric in its $k+1$ variables. Also $k!\,\gamma^{<k>}(x,\ldots,x) = \frac{d^k\gamma}{dx^k}(x) =: \gamma^{(k)}(x)$, for all $x \in U$ (cf. [**21**, § 29] and [**3**, Proposition 6.2]). Let $U^{>k<}$ be the set of all $(x_0,\ldots,x_k) \in U^{k+1}$ such that $x_i \neq x_j$ for all $i \neq j$. Then $U^{>k<}$ is dense in U^{k+1}, which will be useful later.

DEFINITION 1.2. Let E be a topological vector space over a topological field $\mathbb{K}$.

(a) A subset $A \subseteq E$ is called *bounded* if, for each 0-neighbourhood $U \subseteq E$, there exists a 0-neighbourhood $V \subseteq \mathbb{K}$ such that $VA \subseteq U$.

(b) If X is a topological space, then $BC(X,E)$ denotes the set of all continuous maps $\gamma: X \to E$ whose image $\gamma(X)$ is bounded in E. Clearly $BC(X,E)$ is a vector subspace of E^X. We equip $BC(X,E)$ with the topology of uniform convergence.

(c) If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $U \subseteq \mathbb{K}$ is a non-empty subset without isolated points, we let $BC^k(U,E)$ be the space of all C^k-maps $\gamma: U \to E$ such that $\gamma^{<j>} \in BC(U^{j+1},E)$ for all $j \in \mathbb{N}_0$ such that $j \leq k$. We equip $BC^k(U,E)$ with the initial topology with respect to the sequence of mappings $BC^k(U,E) \to BC(U^{j+1},E), \gamma \mapsto \gamma^{<j>}$ (for $j \in \mathbb{N}_0, j \leq k$).

Recall that a topological vector space over a topological field $\mathbb{K}$ is called *complete* if each Cauchy net converges.

LEMMA 1.3. *Let $\mathbb{K}$ a topological field, X be a topological space, $U \subseteq \mathbb{K}$ be a non-empty subset without isolated points, and E a topological $\mathbb{K}$-vector space. Then the following properties hold.*

(a) *$BC(X,E)$ is a topological $\mathbb{K}$-vector space.*

(b) *If E is complete, then also $BC(X,E)$ is complete.*

(c) *For each $k \in \mathbb{N}_0 \cup \{\infty\}$, the map $\theta: BC^k(U,E) \to \prod_j BC(U^{j+1},E)$, $\gamma \mapsto (\gamma^{<j>})_j$ (where $j \in \mathbb{N}_0$ such that $j \leq k$) is linear, a topological embedding and has closed image.*

(d) *$BC^k(U,E)$ is a topological $\mathbb{K}$-vector space, for each $k \in \mathbb{N}_0 \cup \{\infty\}$. If E is complete, then also $BC^k(U,E)$ is complete.*

A topological vector space over a valued field is called *polynormed* if its vector topology can be defined by a family of seminorms. As a replacement for seminorms when dealing with non-polynormed topological vector spaces over a valued field, the more general concept of a *gauge* was introduced in [**13**] (cf. [**15**, § 6.3] for the real case). Using gauges, it is easy to define Lipschitz continuous, Lipschitz differentiable, strictly differentiable, totally differentiable and similar maps between arbitrary topological $\mathbb{K}$-vector spaces ([**11**], [**13**]). We shall slightly generalize the concept of a gauge from [**13**] here, because this will simplify the presentation (See Remarks 1.5 and 1.12.)

DEFINITION 1.4. Let E be a topological vector space over a valued field $(\mathbb{K}, |.|)$. A *gauge on E* is a map $q: E \to [0,\infty[$ (also written $\|.\|_q := q$) satisfying $q(tx) = |t|q(x)$ for all $t \in \mathbb{K}$ and $x \in E$, and such that $B_r^q(0) := q^{-1}([0,r[)$ is a 0-neighbourhood in E, for each $r > 0$.

Note that each gauge is continuous at 0. Sums of gauges and non-negative multiples rq of gauges are gauges.

REMARK 1.5. In [**13**], only upper semicontinuous gauges $q: E \to [0,\infty[$ were considered. Thus, the stronger requirement was made that $B_r^q(0)$ is open in E, for

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each $r > 0$. By the next two remarks, it does not matter for many purposes whether the weaker or the stronger definition is used.

REMARK 1.6. Typical examples of gauges are Minkowski functionals μ_U of balanced, open 0-neighbourhoods U in a topological vector space E over a valued field $\mathbb{K}$ (See [**13**, Remark 1.21].) These are upper semicontinuous. Here $U \subseteq E$ is called *balanced* if $tU \subseteq U$ for all $t \in \mathbb{K}$ such that $|t| \leq 1$. The Minkowski functional is $\mu_U: E \to [0, \infty[$, $x \mapsto \inf\{|t|: t \in \mathbb{K}^\times \text{ with } x \in tU\}$.

REMARK 1.7. If q is a gauge on E, then $q \leq \mu_U$ for the Minkowski functional of some balanced, open 0-neighbourhood U. In fact, we can take any balanced, open 0-neighbourhood $U \subseteq E$ such that $U \subseteq B_1^q(0)$. Given $x \in E$ and $t_n \in \mathbb{K}^\times$ such that $|t_n| \to \mu_U(x)$, we then have $x \in t_n U \subseteq B_{|t_n|}^q(0)$ for each n and thus $q(x) < |t_n|$, from which $q(x) \leq \mu_U(x)$ follows by letting $n \to \infty$.

EXAMPLE 1.8. Given $r \in]0, 1]$, a gauge $q: E \to [0, \infty[$ is called an r-seminorm if $q(x + y)^r \leq q(x)^r + q(y)^r$ for all $x, y \in E$. If, furthermore, $q(x) = 0$ if and only if $x = 0$, then q is called an r-norm (cf. [**15**, § 6.3] for the real case). For examples of r-normed spaces over $\mathbb{R}$ and more general non-locally convex real topological vector spaces, the reader is referred to [**15**, § 6.10] and [**16**]. For $\mathbb{K}$ a valued field, the simplest examples are the spaces $\ell^p(\mathbb{K})$ of all $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ such that $\|x\|_p := \sqrt[p]{\sum_{n=1}^\infty |x_n|^p} < \infty$, for $p \in]0, 1]$. Then $\|.\|_p$ is a p-norm on $\ell^p(\mathbb{K})$ defining a Hausdorff vector topology on this space.

Note that the triangle inequality need not hold for gauges. The following lemma (see [**13**, Lemma 1.29]) provides a certain substitute.

LEMMA 1.9. *If E is a topological vector space over a valued field $\mathbb{K}$ and $U, V \subseteq E$ are balanced open 0-neighborhoods such that $V + V \subseteq U$, then*

$$\mu_U(x + y) \leq \max\{\mu_V(x), \mu_V(y)\} \quad \text{for all } x, y \in E.$$

Hence, for each gauge q on E, there is a gauge p such that $\|x + y\|_q \leq \max\{\|x\|_p, \|y\|_p\}$ and thus $\|x + y\|_q \leq \|x\|_p + \|y\|_p$, for all $x, y \in E$.

DEFINITION 1.10. Let E be a topological vector space over a valued field $\mathbb{K}$. We say that a set Γ of gauges on E is a *fundamental system of gauges* if finite intersections of sets of the form $B_r^q(0)$ with $r > 0$, $q \in \Gamma$ form a basis for the filter of 0-neighbourhoods in E.

Thus, a topological vector space over a valued field is polynormed if and only if it has a fundamental system of gauges which are continuous seminorms. We also mention that, in the real case, the continuous gauges always form a fundamental system (cf. [**15**, § 6.4]). For a more concrete example, consider $\ell^p(\mathbb{K})$ with $p \in]0, 1]$. Then $\{\|.\|_p\}$ is a fundamental system of gauges.

It is useful to know good fundamental systems of gauges for function spaces.

LEMMA 1.11. *Let $\mathbb{K}$ be a valued field, $U \subseteq \mathbb{K}$ a non-empty subset without isolated points, E a topological $\mathbb{K}$-vector space, q a gauge on E, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $j \in \mathbb{N}_0$ such that $j \leq k$. Then*

$$BC^k(U, E) \to [0, \infty[, \quad \gamma \mapsto \|\gamma^{<j>}\|_{q,\infty} := \sup\{\|\gamma^{<j>}(x)\|_q : x \in U^{j+1}\} \tag{2}$$

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is a gauge on $BC^k(U, E)$*. If* Γ *is a fundamental system of gauges for* E*, then the gauges* $\gamma \mapsto \|\gamma^{<j>}\|_{q,\infty}$ *(for* $j \in \mathbb{N}_0$ *such that* $j \leq k$ *and* $q \in \Gamma$*) form a fundamental system of gauges for* $BC^k(U, E)$*.*

If $E = \mathbb{K}$, we simply write $\|\gamma^{<k>}\|_\infty$ instead of $\|\gamma^{<k>}\|_{|.|,\infty}$.

REMARK 1.12. If q in Lemma 1.11 is an upper semicontinuous gauge which does not happen to be a seminorm, then one cannot expect that the gauge $\gamma \mapsto \|\gamma^{<k>}\|_{q,\infty}$ is upper semicontinuous. For this reason, we found it convenient to give up upper semicontinuity in our definition of gauges. Of course, alternatively one might redefine $\|\gamma^{<k>}\|_{q,\infty}$ in a way which enforces upper semicontinuity, but such variants would be more complicated to work with.

We need to know how translations and homotheties affect the gauges from (2).

LEMMA 1.13. *Let* $\mathbb{K}$ *a valued field,* $U \subseteq \mathbb{K}$ *a non-empty subset without isolated points,* E *a topological* $\mathbb{K}$*-vector space,* q *a gauge on* E *and* $k \in \mathbb{N}_0$*.*

(a) *If* $\gamma \in BC^k(U, E)$ *and* $t_0 \in \mathbb{K}$*, then* $\eta\colon U - t_0 \to E$*,* $\eta(t) := \gamma(t + t_0)$ *belongs to* $BC^k(U - t_0, E)$*. Furthermore,* $\|\eta^{<k>}\|_{q,\infty} = \|\gamma^{<k>}\|_{q,\infty}$*.*
(b) *If* $\gamma \in BC^k(U, E)$ *and* $a \in \mathbb{K}^\times$*, then* $\eta\colon a^{-1}U \to E$*,* $\eta(t) := \gamma(at)$ *belongs to* $BC^k(a^{-1}U, E)$*. Furthermore,* $\|\eta^{<k>}\|_{q,\infty} = |a|^k \|\gamma^{<k>}\|_{q,\infty}$*.*
(c) *Let* $V \subseteq U$ *be a non-empty subset without isolated points. Then* $\gamma|_V \in BC^k(V, E)$ *for* $\gamma \in BC^k(U, E)$*, and* $\|(\gamma|_V)^{<k>}\|_{q,\infty} \leq \|\gamma^{<k>}\|_{q,\infty}$*.*

In the real locally convex case, BC^k-maps on intervals are what they should be.

LEMMA 1.14. *Let* E *be a real locally convex space,* $I \subseteq \mathbb{R}$ *a non-singleton interval,* $k \in \mathbb{N}_0$ *and* $\gamma\colon I \to E$ *a map. Then* $\gamma \in C^k(I, E)$ *if and only if* γ *is* C^k *in the usual sense (viz.* $\gamma^{(j)}$ *exists for* $j \in \{0, 1, \ldots, k\}$ *and is continuous). Moreover,* $\gamma \in BC^k(I, E)$ *if and only if* γ *is* C^k *in the usual sense and* $\gamma^{(j)}(I)$ *is bounded in* E *for each* $j \in \{0, 1, \ldots, k\}$*.*

2. Calibrations and absolute convergence. In this section, E is a topological vector space over a valued field $\mathbb{K}$. Our goal is to define a meaningful notion of absolute convergence of series in E. As a tool, calibrations are introduced. These are certain sequences of gauges. Compare [**1**] for the related concept of a "string" in a real vector space.

DEFINITION 2.1. A sequence $(q_n)_{n\in\mathbb{N}_0}$ of gauges on E is called a *calibration* if

$$(\forall n \in \mathbb{N}_0)(\forall x, y \in E) \quad q_n(x + y) \leq q_{n+1}(x) + q_{n+1}(y)\,. \tag{3}$$

The sequence is a *strong calibration* if

$$(\forall n \in \mathbb{N}_0)(\forall x, y \in E) \quad q_n(x + y) \leq \max\{q_{n+1}(x), q_{n+1}(y)\}\,. \tag{4}$$

We shall refer to (3) as the *fake triangle inequality*. Similarly, (4) is called the *fake ultrametric inequality*. If q is a gauge on E, then there always exists a calibration $(q_n)_{n\in\mathbb{N}_0}$ such that $q_0 = q$ (cf. Lemma 1.9). In this situation, we say that q extends to $(q_n)_{n\in\mathbb{N}_0}$.

In this paper, we decided to work entirely with ordinary calibrations. Using strong calibrations instead, one obtains analogous results. For example, variants of

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Lemmas 3.5 and 4.7 hold for strong calibrations (in which case the factors 2^{k-j} in (9) and (20) can be omitted).

REMARK 2.2. If $(q_n)_{n\in\mathbb{N}_0}$ is a calibration, then $q_n \leq q_{n+1}$ for each $n \in \mathbb{N}_0$, because $q_n(x) = q_n(x+0) \leq q_{n+1}(x) + q_{n+1}(0) = q_{n+1}(x)$ for each $x \in E$.

REMARK 2.3. If $q\colon E \to [0, \infty[$ is a continuous seminorm, then $(q)_{n\in\mathbb{N}_0}$ is a calibration (and a strong calibration if q is an ultrametric seminorm). If $(q_n)_{n\in\mathbb{N}_0}$ is any calibration extending the seminorm q, then $q_n \geq q$ for each n, by the preceding remark. Thus $(q)_{n\in\mathbb{N}_0}$ is the smallest calibration extending q.

To illustrate the notion of a calibration, let us look at another example.

EXAMPLE 2.4. *If $r \in]0, 1]$ and q is an r-seminorm on E, define $q_n := 2^{\frac{n}{r}} q$ for $n \in \mathbb{N}_0$. Then $(q_n)_{n\in\mathbb{N}_0}$ is a strong calibration on E. Notably, $(2^{\frac{n}{p}} \|.\|_p)_{n\in\mathbb{N}_0}$ is a strong calibration on $\ell^p(\mathbb{K})$, for each $p \in]0, 1]$. This follows from the observation that $q(x+y)^r \leq q(x)^r + q(y)^r \leq 2\max\{q(x)^r, q(y)^r\}$ for $x, y \in E$ and thus $q(x+y) \leq \sqrt[r]{2\max\{q(x)^r, q(y)^r\}} = \max\{2^{\frac{1}{r}} q(x), 2^{\frac{1}{r}} q(y)\}$.*

The following lemma is obvious.

LEMMA 2.5. *Let $(q_n)_{n\in\mathbb{N}_0}$ be a calibration on E, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $j \in \mathbb{N}_0$ with $j \leq k$. Then the gauges $BC^k(U, E) \to [0, \infty[$, $\gamma \mapsto \|\gamma^{<j>}\|_{q_n,\infty}$, for $n \in \mathbb{N}_0$, form a calibration on $BC^k(U, E)$.* □

Calibrations are valuable tools to establish the convergence of series in topological vector spaces that may fail to be polynormed.

DEFINITION 2.6. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. We say that the series $\sum_{n=1}^{\infty} x_n$ is *absolutely convergent* if each gauge q on E extends to a calibration $(q_n)_{n\in\mathbb{N}_0}$ such that

$$\sum_{n=1}^{\infty} \|x_n\|_{q_n} < \infty.$$

REMARK 2.7. If E is polynormed, then a series $\sum_{n=1}^{\infty} x_n$ in E is absolutely convergent if and only if $\sum_{n=1}^{\infty} \|x_n\|_q < \infty$ for each continuous seminorm q on E (cf. Remark 2.3).

Absolute convergence of series in a topological vector space is a useful concept provided that the latter is *sequentially complete* in the sense that each Cauchy sequence converges.

LEMMA 2.8. *If E is a sequentially complete topological vector space over a valued field $\mathbb{K}$, then every absolutely convergent series in E is convergent.*

Proof. Using (3) repeatedly, we see that $\left\| \sum_{k=m}^{n} x_k \right\|_{q_0} \leq \sum_{k=m}^{n} \|x_k\|_{q_{k-m+1}} \leq \sum_{k=m}^{n} \|x_k\|_{q_k}$ for all $n, m \in \mathbb{N}$ with $n > m$. This ensures that $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ is a Cauchy sequence in E and hence convergent. □

3. Ultrametric General Curve Lemma. In this section, we formulate and prove our first main result.

THEOREM 3.1 *Ultrametric General Curve Lemma. Let E be a topological vector space over an ultrametric field $\mathbb{K}$, $\rho \in \mathbb{K}^\times$ with $|\rho| < 1$ and $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of*

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smooth maps $\gamma_n \in BC^\infty(\rho^n\mathbb{O}, E)$ that become small sufficiently fast in the sense that, for each gauge q on E, there exists a calibration $(q_n)_{n\in\mathbb{N}_0}$ extending q such that

$$(\forall a > 0)\,(\forall k, m \in \mathbb{N}_0) \qquad \lim_{n\to\infty} a^n \|\gamma_n^{<k>}\|_{q_{n+m},\infty} \;=\; 0. \tag{5}$$

Then there exists a smooth map $\gamma \in BC^\infty(\mathbb{K}, E)$ whose image $\mathrm{im}(\gamma)$ is contained in $\{0\} \cup \bigcup_{n\in\mathbb{N}} \mathrm{im}(\gamma_n)$, such that

$$\gamma(\rho^{n-1} + t) \;=\; \gamma_n(t) \quad \textit{for all } n \in \mathbb{N} \textit{ and } t \in \rho^n\mathbb{O}. \tag{6}$$

Remark 3.2. Note that $\rho^n\mathbb{O} = \overline{B}_{|\rho|^n}(0)$ and $\rho^{n-1} + \rho^n\mathbb{O} = \overline{B}_{|\rho|^n}(\rho^{n-1})$ here. Since $|\rho|^n < |\rho^{n-1}|$, we have $|x| = |\rho^{n-1}|$ for each $x \in \overline{B}_{|\rho|^n}(\rho^{n-1})$ (See (1)). As a consequence, the balls $\overline{B}_{|\rho|^n}(\rho^{n-1})$ are pairwise disjoint.

Remark 3.3. If E is polynormed, then the somewhat complicated condition (5) can be simplified. In view of Remark 2.3, condition (5) then amounts to the following. For each $k \in \mathbb{N}_0$ and continuous seminorm q on E, we have

$$(\forall a > 0) \qquad \lim_{n\to\infty} a^n \|\gamma_n^{<k>}\|_{q,\infty} \;=\; 0. \tag{7}$$

Remark 3.4. Let E in Lemma 3.1 be metrizable and suppose that there exists a calibration $(p_n)_{n\in\mathbb{N}_0}$ such that $\{p_n : n \in \mathbb{N}_0\}$ is a fundamental system of gauges, and $C > 0$ such that

$$(\forall k \in \mathbb{N}_0)\,(\forall n \geq k) \quad \|\gamma_n^{<k>}\|_{p_{2n},\infty} \;\leq\; Cn^{-n}. \tag{8}$$

Then the hypothesis (5) of Theorem 3.1 is satisfied: given q, we can extend it to a suitable calibration via $q_n := rp_{n+n_0}$ for $n \in \mathbb{N}$, with $r > 0$ and $n_0 \in \mathbb{N}_0$ sufficiently large. In all our applications, we use this simpler criterion.

The following lemma prepares the way for the proof of Theorem 3.1. As before, $\mathbb{K}$ is an ultrametric field and E a topological $\mathbb{K}$-vector space.

Lemma 3.5. *Let $\gamma \in BC^\infty(U, E)$, where $U := \overline{B}_r^{\mathbb{K}}(0)$ for some $r \in \,]0, \infty[$. Extend γ to a smooth map $\eta : \mathbb{K} \to E$ via $\eta(x) := 0$ for $x \in \mathbb{K} \setminus U$. Then $\eta \in BC^\infty(\mathbb{K}, E)$, and*

$$\|\eta^{<k>}\|_{q_0,\infty} \;\leq\; \max_{j=0,\ldots,k} \left(\frac{2}{r}\right)^{k-j} \|\gamma^{<j>}\|_{q_{k-j},\infty}\,, \tag{9}$$

for each $k \in \mathbb{N}_0$ and calibration $(q_n)_{n\in\mathbb{N}_0}$ on E.

Proof. Note first that η is smooth since smoothness is a local property (see [**3**, Lemma 4.9]) and U is both open and closed. We now show by induction on $k \in \mathbb{N}_0$ that $\eta \in BC^k(\mathbb{K}, E)$ and (9) holds. If $k = 0$, then $\eta \in BC(\mathbb{K}, E)$ and (9) holds because $\sup\{\|\eta(x)\|_{q_0} : x \in \mathbb{K}\} = \sup\{\|\gamma(x)\|_{q_0} : x \in U\} = \|\gamma^{<k>}\|_{q_0,\infty}$. Now suppose that $k \geq 1$ and suppose that (9) holds if k is replaced with $k-1$, for each calibration. Since $U^{>k<}$ is dense in $U^{<k>}$ and $\eta^{<k>}$ is continuous, we only need to show that the right hand side of (9) is an upper bound for $\|\eta^{<k>}(x)\|_{q_0}$, for each $x = (x_0, \ldots, x_k) \in U^{>k<}$. It is convenient to distinguish three cases:

Case 1. If $x_j \in U$ for all $j \in \{0, \ldots, k\}$, then

$$\eta^{<k>}(x_0, \ldots, x_k) \;=\; \gamma^{<k>}(x_0, \ldots, x_k)$$

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and thus $\|\eta^{<k>}(x_0,\ldots,x_k)\|_{q_0} = \|\gamma^{<k>}(x_0,\ldots,x_k)\|_{q_0} \leq \|\gamma^{<k>}\|_{q_0,\infty}$, which does not exceed the right hand side of (9).

Case 2. If $x_j \notin U$ for all $j \in \{0,\ldots,k\}$, then $\eta(x_j) = 0$ for each j and thus $\eta^{<k>}(x_0,\ldots,x_k) = 0$, whence again $\|\eta^{<k>}(x_0,\ldots,x_k)\|_{q_0} = 0$ does not exceed the right hand side of (9).

Case 3. There are $i, j \in \{0,\ldots,k\}$ such that $x_i \in U$ and $x_j \notin U$. By symmetry of $\eta^{<k>}$, we may assume that $i = 0$ and $j = k$. Since $|x_0| \leq r < |x_k|$ and $|.|$ is ultrametric, we have $|x_0 - x_k| = |x_k| > r$. Hence

$$\begin{aligned}\|\eta^{<k>}(x)\|_{q_0} &= \frac{\|\eta^{<k-1>}(x_0,x_1,\ldots,x_{k-1}) - \eta^{<k-1>}(x_k,x_1,\ldots,x_{k-1})\|_{q_0}}{|x_0 - x_k|}\\ &\leq \frac{\|\eta^{<k-1>}(x_0,x_1,\ldots,x_{k-1})\|_{q_1} + \|\eta^{<k-1>}(x_k,x_1,\ldots,x_{k-1})\|_{q_1}}{r}\\ &\leq \frac{2}{r}\cdot \max_{j=0,\ldots,k-1}\left(\frac{2}{r}\right)^{k-1-j}\|\gamma^{<j>}\|_{q_{k-j},\infty}\,, \end{aligned} \tag{10}$$

applying the inductive hypothesis to $\eta^{<k-1>}$ and the calibration $(q_{n+1})_{n\in\mathbb{N}_0}$ to obtain the final inequality. Since the right hand side of (10) does not exceed the right hand side of (9), our inductive proof is complete. □

Proof of Theorem 3.1. For each $n \in \mathbb{N}$, define $\eta_n\colon \mathbb{K} \to E$ via

$$\eta_n(t) := \begin{cases} \gamma_n(t - \rho^{n-1}) & \text{if } |t - \rho^{n-1}| \leq \rho^n;\\ 0 & \text{otherwise.}\end{cases}$$

Then $\eta_n \in BC^\infty(\mathbb{K}, E)$ and

$$\|\eta_n^{<k>}\|_{q_0,\infty} \leq \max_{j=0,\ldots,k}\left(\tfrac{2}{|\rho|^n}\right)^{k-j}\|\gamma_n^{<j>}\|_{q_{k-j},\infty} \tag{11}$$

for each $k \in \mathbb{N}_0$ and calibration $(q_n)_{n\in\mathbb{N}_0}$ on E, by Lemmas 3.5 and 1.13 (a). Define $\gamma(x) := \sum_{n=1}^\infty \eta_n(x)$ for $n \in \mathbb{N}$. Then $\gamma\colon \mathbb{K} \to E$ is smooth on $\mathbb{K}^\times$, using the fact that the supports of the maps η_n form a locally finite family of disjoint subsets of $\mathbb{K}^\times$ (since $\mathrm{supp}(\eta_n) \subseteq \overline{B}_{|\rho|^n}(\rho^{n-1})$).

Step 1. We show that $\sum_{n=1}^\infty \eta_n$ converges in $BC^\infty(\mathbb{K}, \overline{E})$, where $\overline{E}$ is the completion of E. Once this is established, for each $x \in U$ we can apply the continuous linear point evaluation $BC^\infty(\mathbb{K}, \overline{E}) \to \overline{E}$, $\zeta \mapsto \zeta(x)$ to $\sum_{n=1}^\infty \eta_n$, showing that $(\sum_{n=1}^\infty \eta_n)(x) = \gamma(x)$. Since $BC^\infty(\mathbb{K}, \overline{E})$ is complete by Lemma 1.3 (d), to establish convergence we only need to show that the series $\sum_{n=1}^\infty \eta_n$ converges absolutely (See Lemma 2.8). To this end, let q_0 be a gauge on $\overline{E}$ and extend it to a calibration $(q_n)_{n\in\mathbb{N}_0}$ such that (5) holds (and hence also (11)). Since the gauges $BC^\infty(\mathbb{K}, \overline{E}) \to [0,\infty[$, $\zeta \mapsto \|\zeta^{<k>}\|_{q,\infty}$ form a fundamental system of gauges for k ranging through $\mathbb{N}_0$ and q through the gauges of $\overline{E}$ (see Lemma 1.11), $\sum_{n=1}^\infty \eta_n$ will converge absolutely in $BC^\infty(\mathbb{K}, \overline{E})$ if we can show that $\sum_{n=1}^\infty \|\eta_n^{<k>}\|_{q_n,\infty} < \infty$ in the preceding situation, for each $k \in \mathbb{N}_0$. In view of (11), it suffices to show that

$$\sum_{n=1}^\infty \max_{j=1,\ldots,k}\left(\tfrac{2}{|\rho|^n}\right)^{k-j}\|\gamma_n^{<j>}\|_{q_{n+k-j},\infty} < \infty\,.$$

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This will hold if we can show that, for each $j \in \{1, \ldots, k\}$,

$$\sum_{n=1}^{\infty} |\rho|^{-nk} \|\gamma_n^{<j>}\|_{q_{n+k-j},\infty} \; < \; \infty. \tag{12}$$

To prove (12), choose $a > |\rho|^{-k}$ and recall that $a^n \|\gamma_n^{<j>}\|_{q_{n+k-j},\infty} \to 0$, by (5). Thus $A_k := \sup\{a^n \|\gamma_n^{<j>}\|_{q_{n+k-j},\infty} : n \in \mathbb{N}\} < \infty$ and hence $\sum_{n=1}^{\infty} A_k (\frac{|\rho|^{-k}}{a})^n$ is a convergent majorant for $\sum_{n=1}^{\infty} |\rho|^{-nk} \|\gamma_n^{<j>}\|_{q_{n+k-j},\infty}$.

Step 2. We now show by induction on $k \in \mathbb{N}_0$ that $\gamma^{<k>} : \mathbb{K}^{k+1} \to \overline{E}$ actually takes values in E, whence γ is C^k as a map into E. For $k = 0$, this is trivial. Let $k \geq 1$ now and assume that the assertion holds if k is replaced with $k - 1$. Let $x = (x_0, \ldots, x_k) \in \mathbb{K}^{k+1}$. If $x_i \neq x_j$ for certain $i, j \in \{0, \ldots, k\}$, then $\gamma^{<k>}(x)$ is a partial difference quotient of $\gamma^{<k-1>}$ and hence a scalar multiple of two values of $\gamma^{<k-1>}$, which are in E (by induction). Hence also $\gamma^{<k>}(x) \in E$. It remains to show that $\gamma^{<k>}(y, \ldots, y) \in E$ for all $y \in \mathbb{K}$. If $y \in \mathbb{K}^\times$, this follows from the smoothness of $\gamma|_{\mathbb{K}^\times}$. To see that $\gamma^{<k>}(0) \in E$, we exploit the continuity of the map

$$BC^\infty(\mathbb{K}, \overline{E}) \to E\,, \quad \zeta \mapsto \zeta^{<k>}(0, \ldots, 0).$$

Hence $\gamma^{<k>}(0) = \sum_{n=1}^{\infty} \eta_n^{<k>}(0) = 0 \in E$. Thus $\gamma^{<k>}$ has image in E, which completes the induction. □

4. General Curve Lemma for curves in real topological vector spaces. In this section, we prove a version of the General Curve Lemma for curves in arbitrary (not necessarily locally convex) real topological vector spaces.

THEOREM 4.1 *Real Case of General Curve Lemma. Let E be a real topological vector space and $(s_n)_{n\in\mathbb{N}}$ as well as $(r_n)_{n\in\mathbb{N}}$ be sequences of positive reals such that $\sum_{n=1}^{\infty} s_n < \infty$ and $r_n \geq s_n + \frac{2}{n^2}$ for each $n \in \mathbb{N}$. Furthermore, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of smooth maps $\gamma_n : [-r_n, r_n] \to E$ that become small sufficiently fast in the sense that, for each gauge q on E, there exists a calibration $(q_n)_{n\in\mathbb{N}_0}$ extending q such that*

$$(\forall k, \ell, m \in \mathbb{N}_0) \qquad \lim_{n\to\infty} n^\ell \|\gamma_n^{<k>}\|_{q_{n+m},\infty} \; = \; 0\,. \tag{13}$$

Then there exists a curve $\gamma \in BC^\infty(\mathbb{R}, E)$ with $\operatorname{im}(\gamma) \subseteq [0, 1] \cdot \bigcup_{n\in\mathbb{N}} \operatorname{im}(\gamma_n)$ and a convergent sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\gamma(t_n + t) \; = \; \gamma_n(t) \quad \textit{for all } n \in \mathbb{N} \textit{ and } t \in [-s_n, s_n]. \tag{14}$$

Various lemmas are needed to prepare the proof of Theorem 4.1.

LEMMA 4.2. *For each $n \in \mathbb{N}$, there exist integers $N_{i,j} \in \mathbb{N}_0$ indexed by all strictly increasing finite sequences $i = (i_0, \ldots, i_k)$ and $j = (j_0, \ldots, j_\ell)$ with entries in $\{0, 1, \ldots, n\}$, for $k, \ell \in \mathbb{N}_0$ with $k + \ell = n$, such that $\sum_{i,j} N_{i,j} \leq 2^n$ and the following property holds: for each topological field $\mathbb{K}$, non-empty subset $U \subseteq \mathbb{K}$ without isolated points, continuous bilinear map $\beta : E \times F \to H$ between topological $\mathbb{K}$-vector spaces and all C^n-maps*

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$\gamma\colon U \to E$, $\eta\colon U \to F$, we have

$$\begin{aligned}&\big(\beta\circ(\gamma,\eta)\big)^{<n>}(x_0,\ldots,x_n)\\ &= \sum_{k+\ell=n}\ \sum_{\substack{i,j \text{ with}\\ \#i=k,\ \#j=\ell}} N_{i,j}\,\beta\big(\gamma^{<k>}(x_{i_0},\ldots,x_{i_k}),\ \eta^{<\ell>}(x_{j_0},\ldots,x_{j_\ell})\big)\end{aligned} \tag{15}$$

for all $(x_0,\ldots,x_n)\in U^{n+1}$, using the notation $\#(i_0,\ldots,i_k):=k$.

REMARK 4.3. The condition $\sum_{i,j} N_{i,j}\le 2^n$ means that one can consider $\big(\beta\circ(\gamma,\eta)\big)^{<n>}(x_0,\ldots,x_n)$ as a sum of at most $\le 2^n$ summands of the form

$$\beta\big(\gamma^{<k>}(x_{i_0},\ldots,x_{i_k}),\ \eta^{<\ell>}(x_{j_0},\ldots,x_{j_\ell})\big)\,.$$

Proof of Lemma 4.2. The proof is by induction on $n\in\mathbb{N}$. If $n=1$ and $x_0,x_1\in U$ are distinct, then

$$\begin{aligned}&\frac{\beta(\gamma(x_1),\eta(x_1))-\beta(\gamma(x_0),\eta(x_0))}{x_1-x_0}\\ &=\frac{\beta(\gamma(x_1),\eta(x_1))-\beta(\gamma(x_0),\eta(x_1))+\beta(\gamma(x_0),\eta(x_1))-\beta(\gamma(x_0),\eta(x_0))}{x_1-x_0}\\ &=\beta(\gamma^{<1>}(x_0,x_1),\eta(x_1))+\beta(\gamma(x_0),\eta^{<1>}(x_0,x_1))\,.\end{aligned} \tag{16}$$

Since (16) can be used to define a continuous function in $(x_0,x_1)\in U\times U$, we see that $\beta\circ(\gamma,\eta)$ is C^1 with

$$(\beta\circ(\gamma,\eta))^{<1>}(x_0,x_1)=\beta(\gamma^{<1>}(x_0,x_1),\eta(x_1))+\beta(\gamma(x_0),\eta^{<1>}(x_0,x_1)) \tag{17}$$

of the form described in (15).

Induction step. Suppose that the lemma holds for some n and that γ,η are C^{n+1}. For i,j as above with $\#i=k$, $\#j=\ell$ and $k+\ell=n$, abbreviate

$$h_{i,j}(x_0,\ldots,x_n)\ :=\ \beta\big(\gamma^{<k>}(x_{i_0},\ldots,x_{i_k}),\ \eta^{<\ell>}(x_{j_0},\ldots,x_{j_\ell})\big)$$

for $x_0,\ldots,x_n\in U$. The analogue of (15) for $(\beta\circ(\gamma,\eta))^{<n+1>}$ will be apparent from an explicit formula for the continuous extension of the mapping $g\colon\{x=(x_0,\ldots,x_{n+1})\in U^{n+2}\colon x_0\neq x_{n+1}\}\to H$,

$$g(x)\ :=\ \frac{h_{i,j}(x_{n+1},x_1,\ldots,x_n)-h_{i,j}(x_0,x_1,\ldots,x_n)}{x_{n+1}-x_0} \tag{18}$$

to a map $U^{n+2}\to H$, which we now establish. If $i_0\neq 0$ and $j_0\neq 0$, then $h_{i,j}$ does not depend on x_0 and thus $g=0$ has 0 as a continuous extension.

If $i_0=0$ and $j_0\neq 0$, then $\eta^{<\ell>}(x_{j_0},\ldots,x_{j_\ell})$ does not depend on x_0 and thus

$$g(x_0,\ldots,x_{n+1})\ =\ \beta(\gamma^{<k+1>}(x_{i_0},\ldots,x_{i_k},x_{n+1}),\eta^{<\ell>}(x_{j_0},\ldots,x_{j_\ell}))$$

by linearity of β in its first argument, where the right hand side can be used to define a continuous function on U^{n+2}. Similarly, the mapping $U^{n+2}\to H$, $x\mapsto\beta(\gamma^{<k>}(x_{i_0},\ldots,x_{i_k}),\eta^{<\ell+1>}(x_{j_0},\ldots,x_{j_\ell},x_{n+1}))$ provides a continuous extension of g if $i_0\neq 0$ and $j_0=0$.

If $i_0 = j_0 = 0$, then the calculation leading to (16) shows that

$$g(x) = \beta(\gamma^{<k+1>}(x_{i_0}, \ldots, x_{i_k}, x_{n+1}), \eta^{<\ell>}(x_{j_1}, \ldots, x_{j_\ell}, x_{n+1})) \\ +\beta(\gamma^{<k>}(x_{i_0}, \ldots, x_{i_k}), \eta^{<\ell+1>}(x_{j_0}, \ldots, x_{j_\ell}, x_{n+1})),$$

where again the right hand side extends continuously to all of U^{n+2}.

Forming the sum of all contributions just described, we obtain a formula analogous to (15) for $(\beta \circ (\gamma, \eta))^{<n+1>}$. □

REMARK 4.4. For our purposes, we need not know the integers $N_{i,j}$ explicitly.

LEMMA 4.5. *There exist constants $C_k \in \mathbb{N}$ for $k \in \mathbb{N}_0$ such that $\sum_{k=0}^{n} C_k \leq 2^n$ for each $n \in \mathbb{N}$, and the following property holds: For each valued field $\mathbb{K}$, non-empty subset $U \subseteq \mathbb{K}$ without isolated points, $n \in \mathbb{N}$, $\gamma \in BC^n(U, \mathbb{K})$, topological $\mathbb{K}$-vector space E, $\eta \in BC^n(U, E)$ and calibration $(q_k)_{k \in \mathbb{N}_0}$ on E, we have*

$$\|(\gamma \cdot \eta)^{<n>}\|_{q_0,\infty} \;\leq\; \sum_{k=0}^{n} C_k \, \|\gamma^{<k>}\|_\infty \cdot \|\eta^{<n-k>}\|_{q_n,\infty} . \tag{19}$$

Proof. Applying (15) to the scalar multiplication $\beta\colon \mathbb{K} \times E \to E$, we get

$$\|(\gamma \cdot \eta)^{<n>}\|_{q_0,\infty} \;\leq\; \sum_{k=0}^{n} \sum_{i,j} |N_{i,j}| \, \|\gamma^{<k>}\|_\infty \|\eta^{<n-k>}\|_{q_n,\infty} .$$

Here, at most 2^n summands were involved and hence the fake triangle inequality had to be used at most n times, explaining why the gauge q_n occurs. Since $|N_{i,j}| \leq N_{i,j}$, the assertion follows with $C_k := \sum_{i,j} N_{i,j}$, where the sum is taken over all i, j as in Lemma 4.2 such that $\#i = k$ and $\#j = n - k$. □

As a first application of Lemma 4.5, let us construct a family of smooth cut-off functions the size of whose difference quotient maps (of all orders) is well under control. These cut-off functions will be most useful later.

LEMMA 4.6. *There is a sequence $(M_n)_{n \in \mathbb{N}_0}$ of positive real numbers with the following property. For all $a, b > 0$, there exists a smooth function $h\colon \mathbb{R} \to [0, 1]$ with support $\operatorname{supp}(h) \subseteq [-(a+b), a+b]$, such that $h(t) = 1$ for all $t \in [-a, a]$ and*

$$(\forall n \in \mathbb{N}_0) \qquad \|h^{<n>}\|_\infty \;\leq\; M_n \, b^{-n} .$$

Proof. Let $g\colon \mathbb{R} \to [0, 1]$ be a smooth function such that $g(t) = 1$ if $t \leq 0$ and $g(t) = 0$ if $t \geq 1$. Then $g^{(k)}$ is bounded for each $k \in \mathbb{N}_0$ and hence $g \in BC^\infty(\mathbb{R}, \mathbb{R})$, by Lemma 1.14. Given $a, b > 0$, define $h\colon \mathbb{R} \to \mathbb{R}$ via $h(t) := g(\frac{t-a}{b})g(\frac{-t-a}{b})$. Then $h(\mathbb{R}) \subseteq [0, 1]$, $h(t) = 1$ if $|t| \leq a$, and $h(t) = 0$ if $|t| \geq a + b$. By Lemma 4.5, we have $h \in BC^\infty(\mathbb{R}, \mathbb{R})$. Furthermore, combining (19) with Lemma 1.13 (a) and (b), we see that

$$\|h^{<n>}\|_\infty \;\leq\; \sum_{k=0}^{n} C_k \, b^{-k} \|g^{<k>}\|_\infty \cdot b^{-(n-k)} \|g^{<n-k>}\|_\infty \;=\; M_n \, b^{-n}$$

with $M_n := \sum_{k=0}^{n} C_k \|g^{<k>}\|_\infty \cdot \|g^{<n-k>}\|_\infty$ independent of a and b. □

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The following lemma will serve as a substitute for Lemma 3.5 in the real case. Of course, $BC^\infty([a,b],E) = C^\infty([a,b],E)$ here by compactness of $[a,b]$.

LEMMA 4.7. *Let $a < \alpha < \beta < b$ be real numbers, $r := \min\{\alpha - a, b - \beta\}$, E be a real topological vector space, and $\gamma \in BC^\infty([a,b],E)$ be a map such that $\gamma(x) = 0$ if $x \in [a,b] \setminus [\alpha,\beta]$. Define $\eta\colon \mathbb{R} \to E$ via $\eta(x) := \gamma(x)$ if $x \in [a,b]$, $\eta(x) := 0$ elsewhere. Then $\eta \in BC^\infty(\mathbb{R},E)$. Furthermore,*

$$\|\eta^{<k>}\|_{q_0,\infty} \;\leq\; \max_{j=0,\dots,k} \left(\frac{2}{r}\right)^{k-j} \|\gamma^{<j>}\|_{q_{k-j},\infty}\,, \tag{20}$$

for each $k \in \mathbb{N}_0$ and calibration $(q_n)_{n\in\mathbb{N}_0}$ on E.

Proof. We show by induction on $k \in \mathbb{N}_0$ that $\eta \in BC^k(\mathbb{R},E)$ and (20) holds. If $k = 0$, then $\sup\{\|\eta(x)\|_{q_0} : x \in \mathbb{R}\} = \sup\{\|\gamma(x)\|_{q_0} : x \in [a,b]\} = \|\gamma^{<k>}\|_{q_0,\infty}$ for each calibration $(q_n)_{n\in\mathbb{N}_0}$, ensuring that $\eta \in BC(\mathbb{R},E)$ and (20) holds. Now suppose that $k \geq 1$ and suppose that the estimate (20) holds if k is replaced with $k-1$, for each calibration. Since $U^{>k<}$ is dense in $U^{<k>}$ and $\eta^{<k>}$ is continuous, we only need to show that the right hand side of (20) is an upper bound for $\|\eta^{<k>}(x)\|_{q_0}$, for each $x = (x_0,\dots,x_k) \in U^{>k<}$.

Cases 1 *and* 2. If $x_j \in [a,b]$ for all $j \in \{0,\dots,k\}$, or if $x_j \notin [\alpha,\beta]$ for all $j \in \{0,\dots,k\}$, then we see as in Step 1 and 2 of the proof of Lemma 3.5 that $\|\eta^{<k>}(x)\|_{q_0}$ does not exceed the right hand side of (20).

Case 3. Assume that there are $i, j \in \{0,\dots,k\}$ such that $x_i \in [\alpha,\beta]$ and $x_j \notin [a,b]$. Then $|x_i - x_j| \geq r$. By symmetry of $\eta^{<k>}$, without loss of generality $i = 0$ and $j = k$. Now

$$\begin{aligned}
\|\eta^{<k>}(x)\|_{q_0} &= \frac{\|\eta^{<k-1>}(x_0,x_1,\dots,x_{k-1}) - \eta^{<k-1>}(x_k,x_1,\dots,x_{k-1})\|_{q_0}}{|x_0 - x_k|} \\
&\leq \frac{\|\eta^{<k-1>}(x_0,x_1,\dots,x_{k-1})\|_{q_1} + \|\eta^{<k-1>}(x_k,x_1,\dots,x_{k-1})\|_{q_1}}{r} \\
&\leq \frac{2}{r} \cdot \max_{j=0,\dots,k-1} \left(\frac{2}{r}\right)^{k-1-j} \|\gamma^{<j>}\|_{q_{k-j},\infty}\,,
\end{aligned} \tag{21}$$

applying the inductive hypothesis to $\eta^{<k-1>}$ and the calibration $(q_{n+1})_{n\in\mathbb{N}_0}$ to obtain the final inequality. Since the right hand side of (21) does not exceed the right hand side of (20), our inductive proof is complete. □

Proof of Theorem 4.1. After shrinking r_n if necessary, we may assume that $r_n = s_n + \frac{2}{n^2}$ for each $n \in \mathbb{N}$ (cf. Lemma 1.13 (c)). Let $(M_n)_{n\in\mathbb{N}_0}$ be as in Lemma 4.6. Given $n \in \mathbb{N}$, we apply Lemma 4.6 with $a := s_n$ and $b := \frac{1}{n^2}$. We obtain a smooth function $h_n\colon \mathbb{R} \to [0,1]$ such that $h_n(t) = 1$ for all $t \in [-s_n, s_n]$, $\operatorname{supp}(h_n) \subseteq [-s_n - \frac{1}{n^2}, s_n + \frac{1}{n^2}]$, and

$$\|h_n^{<k>}\|_\infty \;\leq\; M_k n^{2k}, \quad \text{for each } k \in \mathbb{N}_0. \tag{22}$$

Set $r_0 := 0$ and define for $n \in \mathbb{N}$

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$$t_n := \sum_{j=1}^{n}(r_j + r_{j-1}).$$

Then $(t_n)_{n\in\mathbb{N}}$ is a monotonically increasing sequence, which converges because $t_\infty := \sum_{j=1}^\infty (r_j + r_{j-1}) \leq 2\sum_{j=1}^\infty s_j + 4\sum_{j=1}^\infty \frac{1}{n^2} < \infty$. By definition,

$$(\forall n \in \mathbb{N}) \qquad t_{n+1} - t_n = r_{n+1} + r_n. \tag{23}$$

Define $\zeta_n : [t_n - r_n, t_n + r_n] \to E$, $\zeta_n(t) := h_n(t - t_n)\gamma_n(t - t_n)$ and let $\eta_n : \mathbb{R} \to E$ be the extension of ζ_n by 0. Then $\operatorname{supp}(\eta_n) \subseteq [t_n - s_n - \frac{1}{n^2}, t_n + s_n + \frac{1}{n^2}] \subseteq]t_n - r_n, t_n + r_n[$, whence the maps η_n have disjoint supports (cf. (23)). Thus $\gamma(t) := \sum_{n=1}^\infty \eta_n(t)$ exists pointwise. To see that γ has the desired properties, let q be a gauge on E and extend it to a calibration $(q_n)_{n\in\mathbb{N}_0}$ such that (13) holds. Then

$$\begin{aligned}\|\zeta_n^{<j>}\|_{q_m,\infty} &\leq \sum_{i=0}^{j} C_i\,\|h_n^{<i>}\|_\infty \cdot \|\gamma_n^{<j-i>}\|_{q_{m+j},\infty}\\ &\leq \sum_{i=0}^{j} n^{2i} C_i M_i \|\gamma_n^{<j-i>}\|_{q_{m+j},\infty},\end{aligned} \tag{24}$$

for all $n \in \mathbb{N}$ and $m, j \in \mathbb{N}_0$, using Lemma 1.13 (a), inequality (19) from Lemma 4.5 and (22). Because ζ_n vanishes outside $[t_n - s_n - \frac{1}{n^2}, t_n + s_n + \frac{1}{n^2}]$ and furthermore $(t_n - s_n - \frac{1}{n^2}) - (t_n - r_n) = \frac{1}{n^2}$ and $(t_n + r_n) - (t_n + s_n + \frac{1}{n^2}) = \frac{1}{n^2}$, Lemma 4.7 and (20) show that $\eta_n \in BC^\infty(\mathbb{R}, E)$, with

$$\begin{aligned}\|\eta_n^{<k>}\|_{q_n,\infty} &\leq \max_{j=0,\ldots,k}\left(\frac{2}{1/n^2}\right)^{k-j}\|\zeta_n^{<j>}\|_{q_{n+k-j},\infty} \leq 2^k n^{2k}\max_{j=0,\ldots,k}\|\zeta_n^{<j>}\|_{q_{n+k},\infty}\\ &\leq n^{4k} A_k \textstyle\sum_{i=0}^{k}\|\gamma_n^{<i>}\|_{q_{n+2k},\infty}\end{aligned}$$

where $A_k := \max\{2^k C_j M_j : j = 0, \ldots, k\}$. Passing to the last line, we used (24) and replaced some terms by larger ones. Since $n^{4k+2}\|\gamma_n^{<i>}\|_{q_{n+2k},\infty}$ converges as $n \to \infty$ for each $i \in \{0, \ldots, k\}$ (by (13)), we have $B_k := \sup\{n^{4k+2}\|\gamma_n^{<i>}\|_{q_{n+2k},\infty} : i \in \{0, \ldots, k\}, n \in \mathbb{N}\} < \infty$. Hence

$$\sum_{n=1}^{\infty}\|\eta_n^{<k>}\|_{q_n,\infty} \leq A_k \sum_{i=0}^{k}\sum_{n=1}^{\infty}\frac{1}{n^2}\underbrace{n^{4k+2}\|\gamma_n^{<i>}\|_{n+2k,\infty}}_{\leq B_k} \leq A_k B_k \sum_{i=0}^{k}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Thus $\sum_{n=1}^\infty \eta_n$ is absolutely convergent and hence convergent in $BC^\infty(\mathbb{R}, \overline{E})$. Pointwise calculation of the limit shows that $\sum_{n=1}^\infty \eta_n = \gamma$ from above. Since $t_\infty \notin \operatorname{supp}(\eta_n)$ for all $n \in \mathbb{N}$, we can argue now as at the end of the proof of Theorem 3.1 to see that $\gamma \in BC^\infty(\mathbb{R}, E)$. By construction, γ has also all other required properties. □

A. Proofs of the lemmas in Section 1. In this appendix, proofs are provided for the lemmas from Section 1.

Proof of Lemma 1.3. (a) For each 0-neighbourhood U in E, we define $\lfloor X, U\rfloor := \{\gamma \in BC(X, E) : \gamma(X) \subseteq U\}$. If $V \subseteq E$ is a 0-neighbourhood with $V = -V$ and $U \supseteq V + V$, then $\lfloor X, V\rfloor + \lfloor X, V\rfloor \subseteq \lfloor X, U\rfloor$ and $-\lfloor X, V\rfloor \subseteq \lfloor X, U\rfloor$, ensuring that there is a unique group topology on $BC(X, E)$ for which the sets $\lfloor X, U\rfloor$ form a basis

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of 0-neighbourhoods. As $\bigcap_U U = \{0\}$, also the sets $\lfloor X, U\rfloor$ have intersection $\{0\}$ and thus $BC(X, E)$ is Hausdorff. To see that the given group topology is a vector topology, it only remains to check conditions ($\mathrm{EVT}'_{\mathrm{I}}$)–($\mathrm{EVT}'_{\mathrm{III}}$) of [**5**, Ch. I, §1, no. 1]. First, given $\gamma_0 \in BC(X, E)$, we show that the map $\mathbb{K} \to BC(X, E)$, $t \mapsto t\gamma_0$ is continuous at 0. Since $\gamma_0(X)$ is bounded, for each 0-neighbourhood $U \subseteq E$ there is a 0-neighbourhood $V \subseteq \mathbb{K}$ such that $V\gamma_0(X) \subseteq U$. Then $V\gamma_0 \subseteq \lfloor X, U\rfloor$, entailing the assertion. Next, given $t_0 \in \mathbb{K}$, let us check that the map $BC(X, E) \to BC(X, E)$, $\gamma \mapsto t_0\gamma$ is continuous at 0. In fact, given U as before, there is a 0-neighbourhood $V \subseteq E$ such that $t_0V \subseteq U$. Then $t_0\lfloor X, V\rfloor \subseteq \lfloor X, U\rfloor$. Also, scalar multiplication $\mathbb{K} \times BC(X, E) \to BC(X, E)$, $(t, \gamma) \mapsto t\gamma$ is continuous at $(0, 0)$. In fact, given U, there are 0-neighbourhoods $V \subseteq \mathbb{K}$ and $W \subseteq E$ such that $VW \subseteq U$. Then $V\lfloor X, W\rfloor \subseteq \lfloor X, U\rfloor$.

(b) If $(\gamma_\alpha)_\alpha$ is a Cauchy net in $BC(X, E)$, then $(\gamma_\alpha(x))_\alpha$ is a Cauchy net in E for each $x \in X$, the point evaluation $BC(X, E) \to E$, $\gamma \mapsto \gamma(x)$ being continuous and linear. Since E is complete, $\gamma_\alpha(x) \to \gamma(x)$ for some $\gamma(x) \in E$. Given a 0-neighbourhood U in E, let $V \subseteq E$ be a 0-neighbourhood such that $V + V + V \subseteq U$, and $W \subseteq V$ a closed, symmetric 0-neighbourhood such that $SW \subseteq V$ for some 0-neighbourhood $S \subseteq \mathbb{K}$. There exists α_0 such that $\gamma_\alpha - \gamma_\beta \in \lfloor X, W\rfloor$ for all $\alpha, \beta \geq \alpha_0$. Then $\gamma_\alpha(x) - \gamma_\beta(x) \in W$ for each $x \in X$. Since W is closed, passage to the limit yields

$$\gamma_\alpha(x) - \gamma(x) \in W \subseteq V \quad \text{for each } x \in X \text{ and } \alpha \geq \alpha_0. \tag{25}$$

Each $x_0 \in X$ has a neighbourhood Q such that $\gamma_{\alpha_0}(x) - \gamma_{\alpha_0}(x_0) \in V$ for all x in Q and hence $\gamma(x) - \gamma(x_0) = (\gamma(x) - \gamma_{\alpha_0}(x)) + (\gamma_{\alpha_0}(x) - \gamma_{\alpha_0}(x_0)) + (\gamma_{\alpha_0}(x_0) - \gamma(x_0)) \in U$. Thus γ is continuous at x_0 and hence continuous. To see that $\gamma(X)$ is bounded, let $T \subseteq S$ be a 0-neighbourhood such that $T\gamma_{\alpha_0}(X) \subseteq V$. Then $t\gamma(x) = t\gamma_{\alpha_0}(x) + t(\gamma(x) - \gamma_{\alpha_0}(x)) \in V + SW \subseteq U$ for each $x \in X$ and $t \in T$, whence $T\gamma(X) \subseteq U$. Thus $\gamma \in BC(X, E)$. Since $\gamma_\alpha - \gamma \in \lfloor X, U\rfloor$ for all $\alpha \geq \alpha_0$, we see that $\gamma_\alpha \to \gamma$.

(c) It is obvious that θ is linear, and it is a topological embedding by definition of the topology on $BC^k(U, E)$. To see that the image is closed, let $(\gamma_\alpha)_\alpha$ be a net in $BC^k(U, E)$ such that $\theta(\gamma_\alpha) \to \eta$ for some $\eta = (\eta_j) \in \prod_j BC(U^{j+1}, E)$. We claim that $\gamma := \eta_0 \in BC^k(U, E)$ and $\theta(\gamma) = \eta$. This will be the case if

$$\eta_{j+1}(x_0, \ldots, x_{j+1}) = \frac{\eta_j(x_0, x_1, \ldots, x_j) - \eta_j(x_{j+1}, x_1, \ldots, x_j)}{x_0 - x_{j+1}} \tag{26}$$

for each $j \in \mathbb{N}_0$ with $j < k$ and each $(x_0, \ldots, x_{j+1}) \in U^{j+2}$ with $x_0 \neq x_{j+1}$. To prove (26), we use the fact that the jth component $\theta_j(\gamma_\alpha)$ converges to η_j, and the continuity of the point evaluation $\varepsilon_1 \colon BC(U^{j+1}, E) \to E$, $\zeta \mapsto \zeta(x_0, \ldots, x_j)$, the point evaluation $\varepsilon_2 \colon BC(U^{j+1}, E) \to E$ at $(x_{j+1}, x_1, \ldots, x_j)$ and the point evaluation $\varepsilon_3 \colon BC(U^{j+2}, E) \to E$ at $(x_0, \ldots, x_{j+1})$. Since

$$\varepsilon_3(\theta_{j+1}(\gamma_\alpha)) = \gamma_\alpha^{<j+1>}(x_0, \ldots, x_{j+1}) = \frac{\varepsilon_1(\theta_j(\gamma_\alpha)) - \varepsilon_2(\theta_j(\gamma_\alpha))}{x_0 - x_{j+1}},$$

passing to the limit we obtain (26).

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(d) It is clear from (c) that $BC^k(U, E)$ is a topological $\mathbb{K}$-vector space. If E is complete, then also $BC^k(U, E)$ is complete as it is isomorphic to a closed vector subspace of a complete topological vector space by (b) and (c). □

Proof of Lemma 1.11. It is clear that the map $\gamma \mapsto \|\gamma^{<j>}\|_{q,\infty}$ is positively homogeneous. Given $r > 0$, pick $s \in]0, r[$. Then the set $V := B^q_s(0)$ is a 0-neighbourhood in E and thus $\lfloor U^{j+1}, V \rfloor$ is a 0-neighbourhood in $BC(U^{j+1}, E)$, entailing that $W := \{\gamma \in BC^k(U, E) : \gamma^{<j>} \in \lfloor U^{j+1}, V \rfloor\}$ is a 0-neighbourhood in $BC^k(U, E)$. As $W \subseteq \{\gamma \in BC^k(U, E) : \|\gamma^{<j>}\|_{q,\infty} \leq s\}$, we see that also the set $\{\gamma \in BC^k(U, E) : \|\gamma^{<j>}\|_{q,\infty} < r\}$ is a 0-neighbourhood. Hence the mappings in contention are gauges.

To see that a fundamental system of gauges is obtained, we use the fact that each 0-neighbourhood in $BC^k(U, E)$ contains a finite intersection of sets of the form

$$W := \{\gamma \in BC^k(U, E) : \gamma^{<j>} \in \lfloor U^{j+1}, V \rfloor\},$$

where $j \in \mathbb{N}_0$ with $j \leq k$ and $V \subseteq E$ is a 0-neighbourhood. There are $r_1, \ldots, r_n > 0$ and $q_1, \ldots, q_n \in \Gamma$ such that $\bigcap_{i=1}^n B^{q_i}_{r_i}(0) \subseteq V$. Consider the gauges $p_i : \gamma \mapsto \|\gamma^{<j>}\|_{q_i,\infty}$ for $i \in \{1, \ldots, n\}$. Then $\bigcap_{i=1}^n B^{p_i}_{r_i}(0) \subseteq W$. □

Proof of Lemma 1.13. (a) A trivial induction on $j \in \{0, 1, \ldots, k\}$ gives $\eta^{<j>}(x_0, \ldots, x_j) = \gamma^{<j>}(x_0 + t_0, \ldots, x_j + t_0)$ for all $(x_0, \ldots, x_j) \in (U - t_0)^{j+1}$. Now $\|\eta^{<k>}\|_{q,\infty} = \|\gamma^{<k>}\|_{q,\infty}$ is an immediate consequence.

(b) A trivial induction on $j \in \{0, 1, \ldots, k\}$ shows that $\eta^{<j>}(x_0, \ldots, x_j) = a^j \gamma^{<j>}(ax_0, \ldots, ax_j)$ for all $(x_0, \ldots, x_j) \in (a^{-1}U)^{j+1}$. Now $\|\eta^{<k>}\|_{q,\infty} = |a|^k \|\gamma^{<k>}\|_{q,\infty}$ is an immediate consequence.

(c) A trivial induction on $j \in \{0, 1, \ldots, k\}$ shows that $\gamma|_V \in BC^j(V, E)$ and $(\gamma|_V)^{<j>} = \gamma^{<j>}|_{V^{j+1}}$. Now $\|(\gamma|_V)^{<k>}\|_{q,\infty} \leq \|\gamma^{<k>}\|_{q,\infty}$ is immediate. □

Proof of Lemma 1.14. If γ is C^k in our sense, then it is easy to show by induction on j that γ is C^j in the usual sense for each $j \in \{0, \ldots, k\}$, with

$$\gamma^{(j)}(x) = j! \gamma^{<j>}(x, \ldots, x) \quad \text{for all } x \in I \tag{27}$$

(cf. [**3**, Proposition 6.2] and [**21**, §29]). Furthermore, it is clear from the preceding formula that $\gamma^{(j)}(I)$ is bounded because $\gamma^{<j>}(I^{j+1})$ is bounded.

Conversely, assume that γ is continuous and assume that the derivatives $\gamma', \gamma'', \ldots, \gamma^{(k)}$ exist and are continuous. Then we have, for all $x_0 \neq x_1$ in I,

$$\frac{\gamma(x_1) - \gamma(x_0)}{x_1 - x_0} = \int_{t_1=0}^{1} \gamma'(x_0 + t_1(x_1 - x_0))\, dt_1$$

by the Fundamental Theorem of Calculus for curves in locally convex spaces (see, e.g., [**14**, Chapter 1]). Hence

$$\gamma^{<1>} : I^2 \to E, \quad (x_0, x_1) \mapsto \int_{t_1=0}^{1} \gamma'((1 - t_1)x_0 + t_1 x_1)\, dt_1$$

is an extension to the difference quotient map, which is continuous by the theorem on parameter-dependence of weak integrals (see [**14**, Chapter 1]. Thus γ is C^1 in the sense of Definition 1.1. Iterating the argument, we find that γ is C^j for each $j \in \{1, \ldots, k\}$

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and $\gamma^{<j>}(x_0, x_1, \ldots, x_j)$ is given by

$$\int_{t_1=0}^{1} \cdots \int_{t_j=0}^{1} (1-t_{j-1}) \cdots (1-t_1) \cdot \\ \gamma^{(j)}(t_1 x_1 + (1-t_1)t_2 x_2 + \cdots + (1-t_1)(1-t_2)\cdots(1-t_{j-1})t_j x_j \\ + (1-t_1)\cdots(1-t_j)x_0)\, dt_j \ldots dt_1. \tag{28}$$

If $\gamma^{(j)}$ is bounded, then so is $\gamma^{<j>}$ by the preceding formula, with

$$\|\gamma^{<j>}\|_{q,\infty} \;\leq\; \|\gamma^{(j)}\|_{q,\infty} \tag{29}$$

for each continuous seminorm q on E. This completes the proof. □

REMARK. By (27) and (29), the seminorms $\gamma \mapsto \|\gamma^{<j>}\|_{q,\infty}$ (for $j \leq k$ and q in the set of continuous seminorms on E) define the same vector topology on $BC^k(U, E)$ as the seminorms $\gamma \mapsto \|\gamma^{(j)}\|_{q,\infty}$ ordinarily used on this space.

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