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Characterisation Results for Steiner Triple Systems and Their Application to Edge-Colourings of Cubic Graphs

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Abstract. It is known that a Steiner triple system is projective if and only if it does not contain the four-triple configuration C_{14}. We find three configurations such that a Steiner triple system is affine if and only if it does not contain one of these configurations. Similarly, we characterise Hall triple systems using two forbidden configurations.

Our characterisations have several interesting corollaries in the area of edge-colourings of graphs. A cubic graph G is S-edge-colourable for a Steiner triple system S if its edges can be coloured with points of S in such a way that the points assigned to three edges sharing a vertex form a triple in S. Among others, we show that all cubic graphs are S-edge-colourable for every non-projective non-affine point-transitive Steiner triple system S.

1 Introduction

Steiner triple systems form a classical notion in combinatorial design theory. Recall that a Steiner triple system S is formed by n points and several triples such that every two distinct points are contained in exactly one common triple. Steiner triple systems are simply-defined though complex and diverse combinatorial designs. A classical result asserts the existence of a Steiner triple system with n points whenever $n = 1, 3 \pmod 6$, $n \geq 3$. The number of results on Steiner triple systems is enormous, and a separate monograph on the topic has recently appeared [2].

There are several prominent classes of Steiner triple systems. Among the most important ones are projective and affine Steiner triple systems. The *projective Steiner triple system* $\mathrm{PG}(d,2)$ is the Steiner triple system with $2^{d+1}-1$ points corresponding to non-zero $(d+1)$-dimensional vectors over $\mathbb{Z}_2$ for $d \geq 1$. Three such vectors form a triple of $\mathrm{PG}(d,2)$ if they sum to the zero vector. The smallest Steiner triple system is the projective system $\mathrm{PG}(1,2)$, comprised of three points forming a single triple. It is referred to as the *trivial* Steiner triple system, while larger Steiner triple systems are called *non-trivial.* The smallest non-trivial projective Steiner triple system is $\mathrm{PG}(2,2)$, the Fano plane, which is denoted by S_7. An *affine Steiner triple system* $\mathrm{AG}(d,3)$ is the Steiner triple system with 3^d points corresponding to d-dimensional vectors over $\mathbb{Z}_3$ for $d \geq 1$. Three such vectors form a triple of $\mathrm{AG}(d,3)$ if they sum to the zero vector. The smallest affine Steiner triple system, $\mathrm{AG}(1,3)$, is isomorphic to the trivial Steiner

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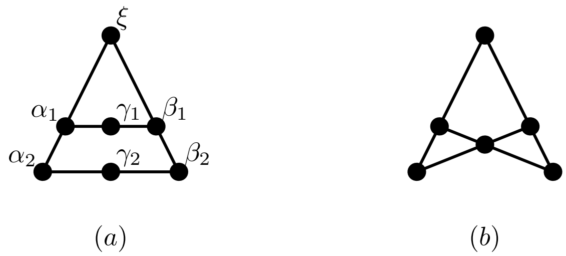

Figure 1: (a) The configuration C_{14} and (b) the Pasch configuration C_{16}.

triple system. The smallest non-trivial affine Steiner triple system, AG(2, 3), is the unique Steiner triple system with nine points and is denoted by S_9.

It is natural to ask whether these two classes of Steiner triple systems can be characterised in terms of well-described forbidden substructures (as for instance, it is known that a graph is planar if and only if it does not contain a subdivision of one of the graphs $K_{3,3}$ or K_5). To be more precise, a *configuration* C is formed by points and triples such that each pair of points is in at most one of the triples, and a Steiner triple system S *contains* C if there is an injective mapping of the points of C to the points of S such that triples of points of C are mapped to triples of points of S.

The solution to the just stated problem is easy to find for the class of projective Steiner triple systems. Two four-triple configurations play important roles in this characterisation: the first one is the *configuration* C_{14} formed by seven distinct points $\xi, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$, and γ_2 together with four triples $\{\xi, \alpha_1, \alpha_2\}$, $\{\xi, \beta_1, \beta_2\}$, $\{\alpha_1, \beta_1, \gamma_1\}$, and $\{\alpha_2, \beta_2, \gamma_2\}$; see Figure 1($a$). The second one is the *configuration* C_{16}, known as the *Pasch configuration*, which is formed by six points and four triples; see Figure 1(b).

Clearly, the configuration C_{14} cannot be contained in a projective Steiner triple system. The converse is also true: Stinson and Wei established that a Steiner triple system S with n points is projective if and only if it contains $\frac{1}{24}n(n-1)(n-3)$ distinct copies of the Pasch configuration [11]. By a counting argument given in [5], if S contains fewer than $\frac{1}{24}n(n-1)(n-3)$ copies of the Pasch configuration, then it must contain a configuration isomorphic to C_{14}. We state this observation as a separate theorem.

Theorem 1.1 **(Grannell et al. [5] and Stinson et al. [11])** *A Steiner triple system S is projective if and only if it contains no configuration C_{14}.*

However, we were not able to find such a simple argument characterising affine Steiner triple systems in the literature. In Section 5, we show that a Steiner triple system is affine unless it contains one of the three configurations depicted in Figure 2, namely the Pasch configuration C_{16}, the *configuration* C_S^1, and the *configuration* C_S^2 (the last two configurations are obtained from the squashed square configuration C_S introduced in Section 2).

Still a finer distinction between affine and non-affine Steiner triple systems can be achieved. Hall triple systems are a prominent class of Steiner triple systems: a

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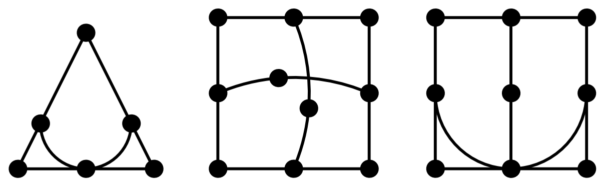

Figure 2: The three forbidden configurations for affine Steiner triple systems: the configurations C_{16}, C_S^1, and C_S^2 (from left to right).

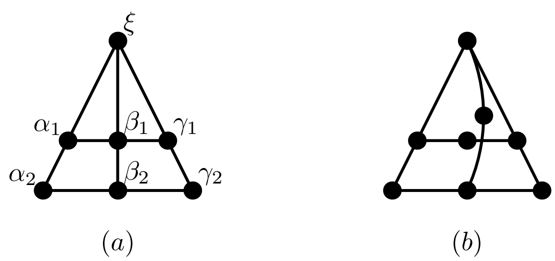

Figure 3: (a) The mitre configuration and (b) the anti-mitre configuration C_A.

Steiner triple system S is a *Hall triple system* if for every point x of S, there exists an automorphism of S that is involution and its only fixed point is x. Hall [6] showed that a Steiner triple system is a Hall triple system if and only if every Steiner triple system induced by the points of two non-disjoint triples of S is isomorphic to S_9. Recall that the Steiner triple system *induced* by a set X of the points of S is the smallest Steiner triple system S' such that S' contains all the points of X and all triples of S' are also triples of S. Hence, Hall triple systems look "locally" like affine Steiner triple systems, and it can seem hard to distinguish these two classes in terms of forbidden substructures. Also note that there are examples of Hall triple systems that are not affine Steiner triple systems for every $n = 3^d$, $d \geq 4$, unlike in the case of projective Steiner triple systems: if every two non-disjoint triples of S induce a Steiner triple system isomorphic to S_7, then S must be projective.

It is known that an n-point Steiner triple system is a Hall triple system if and only if it contains $\frac{n(n-1)(n-3)}{12}$ configurations isomorphic to the mitre configuration [2] which is depicted in Figure 3(a) (the points are labelled for future references). Since no Steiner triple system can contain more than $\frac{n(n-1)(n-3)}{12}$ copies of the mitre configuration, Hall triple systems are those which contains the largest number of configurations isomorphic to the mitre configuration.

Let us turn back to our results. As we mentioned, a Steiner triple system S is affine if and only if it contains none of the configurations C_{16}, C_S^1, and C_S^2. No Hall triple system can contain the configuration C_{16} or the anti-mitre configuration C_A (see Figure 3(b)), as neither of them is contained in the Steiner triple system S_9. The converse is also true as we show in Section 5: a Steiner triple system is a Hall triple system if and only if it does not contain the configuration C_{16} or C_A, and a Hall triple system is affine if and only if it contains neither the configuration C_S^1 nor C_S^2. With the additional assumption that a Steiner triple system S is not projective, we can remove the configuration C_{16} from the list of forbidden configurations both for affine Steiner triple systems and Hall triple systems. We also note that every Steiner triple system containing the configuration C_A also contains one of the configurations C_S^1 and C_S^2 (this seems to be a non-trivial fact which we state as Lemma 5.1).

1.1 Edge-Colourings of Cubic Graphs

Edge-colourings of cubic (bridgeless) graphs form a prominent topic in graph theory because of their close relation to deep and important problems such as the four colour theorem, the cycle double cover conjecture and many others. By Vizing's theorem, the edges of every cubic graph can be coloured with three or four colours in such a way that the edges meeting at the same vertex receive distinct colours [13]. Nontrivially connected cubic graphs (usually the cyclic 4-edge-connectivity is required) such that their edges cannot be coloured with three colours are called *snarks*.

Archdeacon proposed studying edge-colourings of cubic graphs by the points of Steiner triple systems [1]. Steiner triple systems seem to be general enough to "edge-colour" most cubic graphs and still well-structured enough to provide us with new results on cubic graphs. In particular, this notion extends the notion of the ordinary edge-colourings among the lines of well-studied locally injective homomorphisms.

The points of a Steiner triple system S are assigned to the edges of a cubic graph G in such a way that the edges incident with the same vertex are assigned three distinct points that form a triple of S. Edge-colourings with this property are called *S-edge-colourings* and G is said to be *S-edge-colourable*. A natural question is for which cubic graphs G and which Steiner triple systems S, there exists an S-edge-colouring of G. In particular, whether there exists a Steiner triple system S such that every simple cubic graph (bridgeless or not) is S-edge-colourable; such a system S is called *universal*.

Grannell et al. established the existence of a universal Steiner triple system with 381 points [4]. Later, Pál and Škoviera showed that there exists a universal Steiner triple system with 21 points (the system they considered is the direct product of the Fano plane and the trivial Steiner triple system) [10]. One of the corollaries of our results is the existence of a universal Steiner triple system with 13 points. Let us note that no Steiner triple system with fewer than 13 points can be universal [7].

We now survey further results on edge-colourings of cubic graphs with points of Steiner triple systems. Let us emphasise that all cubic graphs that we consider in this paper are connected and they can contain bridges unless stated otherwise. On the other hand, they never contain loops or parallel edges. This assumption does not decrease the generality of our results: a cubic graph with a loop does not have an S-edge-colouring for any Steiner triple system S since the points assigned to the edges incident with the vertex with the loop cannot form a triple in S. If a cubic graph G contains a pair of parallel edges e_1 and e_2 between two vertices v_1 and v_2, let G' be the graph obtained from G by removing e_1 and e_2 and identifying the other edges incident with v_1 and v_2. It is straightforward to verify that G is S-edge-colourable if and only if G' is. Hence, we can subsequently eliminate the pairs of all parallel edges in a cubic graph. Note that loops can appear during this elimination process, yielding that the original graph is not S-edge-colourable for any Steiner triple system S.

Fu showed that every bridgeless cubic graph of order at most 189 and of genus at most 24 is S_7-edge-colourable [3]. A stronger result was obtained by Holroyd and Škoviera who showed that a cubic graph G is S-edge-colourable for a non-trivial projective Steiner triple system S if and only if it is bridgeless [7]. In particular, all bridgeless cubic graphs are S_7-edge-colourable. The condition on G being bridgeless can be easily seen to be necessary since an edge-colouring of G with the points of a projective Steiner triple system $\mathrm{PG}(d,2)$ can be viewed as a nowhere-zero flow over $\mathbb{Z}_2^{d+1}$. It is well known that a graph has a nowhere-zero flow if and only if it is bridgeless.

A characterisation of cubic graphs that are S-edge-colourable for a non-projective Steiner triple system S has been offered as a conjecture (see Conjecture 1). One of the obstacles for the existence of an S-edge-colouring is the notion of a bipartite end which we now introduce. If a cubic graph G has bridges, it can be split along its bridges into 2-connected *blocks*, each incident with one or more bridges. Each bridge is split into two *half-edges*, each half-edge incident with one of the blocks. Note that some blocks can be formed by a single vertex incident with three half-edges; such blocks are called *trivial*. A block incident with a single half-edge is called an *end*. Let H be an end of a cubic graph and H' the graph obtained from H by suppressing the vertex incident with the half-edge. Hence, H' is a bridgeless cubic graph which can contain a single pair of parallel edges (one of those is the contracted edge). We say that H is a *bipartite end* if the graph H' is bipartite, H is a 3-*edge-colourable end* if H' is 3-edge-colourable (in the usual sense), and H is *hamiltonian* if H' has a Hamilton cycle avoiding the edge obtained by suppressing the vertex incident with the half-edge.

We can now state the conjectured characterisation of cubic graphs that are S-edge-colourable with a Steiner triple system S.

Conjecture 1 **(Holroyd and Škoviera [7, Conjecture 1.4])** *Let S be a non-projective Steiner triple system. A cubic graph G is S-edge-colourable unless G has a bipartite end and S is affine.*

If a cubic graph G has a bipartite end H and a Steiner triple system S is affine, an easy linear algebra argument yields that the two edges of H incident with the bridge must be coloured with the same point of S [7]. Hence, G cannot be S-edge-coloured. The conjecture of Holroyd and Škoviera asserts this to be the only obstacle for the existence of an S-edge-colouring unless S is projective. A counterexample to Conjecture 1 based on altering a projective Steiner triple system has been found very recently by Griggs and Macajova.[1] Its structure suggests that perhaps Conjecture 1 is not far from the truth, since the constructed counterexample is obtained through a simple modification of a projective Steiner triple system (an exceptional system in the conjecture).

[1] Personal communication.

As a corollary of our characterisation results on Steiner triple systems, we can show that Conjecture 1 is true when restricted to point-transitive Steiner triple systems. A Steiner triple system is *point-transitive* if for every two points x and y of S, there exists an automorphism of S that maps x to y. In fact, for non-trivial point-transitive Steiner triple systems S, we can characterise cubic graphs G that are S-edge-colourable: if S is projective, G is S-edge-colourable if and only if G is bridgeless (this follows from the results of [7]). If S is affine, G is S-edge-colourable if and only if G has no bipartite end (this solves an open problem from [7, 10] whether every cubic graph with no bipartite end is $\mathrm{AG}(2,3)$-edge-colourable). Finally, if S is a non-trivial point-transitive Steiner triple system that is neither projective nor affine, then G is always S-edge-colourable.

Since there exists a point-transitive Steiner triple system with 13 vertices that is neither projective nor affine, we can infer from our results the existence of a universal 13-point Steiner triple system. Since the only smaller Steiner triple systems are the trivial Steiner triple system, the projective Steiner triple system $S_7 = \mathrm{PG}(2,2)$ and the affine Steiner triple system $S_9 = \mathrm{AG}(2,3)$, the point-transitive Steiner triple system with 13 points is the universal Steiner triple system with the smallest number of points (this solves an open problem from [4] to determine the number of points of the smallest universal Steiner triple system).

2 Notation

In this section, we introduce some additional notation related to Steiner triple systems. If S is a Steiner triple system, we find it convenient to have a special notation for the point z forming a triple with two given distinct points x and y: such point z is denoted by $x \oplus y$ throughout the paper. For instance, in C_{14} depicted in Figure 1(a), $\xi = \beta_1 \oplus \beta_2$. Note that the operation $\oplus$ is commutative and need not be associative, *i.e.*, the points $x_1 \oplus (x_2 \oplus x_3)$ and $(x_1 \oplus x_2) \oplus x_3$ could be distinct.

The set X of points of a Steiner triple system S is said to be *independent* if for every $x \in X$, the Steiner triple system S' induced by $X \setminus \{x\}$ in S does not contain x. For instance, a set X of points of an affine Steiner triple system is independent if and only if X is affinely independent over $\mathbb{Z}_3$.

We now introduce another notion of containment of configurations in Steiner triple systems and relate it to the standard notion. Let C be a configuration, as defined earlier, with a distinguished pair (a, b) of its points. As an example, consider the squashed square configuration C_S depicted in Figure 4 with $a = x_{\alpha\beta}$ and $b = x_{\gamma\delta}$. We say that a Steiner triple system S *homomorphically contains* the configuration C if there exists a mapping φ of points of C to the points of S such that every triple of C is injectively mapped onto a triple of S and $\varphi(a) \neq \varphi(b)$. Note that we do not require the mapping φ to be injective but we require that no two points of the same triple of C are mapped to the same point of S and that the points a and b are mapped to distinct points of S.

In Sections 3 and 4, when proving our characterisation theorems, it will be more convenient to show that a given Steiner triple system homomorphically contains C_S, rather than that it contains one of the configurations C_S^1 and C_S^2. Let us realise that the two statements are equivalent.

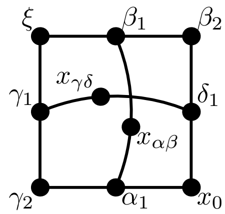

Figure 4: The squashed square configuration C_S. Note that it is required that $\varphi(x_{\alpha\beta}) \neq \varphi(x_{\gamma\delta})$, but the other pairs of points can coincide in a Steiner triple system.

Lemma 2.1 *A Steiner triple system S homomorphically contains the squashed square configuration C_S if and only if S contains the configuration C_S^1 or the configuration C_S^2.*

Proof If S contains the configuration C_S^1 or the configuration C_S^2, then S homomorphically contains C_S. In the rest, we focus on proving the converse implication. Let X be the set of points of the squashed square configuration C_S; the points of C_S are denoted as in Figure 4. Assume that S homomorphically contains the squashed square configuration C_S and let φ be the mapping from X to the points of C_S as in the definition of homomorphical containment.

We first show that $\varphi(\xi)$ is distinct from $\varphi(x)$ for every $x \in X \setminus \{\xi\}$. By symmetry, it is enough to consider that $\varphi(\xi)$ would be equal to $\varphi(\alpha_1)$, $\varphi(x_0)$ or $\varphi(x_{\alpha\beta})$ and obtain a contradiction. If $\varphi(\xi) = \varphi(\alpha_1)$, then $\varphi(x_0) = \varphi(\gamma_1)$, since $\gamma_2 = \xi \oplus \gamma_1 = \alpha_1 \oplus x_0$ (in more detail, since $\varphi(\xi) = \varphi(\alpha_1)$ and S contains the triples $\{\varphi(\gamma_2), \varphi(\xi), \varphi(\alpha_1)\}$ and $\{\varphi(\gamma_2), \varphi(\alpha_1), \varphi(x_0)\}$, it must also hold that $\varphi(\alpha_1) = \varphi(x_0)$). Similarly, the equality $\beta_1 = \xi \oplus \beta_2 = \alpha_1 \oplus x_{\alpha\beta}$ implies that $\varphi(\beta_2) = \varphi(x_{\alpha\beta})$, and $\delta_1 = x_0 \oplus \beta_2 = \gamma_1 \oplus x_{\gamma\delta}$ yields that $\varphi(\beta_2) = \varphi(x_{\gamma\delta})$. We infer that $\varphi(x_{\alpha\beta}) = \varphi(x_{\gamma\delta}) = \varphi(\beta_2)$, which is impossible.

If it held $\varphi(\xi) = \varphi(x_0)$, then it would also hold that $\varphi(\alpha_1) = \varphi(\gamma_1)$ and $\varphi(\beta_1) = \varphi(\delta_1)$. Consequently, $\varphi(x_{\alpha\beta}) = \varphi(x_{\gamma\delta})$, contrary to the definition of homomorphical containment.

Finally, if $\varphi(\xi) = \varphi(x_{\alpha\beta})$, we obtain $\varphi(\alpha_1) = \varphi(\beta_2)$. From this it follows that $\varphi(x_{\alpha\beta}) = \varphi(x_{\gamma\delta})$, similarly to the case where $\varphi(\xi) = \varphi(\alpha_1)$ with β_2 playing the role of ξ.

If φ maps X to ten distinct points of S, *i.e.,* φ is injective, then S contains the configuration C_S^1. Otherwise, two of the points of X are mapped to the same point of S. As the points ξ, x_0, β_2, and γ_2 are mapped to distinct points of S, $\varphi(\alpha_1) \neq \varphi(\delta_1)$ (otherwise, $x_0 = \alpha_1 \oplus \gamma_2 = \delta_1 \oplus \beta_2$ implies that $\varphi(\beta_2) = \varphi(\gamma_2)$). By the symmetry, we can assume that $\varphi(\alpha_1) = \varphi(x_{\gamma\delta})$. It is now straightforward to check that no other two points of X can be mapped by φ to the same point of S and thus S contains the configuration C_S^2. We conclude that if S homomorphically contains the squashed-square configuration C_S, then S contains one of the configurations C_S^1 and C_S^2. ∎

3 Forbidden Configurations in Hall Triple Systems

We split the proof of our characterisation results into two parts. In this section, we deal with Hall triple systems and in the next section, we focus on non-Hall triple systems.

Lemma 3.1 *Every Hall triple system S that is not affine homomorphically contains the squashed-square configuration C_S.*

Proof Let us first introduce some additional notation that we use throughout the proof. Fix an arbitrary point of S; we refer to this point as to the *zero* and it is denoted by $\underline{0}$. The other points of S are said to be *non-zero*.

For a set M of non-zero points of S, $L(M)$ is the Steiner triple system induced by $M \cup \{\underline{0}\}$. If M contains a single point, then $L(M)$ is isomorphic to the trivial Steiner triple system. Observe that if a set M of non-zero points of S is such that $a \notin L(M \setminus \{a\})$ for every $a \in M$, then it is independent, as defined in Section 2. Since S is Hall, $L(M)$ is isomorphic to S_9 for every two-point set M such that $M \cup \{\underline{0}\}$ is independent. On the other hand, since S is not affine there exists an independent set M such that $L(M)$ is not an affine Steiner triple system. Let $M = \{e_1, \ldots, e_d\}$ be an inclusion-wise minimal independent set for which $L(M)$ is not affine. Note that $d \geq 3$.

Let $V = \mathbb{Z}_3^d$ and let V^0 be the set of the vectors of V with at least one coordinate equal to zero. Define a mapping $\phi\colon V^0 \to L(M)$ as follows: the zero-vector is mapped to $\underline{0}$ and the unit vectors are mapped to the points of M. The mapping ϕ is then extended to the remaining vectors of V^0 in such a way that whenever the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ have at least one common coordinate equal to zero and sum to zero, then $\phi(\vec{a})$, $\phi(\vec{b})$, and $\phi(\vec{c})$ form a triple in $L(M)$. Since each of the systems $L(M \setminus \{e_i\})$, $i = 1, \ldots, d$, is affine, such an extension of ϕ to V^0 is well defined and unique.

Since $L(M)$ is not affine, the mapping cannot be extended to the whole set V in such a way that each triple of vectors summing to zero is mapped to a triple of $L(M)$. There could be several obstacles for the existence of such an extension. The simplest one is that there exist three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ of V^0 such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ but the points $\phi(\vec{a})$, $\phi(\vec{b})$, and $\phi(\vec{c})$ do not form a triple in $L(M)$. By the definition of ϕ, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ cannot have a common coordinate equal to zero. Hence, there exist three distinct indices i, j, and k such that $\vec{a}_i = \vec{b}_j = \vec{c}_k = 0$ but the coordinates $\vec{a}_j$, $\vec{a}_k$, $\vec{b}_i$, $\vec{b}_k$, $\vec{c}_i$, and $\vec{c}_j$ are non-zero.

We now define four vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ based on the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. The i-th coordinates of the vectors are $\vec{x}_i = \vec{y}_i = \vec{z}_i = 0$ and $\vec{w}_i = -\vec{c}_i$; the j-th coordinates are $\vec{x}_j = \vec{w}_j = 0$, $\vec{y}_j = -\vec{a}_j$, and $\vec{z}_j = -\vec{c}_j$; and the k-th coordinates are $\vec{x}_k = \vec{y}_k = \vec{a}_k$ and $\vec{w}_k = \vec{z}_k = 0$. The remaining coordinates are set as $\vec{x}_t = -\vec{a}_t$, $\vec{y}_t = 0$, and $\vec{w}_t = \vec{z}_t = \vec{c}_t$ for $t \notin \{i, j, k\}$. Note that $\vec{a} + \vec{x} + \vec{y} = \vec{0}$ and $\vec{c} + \vec{w} + \vec{z} = \vec{0}$. In addition, the vectors $\vec{x}$ and $\vec{w}$ have a common coordinate equal to zero, and similarly the pairs of vectors $\vec{x}$ and $\vec{y}$, $\vec{y}$ and $\vec{z}$, and $\vec{w}$ and $\vec{z}$. Observe also that the j-th coordinate of $\vec{b}$ and $-(\vec{x} + \vec{w})$ is zero, and $\vec{b} + (-\vec{x} - \vec{w}) + (-\vec{y} - \vec{z}) = \vec{0}$. Since $\phi(\vec{b}) \neq \phi(\vec{a}) \oplus \phi(\vec{c})$ by our assumption, the configuration is as depicted in Figure 5(a). Hence, we have exhibited the squashed-square configuration in S.

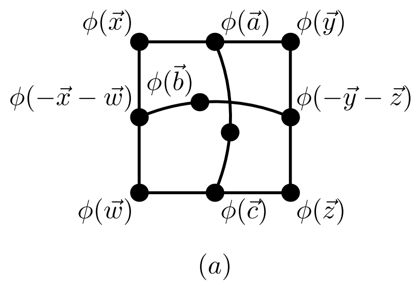

Figure 5: Two squashed-square configurations constructed in the proof of Lemma 3.1.

Assume now that for every triple $\vec{a}$, $\vec{b}$, and $\vec{c}$ of vectors of V^0 with $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ the points $\phi(\vec{a})$, $\phi(\vec{b})$, and $\phi(\vec{c})$ form a triple in $L(M)$. Our next aim is to try to extend ϕ to V by defining $\phi(\vec{a})=\phi(\vec{a}')\oplus\phi(\vec{a}'')$ for vectors $\vec{a}\in V\setminus V^0$ and $\vec{a}',\vec{a}''\in V^0$ such that $\vec{a}+\vec{a}'+\vec{a}''=\vec{0}$. This extension would be possible unless there existed four vectors $\vec{a}',\vec{a}'',\vec{b}',\vec{b}''\in V^0$ such that $\vec{a}'+\vec{a}''=\vec{b}'+\vec{b}''$ and $\phi(\vec{a}')\oplus\phi(\vec{a}'')\neq\phi(\vec{b}')\oplus\phi(\vec{b}'')$. Assume the existence of such vectors $\vec{a}'$, $\vec{a}''$, $\vec{b}'$, and $\vec{b}''$.

We next find four vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ in V^0 such that $\vec{a}'+\vec{x}+\vec{y}=\vec{0}$, $\vec{b}'+\vec{x}+\vec{z}=\vec{0}$, $\vec{a}''+\vec{w}+\vec{z}=\vec{0}$, and $\vec{b}''+\vec{y}+\vec{w}=\vec{0}$; see Figure 5($b$). Thus S contains the squashed-square configuration. Note that $\vec{a}'+\vec{a}''=\vec{b}'+b''\notin V^0$ by our assumptions. In particular, the vectors $\vec{a}'$ and $\vec{a}''$ do not have the same coordinate equal to zero and the same also holds for $\vec{b}'$ and $\vec{b}''$. If we fix arbitrarily one of the coordinates of $\vec{w}$, say $\vec{w}_i$, then the i-th coordinates of the vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ are uniquely determined: $\vec{y}_i=-(\vec{b}''_i+\vec{w}_i)$, $\vec{z}_i=-(\vec{a}''_i+\vec{w}_i)$, and $\vec{x}_i=-(\vec{a}'_i+\vec{y}_i)=-(\vec{b}'_i+\vec{z}_i)$ (the last two expressions are the same, since $\vec{a}'_i+\vec{a}''_i=\vec{b}'_i+\vec{b}''_i$). Similarly, fixing the i-th coordinate of $\vec{x}$ determines the i-th coordinate of the vectors $\vec{y}$, $\vec{z}$, and $\vec{w}$.

First we assume that the vectors $\vec{a}'$ and $\vec{b}'$ have the same coordinate equal to zero, say $\vec{a}'_i=\vec{b}'_i=0$. We set $\vec{x}_i=0$. This determines the i-th coordinates of the vectors $\vec{w}$, $\vec{y}$, and $\vec{z}$. In particular, $\vec{y}_i=0$ and $\vec{z}_i=0$. Since $\vec{a}'',\vec{b}''\in V^0$, there exist i' and i'' such that $\vec{a}''_{i'}=0$ and $\vec{b}''_{i''}=0$ (the indices i' and i'' need not be distinct). Since the vectors $\vec{a}'$ and $\vec{a}''$ do not have a common coordinate equal to zero, $i\neq i'$. Analogously, $i\neq i''$. Set $\vec{w}_{i'}=0$ and $\vec{w}_{i''}=0$. The remaining coordinates of the vector $\vec{w}$ are chosen arbitrarily. This completely determines all the vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$. All the vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ are in V^0 and they satisfy the above constraints. Analogously, we can handle the cases where the vectors $\vec{a}'$ and $\vec{b}''$, $\vec{a}''$ and $\vec{b}'$, or $\vec{a}''$ and $\vec{b}''$ have the same coordinate equal to zero.

We now assume that no pair of the vectors $\vec{a}'$, $\vec{a}''$, $\vec{b}'$, and $\vec{b}''$ have the same coordinate equal to zero. Let i', i'', j', and j'' be the indices such that $\vec{a}'_{i'}=0$, $\vec{a}''_{i''}=0$, $\vec{b}'_{j'}=0$, and $\vec{b}''_{j''}=0$. Choose $\vec{x}_{i'}=\vec{x}_{j'}=0$ and $\vec{w}_{i''}=\vec{w}_{j''}=0$. This determines the i'-th, i''-th, j'-th, and j''-th coordinate of all the vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$. The remaining coordinates of the vector $\vec{x}$ are chosen arbitrarily and the remaining coordinates of the other vectors are determined by this choice. Again, we have obtained the configuration depicted in Figure 5(b).

Hence, we have excluded the existence of the vectors $\vec{a}'$, $\vec{a}''$, $\vec{b}'$, and $\vec{b}''$. Therefore, we conclude that the mapping ϕ can be extended to $V \setminus V^0$ in such a way that $\phi(\vec{x}) = \phi(\vec{x}') \oplus \phi(\vec{x}'')$ for all vectors $\vec{x} \in V$ and $\vec{x}', \vec{x}'' \in V^0$ with $\vec{x}+\vec{x}'+\vec{x}'' = \vec{0}$. Since $L(M)$ is not affine, there must exist three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in V such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ and $\phi(\vec{c}) \neq \phi(\vec{a}) \oplus \phi(\vec{b})$ for this (uniquely defined) mapping ϕ.

If $\vec{a} = \vec{0}$, choose arbitrarily two vectors $\vec{b}'$ and $\vec{b}''$ in V^0 such that $\vec{b} + \vec{b}' + \vec{b}'' = \vec{0}$. Hence, $\vec{c} + (\vec{a} + \vec{b}') + (\vec{a} + \vec{b}'') = \vec{0}$ and both $\vec{a} + \vec{b}'$ and $\vec{a} + \vec{b}''$ are contained in V^0. Since S is Hall and $\phi(\vec{x}) = \phi(\vec{x}') \oplus \phi(\vec{x}'')$ for all vectors $\vec{x} \in V$ and $\vec{x}', \vec{x}'' \in V^0$ with $\vec{x} + \vec{x}' + \vec{x}'' = \vec{0}$, it follows that $\phi(\vec{c}) = \phi(\vec{a}) \oplus \phi(\vec{b})$, a contradiction. Hence, $\vec{a} \neq \vec{0}$.

Since the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are distinct, there exists i such that $\vec{a}_i$, $\vec{b}_i$ and $\vec{c}_i$ are distinct. By symmetry, we can assume that $\vec{a}_i = 0$. We aim to define four vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ that would form the configuration depicted in Figure 5(a). Set $\vec{x}_i = \vec{y}_i = \vec{z}_i = 0$ and $\vec{w}_i = -\vec{c}_i$. Let j be the index such that $\vec{a}_j \neq 0$ and set $\vec{x}_j = \vec{w}_j = 0$, $\vec{y}_j = -\vec{a}_j$, and $\vec{z}_j = -\vec{c}_j$. The remaining coordinates of the vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ are chosen in such a way that $\vec{a}+\vec{x}+\vec{y} = \vec{0}$ and $\vec{c}+\vec{w}+\vec{z} = \vec{0}$. Observe that $\vec{x}+\vec{w} \in V^0$ and $\vec{y}+\vec{z} \in V^0$. Hence, $\phi(\vec{b}) = \phi(-\vec{x}-\vec{w}) \oplus \phi(-\vec{y}-\vec{z})$ since $\vec{b}+(-\vec{x}-\vec{w})+(-\vec{y}-\vec{z}) = \vec{0}$. We conclude that the constructed configuration matches that depicted in Figure 5(a). This finishes the proof that if the mapping ϕ cannot be extended to the whole set V, then S homomorphically contains the squashed-square configuration. Hence, the proof of the lemma is complete. ■

4 Forbidden Configurations in Other Steiner Triple Systems

In the previous section, we have dealt with Hall triple systems, and in this section we consider non-Hall triple systems. In the next section, the results of the previous and this section are combined to obtain characterisations of affine Steiner triple systems and Hall triple systems in terms of forbidden subconfigurations.

Our aim is to show that every non-projective Steiner triple system that is not Hall contains the anti-mitre configuration C_A, depicted in Figure 3(b). As the first step, we establish that if a Steiner triple system S does not contain C_A, then every three independent points of S induce a system isomorphic to S_7 or S_9.

Lemma 4.1 *If a Steiner triple system S does not contain the anti-mitre configuration, then every three independent points of S induce a Steiner triple system isomorphic to S_7 or S_9.*

Proof Let A, B, and C be three independent points of S and let $a = B \oplus C$, $b = A \oplus C$, and $c = A \oplus B$; see Figure 6(a). Note that all the points A, B, C, a, b, and c are mutually distinct. We next consider two cases.

- **The points a, b, and c form a triple of S.**
  Let $m = A \oplus a$. If $m \oplus c$ is neither b nor C, we obtain a configuration isomorphic to the anti-mitre configuration. Since the points a, b, and c form a triple in S, $m \oplus c$ cannot be b. Hence, $m \oplus c = C$. A symmetric argument yields that $m \oplus b = B$. So the points A, B, and C induce a Steiner triple system isomorphic to S_7.

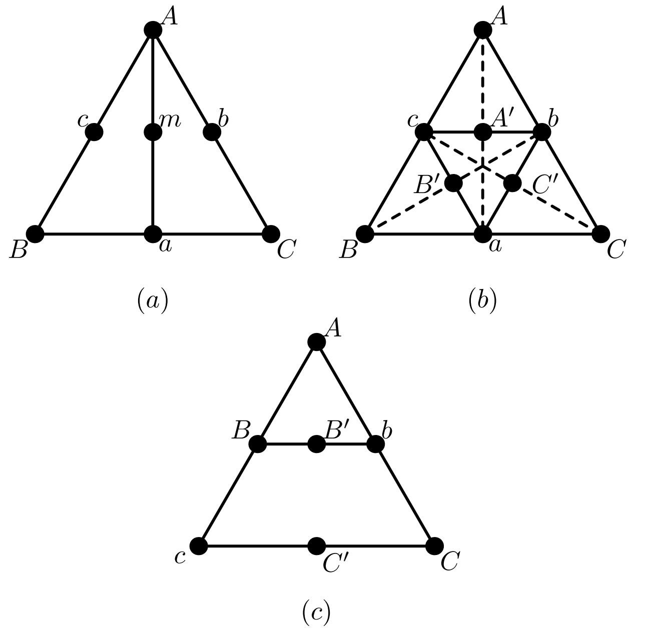

Figure 6: The configurations considered in the proof of Lemma 4.1.

- **The points a, b, and c are independent.**
  Let $A' = b \oplus c$, $B' = a \oplus c$, and $C' = a \oplus b$. Since a, b, and c are independent, all the points A, B, C, a, b, c, A', B', and C' are mutually distinct. Since S contains no anti-mitre configuration, the point A' is either a or $A \oplus a$. We have just excluded the former case. Hence, $A' = A \oplus a$. By symmetry, we deduce that $B' = B \oplus b$ and $C' = C \oplus c$.
  Therefore we obtain the configurations depicted in Figure 6(b) and (c). Again, since S does not contain the anti-mitre configuration, the point B' is equal to either C' or $A \oplus C'$. Since the points B' and C' are distinct, it holds that $B' = A \oplus C'$. By symmetry, $B = A' \oplus C'$ and $C = A' \oplus B'$. We conclude that the points A, B, C, a, b, c, A', B', and C' induce a Steiner triple system isomorphic to S_9. ■

Before going further, we state the following result obtained by Teirlinck [12].

Lemma 4.2 *Let S be a Steiner triple system. If every three independent points of S induce a system isomorphic to S_7 or S_9, then either every such triple of S induces a system isomorphic to S_7, or every such triple induces a system isomorphic to S_9.*

Using Lemmas 4.1 and 4.2, we can now show that every non-projective Steiner triple that is not Hall contains the anti-mitre configuration.

Lemma 4.3 *Every Steiner triple system S that is not projective and that is not a Hall triple system contains the anti-mitre configuration C_A.*

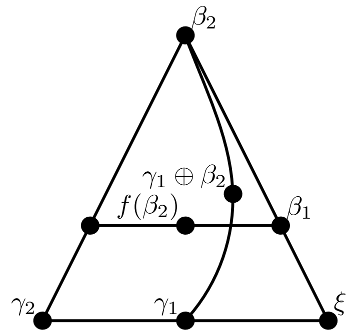

Figure 7: The notation used in the proof of Lemma 5.1.

Proof Assume for the sake of contradiction that S is neither a projective Steiner triple system nor a Hall triple system, and yet does not contain the configuration C_A. By Lemma 4.1, every three independent points induce a Steiner triple system isomorphic to S_7 or S_9. By Lemma 4.2, either all such triples induce systems isomorphic to S_7 or all such triples induce systems isomorphic to S_9. The latter is excluded by our assumption that S is not a Hall triple system. Hence, Theorem 1.1 yields that S is projective, which contradicts our original assumptions on S. ∎

5 Characterisations of Affine Steiner Triple Systems and Hall Triple Systems

Before we can state our characterisation results, we have to show that every Steiner triple system containing C_A also contains one of the configurations C_S^1 and C_S^2. We prove instead that it homomorphically contains C_S which is equivalent by Lemma 2.1.

Lemma 5.1 *Every Steiner triple system S containing the anti-mitre configuration C_A also homomorphically contains the squashed-square configuration C_S.*

Proof Choose the points ξ, β_1, β_2, γ_1, and γ_2 to be the points of the anti-mitre configuration as depicted in Figure 7. Note that the five points are distinct and $\xi = \beta_1 \oplus \beta_2 = \gamma_1 \oplus \gamma_2$.

For a point z of a Steiner triple system S, let $f(z) = (z \oplus \gamma_2) \oplus \beta_1$ and $h(z) = (z \oplus \beta_2) \oplus \gamma_1$. Note that the functions f and h are not defined for all the points of S. More precisely, the function $f(z)$ is a bijection between its domain $D(f) = S \setminus \{\gamma_2, \gamma_2 \oplus \beta_1\}$ and its image set $R(f) = S \setminus \{\beta_1, \gamma_2 \oplus \beta_1\}$. Similarly, $h(z)$ is a bijection between $D(h) = S \setminus \{\beta_2, \beta_2 \oplus \gamma_1\}$ and $R(h) = S \setminus \{\gamma_1, \gamma_1 \oplus \beta_2\}$.

Observe that if there exists a point z of S such that both $f(z)$ and $h(z)$ are defined and $f(z) \neq h(z)$, we are able to construct the squashed-square configuration in S. Indeed, we set $x_0 = z$, $\alpha_1 = z \oplus \gamma_2$, $\delta_1 = z \oplus \beta_2$, $x_{\alpha\beta} = f(z) = \alpha_1 \oplus \beta_1 = (z \oplus \gamma_2) \oplus \beta_1$, and $x_{\gamma\delta} = h(z) = \delta_1 \oplus \gamma_1 = (z \oplus \beta_2) \oplus \gamma_1$; see Figure 4. Hence, in order to establish the statement of the lemma it is enough to show that there exists a point z of S such that $f(z) \neq h(z)$.

Recall that f and h are defined on all the points of S except $\beta_2, \gamma_2, \beta_1 \oplus \gamma_2$, and $\beta_2 \oplus \gamma_1$. The points β_2 and γ_2 are distinct from the remaining exceptional points, but the points $\beta_1 \oplus \gamma_2$ and $\beta_2 \oplus \gamma_1$ might be the same. Analogously, the set of possible common values of f and h are the points of S except for $\beta_1, \gamma_1, \beta_1 \oplus \gamma_2$, and $\beta_2 \oplus \gamma_1$. Again, these points are distinct with a possible exception of the pair $\beta_1 \oplus \gamma_2$ and $\beta_2 \oplus \gamma_1$. Since the pairs of points that could be the same point are the same pairs both for the domains and image sets of f and h, it holds that $|D(f) \cap D(h)| = |R(f) \cap R(h)|$.

Assume to the contrary that there is no point z such that $z \in D(f) \cap D(h)$ and $f(z) \neq h(z)$. In particular, f and h restricted to $D(f) \cap D(h)$ are the same bijective mapping onto $R(f) \cap R(h)$. Therefore, the points of $D(f) \setminus D(h)$ are mapped by f to $R(f) \setminus R(h)$. We conclude that $f(\beta_2)$ is one of the points γ_1 and $\gamma_1 \oplus \beta_2$. However, $f(\beta_2)$ cannot be either of these two points; see Figure 7. ■

We can now state the theorem characterising affine Steiner triple systems and Hall Steiner triple systems.

Theorem 5.2 *A Steiner triple system S is affine if and only if it contains none of the configurations C_{16}, C_S^1, or C_S^2. Moreover, if S is known to be non-projective, then S is affine if and only if it contains neither of the configurations C_S^1 and C_S^2.*

Similarly, S is a Hall triple system if and only if it contains neither of the configurations C_{16} and C_A, and if S is known to be non-projective, then S is Hall if and only if it does not contain the configuration C_A.

Proof It is easy to see that an affine Steiner triple system S cannot contain any of the configurations C_{16}, C_S^1, and C_S^2. Suppose now that S is not affine. If S is projective, then it contains C_{16}. If S is a Hall triple system, then it contains C_S^1 or C_S^2 by Lemmas 2.1 and 3.1. Finally, if S is not projective and not a Hall triple system, then it contains C_A by Lemma 4.3, and hence it contains one of the configurations C_S^1 or C_S^2 by Lemmas 2.1 and 5.1.

Similarly, a Hall triple system S contains neither C_{16} nor C_A as neither of these two configurations is contained in S_9. Analogously to the above argument, a projective Steiner triple system S contains the configuration C_{16}, and a non-projective Steiner triple system that is not a Hall triple system contains the configuration C_A by Lemma 4.3. ■

6 Preliminary Results on Colourings with Points of Steiner Triple Systems

We now turn our attention to applying our characterisation results to edge-colourings of cubic graphs. We start by recalling several results on colourings of cubic graphs with Steiner triple systems. As the first, a theorem on edge-colourings of bridgeless cubic graphs with the points of Steiner triple systems from [7] is stated and proved. Though the main idea follows that of [7], we provide its proof for several reasons. The most important is that the proof is later altered to obtain edge-colourings in more special scenarios, and we want to avoid extensive referral to a different paper. Another reason is that we want to present the proof using our notation. Given a subgraph C of a graph G, let G/C be the graph obtained by contracting the components

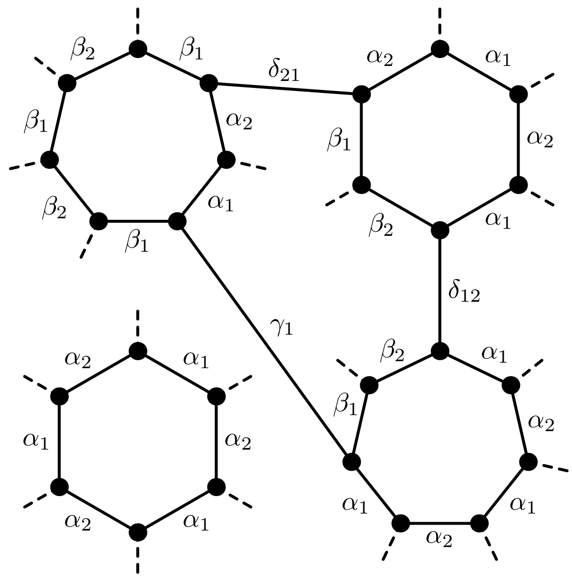

Figure 8: The colouring constructed in the proof of Theorem 6.1. The edges of $M \setminus (T_1 \cup T_2)$ are drawn as dashed and coloured with the point ξ.

of C to single vertices and removing the loops that may arise (but not the parallel edges).

Theorem 6.1 **(Holroyd and Škoviera [7])** *Let S be a non-trivial Steiner triple system. Every bridgeless cubic graph is S-edge-colourable. Moreover, for every edge e of G and every point ξ of S, there exists an S-edge-colouring that assigns ξ to e.*

Proof It can be shown using standard graph theory arguments that G contains a perfect matching M with the following properties (see [8,9] for instance): M contains the edge e and the graph $H = G/C$, where C is the complement of M in G, contains two edge-disjoint spanning forests T_1 and T_2, $e \notin T_1 \cup T_2$, such that the degree of a vertex v is odd in T_i, $i = 1, 2$, if and only if the degree of v in H is odd. Such spanning forests are called *parity forests*.

Choose an arbitrary point α_1 of S distinct from ξ and set $\alpha_2 = \xi \oplus \alpha_1$. Let β_1 be a point of S distinct from ξ, α_1, and α_2 and let $\beta_2 = \xi \oplus \beta_1$. Further, let $\gamma_i = \alpha_i \oplus \beta_i$ for $i = 1, 2$. Finally, let $\delta_{12} = \alpha_2 \oplus \beta_1$ and $\delta_{21} = \alpha_1 \oplus \beta_2$; see Figure 9.

If all the cycles of C were even, we would colour the edges of M with ξ and the edges of the cycles with α_1 and α_2 alternately. This would give us the desired colouring of the edges of G. However, some cycles of C could be odd. We cope with this in the rest of the proof.

The edges of cycles will be coloured with α's and β's in such a way that the indices of the colours alternate precisely at vertices not incident with the edges of T_1. Once we choose which edges of a cycle C_1 of C are coloured with α_1 or β_1 and which with

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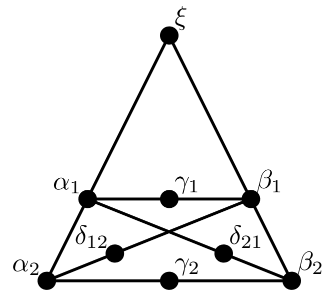

Figure 9: The notation used in the proof of Theorem 6.1.

α_2 or β_2, the points assigned to the edges of T_1 incident with C_1 are determined to be either γ_1 or γ_2 depending on the colours of the two edges incident with that edge of T_1. The choice of a colouring of a single cycle corresponding to a vertex of a component of T_1 determines the colourings of the edges of all cycles corresponding to the vertices of the same component of T_1. Since T_1 is acyclic and each cycle is incident with the number of edges of T_1 matching its parity, the edges of T_1 can be coloured with γ_1 and γ_2 in such a way that the colouring of T_1 can be extended to all cycles.

Similarly, we can partition the edges of C into two classes such that the edges of one of the classes will be coloured with α_1 or β_2, and the edges of the other class with α_2 or β_1. The classes alternate at vertices not incident with an edge of T_2 and the edges of T_2 get the colours δ_{12} and δ_{21}.

In the way just described, we have defined colourings of the edges of T_1 and T_2. The edges of M not contained in T_1 or T_2 are coloured by ξ. As discussed above, the colouring of the edges of M extends to the edges of C. Indeed, the colours of the edges of T_1 determine the indices and those of T_2 then completely fix the colours assigned to the edges of C. Since $e \in M \setminus (T_1 \cup T_2)$, the colour of e is ξ, as desired. ■

Note that in addition to prescribing the colour of e to be ξ, we can also assume in the proof of Theorem 6.1 that the colours of the edges of the even cycles corresponding to vertices that are isolated both in T_1 and T_2 are coloured with α_1 and α_2 alternately, and fix a colouring of one odd cycle.

We now sketch an alternative proof of the following result of Pál and Škoviera [10]: we focus on the aspects that are different and used in our further arguments. The proof was obtained during discussions between Zdeněk Dvořák and the first author.

Theorem 6.2 **(Pál and Škoviera [10])** *Every cubic graph with no bipartite end is* AG$(d,3)$*-edge-colourable for all* $d \geq 3$.

The graph is first split into blocks with half-edges corresponding to bridges incident with the blocks. We next construct an edge-colouring of each block. It is required that the points assigned to each triple of edges sharing the same vertex form a triple in the system. In particular, the points assigned to a half-edge and the two edges incident with it form a triple. As stated in the next lemma, it is enough to be able to construct edge-colourings of non-bipartite ends and bridgeless cubic graphs since the half-edges of non-trivial blocks incident with more bridges can be merged to obtain graphs only of these two kinds.

Lemma 6.3 *Let H be a non-trivial block of a cubic graph incident with at least two half-edges. If H is incident with an odd number of half-edges and H is not a triangle with three half-edges, then the half-edges can be identified in such a way that the resulting graph is a non-bipartite end. On the other hand, if H is incident with an even number of half-edges, then the half-edges of H can be identified in such a way that the resulting graph (after a possible successive removal of pairs of parallel edges) is either a simple bridgeless cubic graph or a graph formed by a triple of parallel edges.*

Proof If H is incident with four or more half-edges, than H contains two half-edges f_1 and f_2 which are not incident with adjacent vertices. We identify the half-edges f_1 and f_2 and obtain a block with fewer half-edges and without parallel edges that also satisfies the assumption of the lemma. Hence, we assume in the rest that the block is incident with two or three half-edges.

If H is incident with two half-edges, we identify the two half-edges and obtain a bridgeless cubic graph H', which need not be simple. We next remove pairs of parallel edges as we describe in the Introduction: remove the pair of parallel edges and identify the two edges incident with the pair of parallel edges that remain in the graph. Note that the graph is kept bridgeless and cubic in this way and thus we cannot obtain a loop. Hence, we either end with a simple bridgeless cubic graph or a triple of parallel edges as claimed in the statement of the lemma.

The case where H is incident with three half-edges is more involved. We distinguish two cases based on whether or not the vertices of H can be coloured with two colours. If the vertices can be coloured with two colours, then identify two half-edges incident with vertices of the same colour. This does not create a pair of parallel edges. In addition, the resulting graph is not a bipartite end, since we have just created an odd cycle.

If the vertices of H cannot be 2-coloured, then identifying any pair of half-edges does not create a bipartite end. Hence, we have only to avoid creating a pair of parallel edges, *i.e.,* we have to identify half-edges incident with non-adjacent vertices. This is possible unless H is a triangle with incident three half-edges. ■

Every bridgeless cubic graph as well as a trivial block or a block formed by a triangle with three incident half-edges is $\mathrm{AG}(d,3)$-edge-colourable [7]. Therefore, we only have to establish that each non-bipartite end is $\mathrm{AG}(d,3)$-edge-colourable for all $d \geq 3$ to finish the proof of Theorem 6.2. We next show that if the non-bipartite end is 3-edge-colourable, then it is indeed $\mathrm{AG}(d,3)$-edge-colourable for every $d \geq 2$.

Lemma 6.4 *Let S be a non-trivial affine Steiner triple system. Every 3-edge-colourable non-bipartite end G_0 is S-edge-colourable.*

Proof Let S be an affine Steiner triple system $\mathrm{AG}(d,3)$, $d \geq 2$. The points of S are d-dimensional vectors $(x_1,\dots,x_d)$ with $x_i \in \{0,1,2\}$. Let G be the graph obtained from G_0 by suppressing the vertex incident with the half-edge and let e be the new edge of G. For the rest of the proof, we fix a 3-edge-colouring of G.

We claim that G contains an odd cycle C_0 passing through the edge e. Since G is not bipartite, it contains an odd cycle C. If $e \in C_0$, the claim holds. Assume that $e \notin C_0$. Since G is bridgeless, there exist two edge-disjoint paths in G from the end-vertices of e to C. The two paths combine with one of the two parts of C delimited by the final vertices of the paths to an odd cycle C_0 containing the edge e.

We are now ready to define the S-edge-colouring of G_0. The first coordinate x_1 is equal to 0, 1, or 2 depending on the colour of the edge of G in the fixed colouring. Note that the first coordinates of every two incident edges, except for the two edges incident with the half-edge, are distinct. The second coordinate x_2 is equal to 0 for all the edges not contained in C_0. For the other edges which form an even cycle in G_0, the second coordinate alternates between 1 and 2. The remaining coordinates x_i, $i > 2$, are equal to 0. It is easy to verify that the defined colouring is an S-edge-colouring of G_0. ■

The proof that each non-3-edge-colourable (and thus non-bipartite) end is $\mathrm{AG}(d, 3)$-edge-colourable if $d \geq 3$ is obtained by combining arguments used in the proofs of Theorem 6.1 and Lemma 6.4. The first two coordinates are used to mimic the edge-colouring from the proof of Theorem 6.1, and the third coordinate is used to distinguish the colours assigned to the edges incident with the half-edge (in the same way as the second coordinate in the proof of Lemma 6.4). Since we do not use Theorem 6.2 in our further considerations and only use Lemmas 6.3 and 6.4, we leave to the reader the details missing to complete the proof of Theorem 6.2.

7 Non-3-Edge-Colourable Ends

By Lemma 6.3, the problem of an edge-colouring of a cubic graph can be decomposed into several problems dealing with edge-colourings of ends. In this section, we consider ends that are not 3-edge-colourable.

Lemma 7.1 *Let S be a Steiner triple system containing the configuration C_{14}. Every non-3-edge-colourable end G_0 is S-edge-colourable.*

Proof We alter the proof of Theorem 6.1. The points of C_{14} are ξ, α_i, β_i, and γ_i as in Figure 1(a). Note that this is consistent with the notation used in the proof of Theorem 6.1.

Let e_1 and e_2 be the two edges incident with the half-edge and G be the bridgeless cubic graph obtained by suppressing the vertex incident with the half-edge. Let e be the resulting edge of G. As in Theorem 6.1, we consider a perfect matching M, $e \in M$, and two disjoint parity forests T_1 and T_2 of G/C where C is the complement of M and $e \notin T_1 \cup T_2$. Let v_0 be the vertex of G/C corresponding to a cycle incident with e_1.

Since the end is not 3-edge-colourable, G/C contains a vertex of odd degree. Hence, both T_1 and T_2 contain some edges. Let $v_0 \cdots v_k$ be a shortest path in G/C from the vertex v_0 to a vertex v_k that is not isolated in T_1 or T_2. Note that k may be 0. Further, let C_i be the cycle of C corresponding to the vertex v_i, $i = 0, \ldots, k$, and let P be the path in G_0 comprised of the edge e_1, the edges corresponding to the path $v_0 \cdots v_k$ and parts of the cycles $C_0, \ldots, C_k$. By symmetry, we can assume that e_2 does

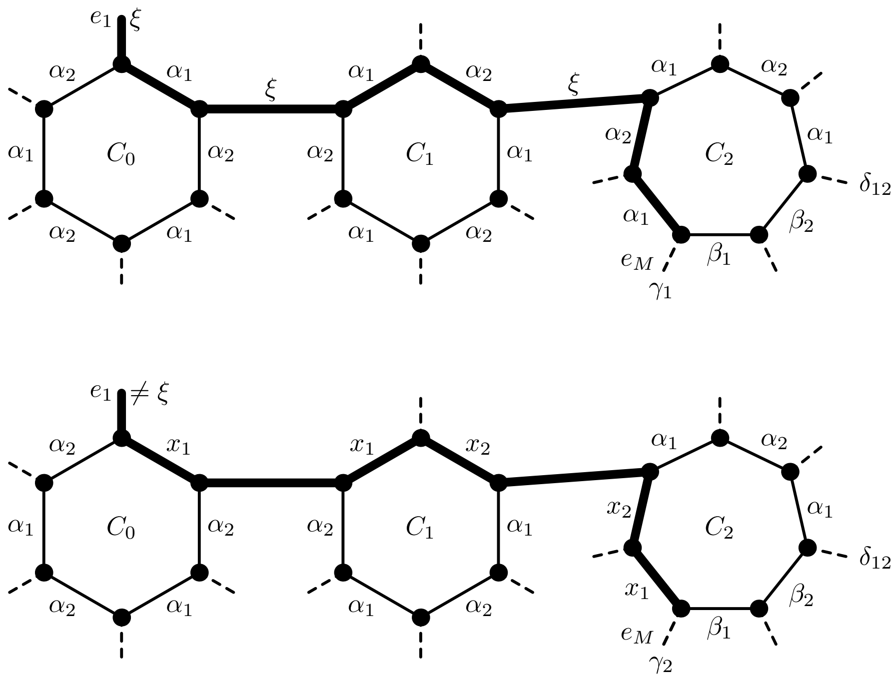

Figure 10: The modification of the colouring performed in the proof of Lemma 7.1: the original colouring is in the top and the modified in the bottom of the figure. The edges of the path P are drawn solid bold.

not belong to P, see Figure 10. The last vertex of P is the only vertex of P incident with an edge contained in $T_1 \cup T_2$.

Next, we modify the colouring obtained in the proof of Theorem 6.1. As mentioned after its proof, we can assume that the edges of $P \cap C$ are coloured with α_1 and α_2 only. In addition, we can assume that the colours of edges of $P \cap C$ incident with the same edge of $P \cap M$ are the same, *i.e.,* they both are either α_1 or α_2, the last edge of P is coloured with α_1 and the edge e_M of M incident with the last vertex of P is contained in T_1 (see the figure). In particular, the colour of e_M is γ_1. Note that all the edges of $P \cap M$ are coloured with ξ.

Let x_1 be the point $\beta_1 \oplus \gamma_2$. Note that x_1 is distinct from the points ξ, α_1, and α_2. Further, let $x_2 = \xi \oplus x_1$. Analogously, x_2 is different from α_1 and α_2. Note that $\alpha_2 + x_i \neq \xi$ for each $i \in \{1, 2\}$.

We now alter the constructed colouring. Recolour the edge e_M to γ_2 and swap the indices of the colours of the edges of the subtree of T_1 separated by e_M from v_k, *i.e.,* the edges of the subtree coloured with γ_1 are now coloured with γ_2 and vice versa. This results in a change of the colouring of the edges of C contained in the cycles incident with edges of swapped colours. By the choice of the path $v_0 \cdots v_k$, the colours of all the edges of P as well as edges incident with them, except for the edge e_M, have been preserved.

We now modify the colouring of the edges of P in such a way that the obtained colouring is an S-edge-colouring of G_0. The colours of the edges of $P \cap C$ that are originally coloured with α_i are changed to x_i both for $i = 1$ and $i = 2$. Consider now an edge $f \in P \cap M$. Note that both the ends of f are either incident with edges coloured α_1 and x_2, or with edges coloured α_2 and x_1. Hence, we can extend the colouring to the edges of $P \cap M$. Since the colour of e_1 is not equal to ξ, and the colour of e_2 is ξ, the colouring can also be extended to the half-edge of G_0 which is incident with e_1 and e_2. This finishes the proof of the lemma. ■

8 3-Edge-Colourable Ends

In Section 7, we constructed colourings of non-3-edge-colourable ends. It remains to consider 3-edge-colourable ends. It seems that the core of the problem lies in hamiltonian ends. More precisely, we first construct the desired colourings for hamiltonian ends and we later show how to reduce the problem of colouring general 3-edge-colourable ends to hamiltonian ones. The first cases we consider are hamiltonian non-bipartite ends.

Lemma 8.1 *Let S be a Steiner triple system containing the configuration C_{14}. Every hamiltonian non-bipartite end G_0 is S-edge-colourable. Moreover, there exists an S-edge-colouring such that the edges not contained in the Hamilton cycle receive at most three distinct colours.*

Proof Let G be the graph obtained from G_0 by suppressing the vertex incident with the half-edge, e the resulting edge of G, and e_1 and e_2 the two edges of G_0 incident with the half-edge. Further, let C_H be the Hamilton cycle with $e \notin C_H$ and for every chord f of C_H, let C_f be one of the cycles formed by f and a part of the cycle C_H. Since the cycles C_H and C_f form a base of the cycle space of G and G is not bipartite, one of these cycles is odd. Since G is cubic, the length of C_H is even and thus one of the cycles C_f is odd.

We first consider the case that the cycle C_e is odd (in the graph G). The edges e_1 and e_2 are coloured with γ_1 and γ_2, the other chords of C_H are coloured with ξ. The edges of one of the parts of C_H delimited by e are coloured with α_1 and α_2 and the edges of the other part with β_1 and β_2 (see Figure 11). Let us recall that the points of C_{14} are ξ, α_i, β_i, and γ_i as in Figure 1(a).

In the rest of the proof, we assume that the cycle C_e is even and consider a cycle C_f, $f \neq e$, that is odd. Before we proceed further, let us introduce some additional notation: α_1' is the point $\alpha_2 \oplus \gamma_1$, α_2' is $\xi \oplus \alpha_1'$, and ξ' is $\alpha_2' \oplus \beta_2$. Since $\alpha_1' \neq \xi$, the point α_2' is well defined. Moreover $\alpha_2' \neq \beta_2$, *i.e.*, the point ξ' is also well defined.

We now distinguish two cases based on whether the chords e and f cross. If the chords e and f cross (see Figure 12(a)), colour f with γ_1 and one of the parts of C_H delimited by f, say one incident with e_1, with α_1 and α_2 alternately. The other part of C_H is split by e_2 into two parts: the edges of one of the parts are coloured with β_1 and β_2 alternately and the edges of the other with α_1' and α_2' alternately. Finally, the edges e_1 and e_2 are coloured with ξ and ξ', and the remaining chords of C_H are coloured with ξ. It is straightforward to verify that the colouring is an S-edge-colouring. Since $\xi \neq \xi'$, the colouring can be extended to the half-edge.

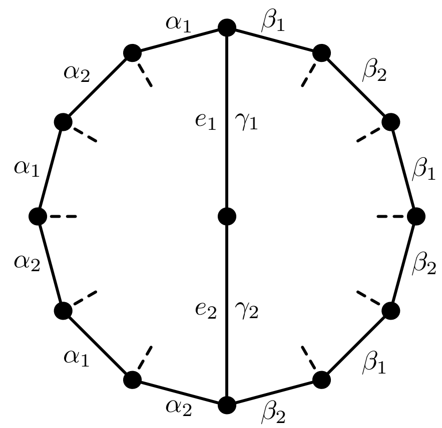

Figure 11: The colouring of G_0 constructed in the proof of Lemma 8.1 in the case where the cycle C_e is odd in G. The dashed chords are labelled with ξ.

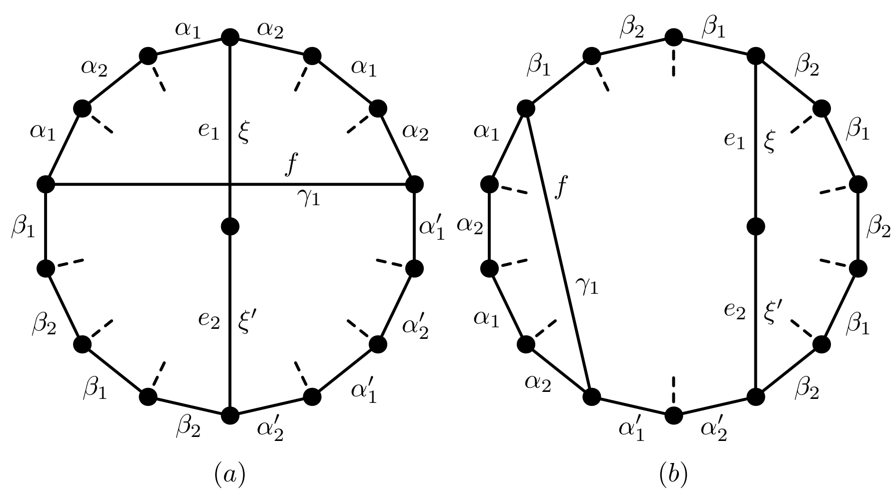

Figure 12: The colourings of G_0 constructed in the proof of Lemma 8.1 in the case where the cycle C_e is even and (a) the chords e and f cross, (b) the chords e and f do not cross. The dashed chords are labelled with ξ.

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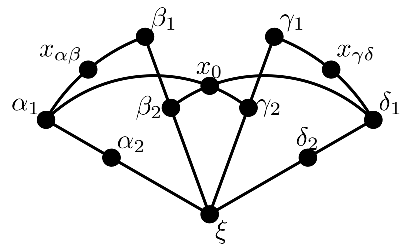


Figure 13: The squashed-square configuration C_S enhanced by two new points $\alpha_2 = \xi \oplus \alpha_1$ and $\delta_2 = \xi \oplus \delta_1$. Note that it is required that $x_{\alpha\beta} \neq x_{\gamma\delta}$ but the other pairs of points can coincide.

The final case to consider is that the chords e and f do not cross. By symmetry, we can assume that the part of C_H delimited by e_1 and f is odd and the part delimited by e_2 and f is even; see Figure 12(b). The edges of the part of C_H between f and e_1, and e_1 and e_2, are coloured with β_1 and β_2 alternately. The edge f is assigned γ_1 and the edges of the part of C_H delimited by f are assigned α_1 and α_2 alternately. The remaining edges of C_H are then coloured with α_1' and α_2'. Finally, the edge e_2 is coloured with ξ' and the remaining chords of C_H with ξ. Since $\xi \neq \xi'$, this colouring can also be extended to the half-edge.

Note that, regardless whether the cycle C_e is odd or even, there are three distinct points assigned to the edges not contained in the Hamilton cycle, namely, the points γ_1, ξ, and ξ'. ■

It remains to consider hamiltonian bipartite ends. This is the point where we will utilise our characterisation results since such ends cannot be edge-coloured by affine Steiner triple systems.

Lemma 8.2 *Let S be a Steiner triple system homomorphically containing the squashed-square configuration C_S. Every hamiltonian bipartite end G_0 is S-edge-colourable. Moreover, there exists an S-edge-colouring such that the edges not contained in the Hamilton cycle receive at most five distinct colours.*

Proof Let us enhance the squashed-square configuration C_S by introducing two new points $\alpha_2 = \xi \oplus \alpha_1$ and $\delta_2 = \xi \oplus \delta_1$; see Figure 13. Let G be the graph obtained from G_0 by suppressing the vertex incident with the half-edge, e the resulting edge of G, and e_1 and e_2 the two edges of G_0 incident with the half-edge. Further, let C_H be the Hamilton cycle of G with $e \notin C_H$.

Assume first that G contains a chord f that crosses e as depicted in Figure 14. The edge e_1 is coloured with $x_{\alpha\beta}$, e_2 with $x_{\gamma\delta}$, and f with x_0. The other chords of C_H are coloured with ξ. The edges of the parts of C_H delimited by e and f are coloured with α_1 and α_2, β_1 and β_2, γ_1 and γ_2, and δ_1 and δ_2, each part with one of the pairs of the colours alternately. Since $x_{\alpha\beta} \neq x_{\gamma\delta}$, the colouring can be extended to the half-edge. Note that only the points ξ, x_0, $x_{\alpha\beta}$, and $x_{\gamma\delta}$ are assigned to the edges not contained in the Hamilton cycle.

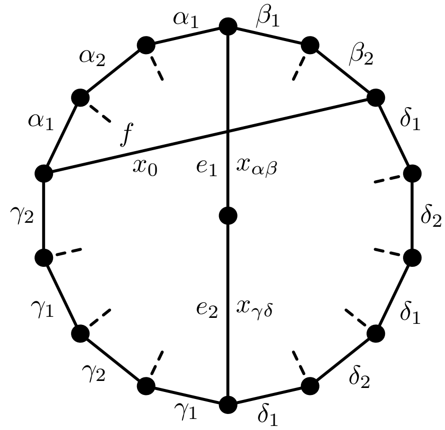


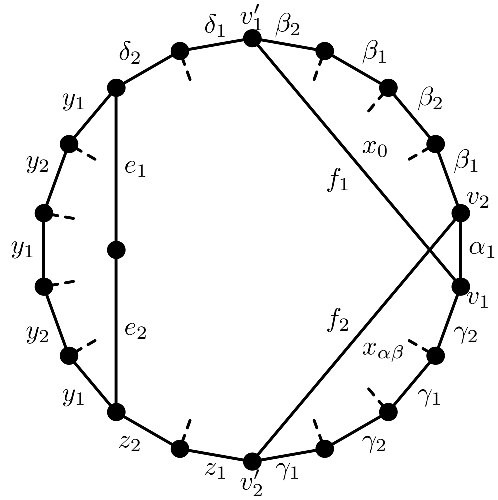
Figure 14: The colouring of G_0 constructed in the proof of Lemma 8.2 in the case where there is a chord crossing e. The dashed chords are labelled with ξ.

A more involved case is when no chord of C_H crosses e. Since G has no parallel edges (with a possible exception of e), it contains a pair of crossing chords. Each pair of crossing chords split C_H into four parts, out of which one is incident with both the end-vertices of e. Choose among all pairs of chords f_1 and f_2 the one with distance between the end-vertices of f_1 and f_2 delimiting the part opposite to the part incident with e the smallest possible. It is not hard to show that this distance must be equal to one, otherwise, there would exist a pair of crossing chords with closer end-vertices. Let v_1 and v_2 be the end-vertices of f_1 and f_2 that are adjacent and v_1' and v_2' the other end-vertices of f_1 and f_2.

Before we proceed further, we show that it can be assumed that $x_{\alpha\beta} \neq \gamma_1$ or $x_{\gamma\delta} \neq \beta_1$ (note that the two inequalities are symmetric). If $x_{\alpha\beta} = \gamma_1$ and $x_{\gamma\delta} = \beta_1$, then $\alpha_1 = \delta_1$; see Figure 13. Consequently, $\beta_2 = \gamma_2$ and $\beta_1 = \gamma_1$. However, this implies $x_{\alpha\beta} = x_{\gamma\delta}$ which is inconsistent with the definition of the configuration C_S. Hence, we can assume that $x_{\alpha\beta} \neq \gamma_1$ in the rest of the proof. Since $x_{\alpha\beta} \neq \gamma_1$, there exists a point z_1 that forms a triple with $x_{\alpha\beta}$ and γ_1. Observe that $z_1 \neq \xi$ (otherwise, $x_{\alpha\beta}$ would be equal to γ_2, x_0 would be β_1, and ξ would be δ_1, which is impossible). Consequently, there exists a point $z_2 = \xi \oplus z_1$.

We are now ready to colour the edges of G. The edge v_1v_2 is coloured with α_1, the edge f_1 with x_0, and the edge f_2 with $x_{\alpha\beta}$. The chords of C_H distinct from e_1, e_2, f_1, and f_2 are coloured with ξ. The edges of the part of C_H delimited by v_1 and v_2' are coloured with γ_1 and γ_2 alternately and the edges of the part delimited by v_2 and v_1' are coloured with β_1 and β_2 as in Figure 15. This colouring switches at the vertex incident with f_1 to an alternating colouring with δ_1 and δ_2 until it hits the end-vertex of the edge e_1. The edges of C_H between the end-vertices of e_2 and f_2 are coloured with z_1 and z_2 alternately; see Figure 15.

It remains to colour the edges of the part of C_H delimited by e and the edges e_1 and e_2 themselves. The coloured edge incident with e_1 is coloured with δ_k and the



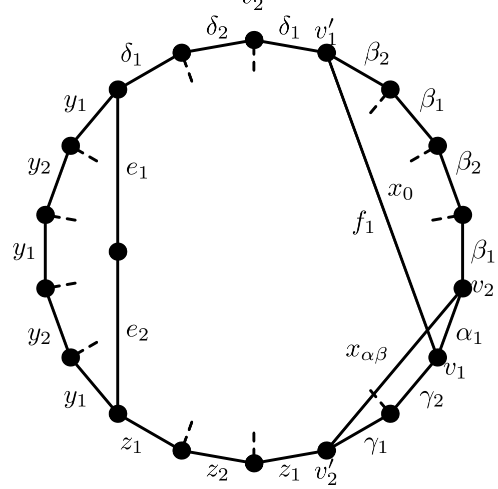


Figure 15: The colourings of G_0 constructed in the proof of Lemma 8.2 in the case where there is no chord crossing e. The dashed chords are labelled with ξ.

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coloured edge incident with e_2 is coloured with z_k (note that the indices of the two colours are the same by the parity constraints). We assert that $\delta_k \neq z_k$: if $z_1 = \delta_1$, then $x_{\alpha\beta} = x_{\gamma\delta}$ which is impossible. If $z_2 = \delta_2$, then $z_1 = \delta_1$, which we have just excluded.

Choose arbitrarily a point y_1 of S that is distinct from ξ, δ_k and z_k, and let $y_2 = \xi \oplus y_1$. The remaining edges of H_C are coloured with y_1 and y_2 alternately in such a way that the edges e_1 and e_2 are incident with edges coloured with y_1. Since $y_1 \notin \{z_k, \delta_k\}$, the colouring can be extended to e_1 and e_2. Moreover, since $z_k \neq \delta_k$, the colours of e_1 and e_2 are distinct. Hence, the colouring can also be extended to the half-edge.

It remains to verify that there are at most five points used to colour the edges not contained in the Hamilton cycle: indeed, only the points ξ, x_0, $x_{\alpha\beta}$ and the points assigned to e_1 and e_2 are such points. The proof of the lemma is now finished. ■

We have constructed colourings of hamiltonian ends. A 3-edge-colourable end does not need to be hamiltonian in general, but we can reduce the problem to hamiltonian ends as described in the next lemma.

Lemma 8.3 *Let S be a Steiner triple system containing the configuration C_{14} and homomorphically containing the squashed-square configuration C_S. Every 3-edge-colourable end G_0 is S-edge-colourable.*

Proof Let G be the graph obtained by suppressing the vertex u incident with the half-edge and $e = v_0v_1$ the resulting edge. Fix a colouring of the edges of G with three colours, say red, green and blue, such that the edge e is red and the cycle formed by green and blue edges that contains v_0 is the longest possible.

Let $w_1, \dots, w_k$ be the vertices of the cycle formed by green and blue edges that contains the vertex v_0 and assume that $w_1 = v_0$. We construct an auxiliary end H_0 with the vertices $w_0, \dots, w_k$, where w_0 is a new vertex which will be incident with the half-edge of H_0. The cycle $w_1 \cdots w_k$ is contained in H_0. In addition, the vertices w_i and w_j are joined in H_0 by an edge e_{ij} if G contains a path P_{ij} formed by red and blue edges such that w_i and w_j are the only vertices of P_{ij} contained in the cycle $w_1 \cdots w_k$. Note that for each vertex w_i, there exists a unique vertex w_j with this property. The chord incident with w_1 is replaced by a path containing the vertex w_0. Observe that the resulting graph H_0 is a hamiltonian end. In addition, H_0 has no parallel edges: if two vertices w_i and w_j adjacent in the cycle were joined by a chord, then we could assume by the symmetry that the edge w_iw_j of the cycle is blue and switch the red and blue colours on the cycle $P_{ij} \cup \{w_iw_j\}$ thereby obtaining a longer green-blue cycle with v_0. This would contradict our choice of the 3-edge-colouring.

By Lemma 8.1 or 8.2 (depending on whether H_0 is bipartite), H_0 has an S-edge-colouring such that at most five distinct points of S are used to colour the chords of the Hamilton cycle of H_0. Choose a point ξ_0 of S distinct from the five points used on edges not contained in the Hamilton cycle. In addition, choose another point α_0 distinct from ξ_0. We now construct an S-edge-colouring of the edges of G_0. The edges of the cycle $w_1 \cdots w_k$ keep their colours. The red edges of the path P_{ij} get the colour of the edge e_{ij}; the edge w_1u gets the colour of w_0w_1, and the red edges on the path from w_1 as well as the edge uv_1 get the colour of the other edge incident with

w_0. The remaining red edges of G_0 are coloured with α_0. The green edges of G_0 not contained in the cycle $w_1 \cdots w_k$ are coloured with ξ_0.

The only edges that are not yet coloured are blue edges not contained in the cycle $w_1 \cdots w_k$. Now observe that each end-vertex of every uncoloured blue edge is incident with two edges coloured with the same pair of points of S (one of these two colours being ξ_0). Hence, the colouring can be extended to all the edges of G_0. This finishes the proof of the lemma. ■

9 Characterisation of Edge-Colourability of Cubic Graphs

In Sections 7 and 8, we constructed S-edge-colourings of ends of cubic graphs. It remains to combine these colourings to a colouring of the whole cubic graph.

Theorem 9.1 *Let G be a cubic graph and S a non-trivial point-transitive Steiner triple system.*

- *If S is projective, then G is S-edge-colourable if and only if G is bridgeless.*
- *If S is affine, then G is S-edge-colourable if and only if G has no bipartite end.*
- *If S is neither projective nor affine, then G is always S-edge-colourable.*

Proof Fix a cubic graph G and a non-trivial point-transitive Steiner triple system S. If S is projective, then G is S-edge-colourable if and only if G is bridgeless [7].

Next, we assume S is affine. If G has a bipartite end, then G is not S-edge-colourable [7]. Hence, assume G has no bipartite ends. If G is bridgeless, G is S-edge-colourable since all bridgeless cubic graphs are edge-colourable by all non-trivial Steiner triple systems; see Theorem 6.1. Hence, we also assume G has one or more bridges.

Now split the graph G along its bridges into several blocks. We colour each block with its incident half-edges. Since S is affine, it contains the configuration C_{14}. Hence, all the ends can be S-edge-coloured either by Lemma 6.4 or by Lemma 7.1. Let us now consider a block incident with two or more half-edges. The trivial blocks and the blocks formed by a triangle with three half-edges are clearly S-edge-colourable. By Lemma 6.3, the half-edges of the remaining blocks of G can be identified in such a way that the resulting graph (after a possible removal of parallel edges) is a bridgeless cubic graph, a triple edge, or a non-bipartite end. All such graphs are S-edge-colourable as argued before. Since the obtained S-edge-colouring can be extended to the removed parallel edges, the original block of G is also S-edge-colourable.

It remains to combine the colourings of the blocks. After contracting the blocks of G into single vertices, the graph G becomes a tree. We further refer to the vertices as nodes in order to distinguish them from the vertices of G. Root this tree at any of its nodes. We fix the S-edge-colourings of the blocks from the root of the tree to its leaves. The block corresponding to the root keeps its original colouring. Since S is point-transitive, we can permute the colourings of the blocks corresponding to the nodes adjacent to the root (its children) in such a way that they match on the half-edges. Next, we permute the edge-colourings of the blocks corresponding to the grandchildren in such a way that they match the colours of edges leading from the blocks corresponding to the children. We proceed in this way until we obtain an S-edge-colouring of the entire graph G.

It remains to consider the case where S is neither projective nor affine. By Theorems 1.1 and 5.2, and Lemma 2.1, we deduce that S contains the configuration C_{14} as well as it homomorphically contains the squashed-square configuration C_S. If G is bridgeless, G is S-edge-colourable by Theorem 6.1. Otherwise, we also split G along its bridges to blocks. The ends can be S-edge-coloured by Lemmas 7.1 and 8.3. The trivial blocks and the blocks formed by triangles with three half-edges are S-edge-colourable from trivial reasons. The half-edges of the remaining blocks can be identified to get bridgeless graphs and (non-bipartite) ends. Hence, such blocks are also S-edge-colourable. The obtained colourings of the blocks of G can be combined to an S-edge-colouring of the whole graph G in the same way as in the case of affine Steiner triple systems. ■

As a corollary of Theorem 9.1, we derive the conjecture of Holroyd and Škoviera (Conjecture 1) for point-transitive Steiner triple systems.

Corollary 9.2 *Let G be a cubic graph and S a non-projective (and thus non-trivial) point-transitive Steiner triple system. If G has no bipartite end or S is not affine, then G is S-edge-colourable.*

10 Concluding Remarks

In the first part of the paper, we characterised several important classes of Steiner triple systems in terms of forbidden subconfigurations: we have shown that a Steiner triple system is affine if and only if it contains none of the configurations C_{16}, C_S^1, or C_S^2, and S is a Hall triple system if and only if it contains neither the configuration C_{16} nor C_A. It would be interesting to provide such characterisations of other important classes of Steiner triple systems.

Let us now turn our attention to edge-colourings of cubic graphs with elements of Steiner triple systems. Conjecture 1 is now proven for affine Steiner triple systems since all affine Steiner triple systems are point-transitive. Hence, it is only open whether every non-projective non-affine Steiner triple system that is not point-transitive is universal. The smallest such Steiner triple system has 13-vertices; let S'_{13} be this Steiner triple system. Note that there are only two Steiner triple systems with 13 vertices, namely the point-transitive system S_{13} mentioned in Section 1, and the system S'_{13}.

Along the lines of the proof of Theorem 9.1, we have tried to establish that S'_{13} is also universal. First we identified several configurations such that if all of them are present in S'_{13}, all the ends can be coloured. As in the proof of the lemmas in Sections 7 and 8, several cases based on whether the end is 3-colourable, bipartite, etc., need to be considered. Next, we prepared a computer program to check the presence of these configurations in S'_{13} with the additional requirement that one of their points, the one to be assigned to the half-edge, is a prescribed point of S'_{13}. In this way, the presence of all the needed configurations in S'_{13} with the special point being any of the points of S'_{13} was established.

Based on this, we concluded that each block of every cubic graph can be S'_{13}-edge-coloured if one of the half-edges incident with it is precoloured with a point of S'_{13}. Hence, the system S'_{13} is universal. We believe that an analogous argument can be used to establish that other small non-point-transitive Steiner triple systems are universal but we have not yet managed to develop a theoretical background that would give us any hope of extending our results to Steiner triple systems that need not be point-transitive.

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