# A NOTE ON DIRICHLET CONVOLUTIONS

S. L. Segal*

(received March 9, 1966)

In [3] Rubel proved that if $h(n)$ is an arithmetic function such that $\lim_{n\to\infty} h(n) = L$, $L$ finite, then $\sum_{n=1}^{\infty} 1/n \sum_{d|n} \mu(d)h(n/d) = L$, where $\mu(n)$ is the Möbius function. This result was extended to functions other than $\mu(n)$ in [4]; however, (as first pointed out to the author by Benjamin Volk), the order condition imposed there is unnecessary; in fact, utilizing the result of [3], the following slightly more general theorem has an almost trivial proof.

THEOREM 1. Suppose $f(n)$ is an arithmetic function such that $\sum_{n=1}^{\infty} 1/n \sum_{d|n} f(d)$ converges absolutely to $A$; and $h(n)$ any arithmetic function such that $\lim_{n\to\infty} h(n) = L$, $L$ finite, then

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} f(d)h\left(\frac{n}{d}\right) = AL . \tag{1}$$

Proof. Let $\mathcal{F}(s)$ and $\mathcal{H}(s)$ be the ordinary Dirichlet series which are generating functions for $f(n)$ and $h(n)$ respectively and let $\zeta(s)$ be the Riemann zeta-function $(s=\sigma+it)$. Then the hypothesis of the theorem is that $\mathcal{F}(s)\zeta(s)$ converges absolutely at $s = 1$ to $A$. By Rubel's result [3], $\mathcal{H}(s)/\zeta(s)$ converges at $s = 1$ to $L$. Hence by the Dirichlet-series analogue of Mertens' theorem for power-series, $\mathcal{F}(s)\mathcal{H}(s)$

---

* This note was written while the author was a Fulbright Research Fellow at the University of Vienna.



https://doi.org/10.4153/CMB-1966-055-8 Published online by Cambridge University Press

converges at $s = 1$ to $AL$, which is the theorem.

The case $A = 1$ effectively corresponds to eliminating the order condition in [4].

The question arises whether this theorem is a complete characterization of such functions; precisely if $\mathcal{F}(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $\mathcal{H}(s) = \sum_{n=1}^{\infty} h(n)n^{-s}$, and $\mathcal{F}(s)\mathcal{H}(s)$ converges at $s = 1$ whenever $\lim_{n\to\infty} h(n) = L$, $L$ finite; then does $\zeta(s)\mathcal{F}(s)$ converge absolutely at $s = 1$? It might be expected that the answer is no, even if we require that $\zeta(s)\mathcal{F}(s)$ converge to a non-zero value at $s = 1$ (clearly $\zeta(s)\mathcal{F}(s)$ converges at $s = 1$, by the case $h(n) = 1$ for all positive integers $n$). This is in fact true on the assumption of some sort of "quasi-Riemann" hypothesis that there exists a fixed $\epsilon > 0$ such that $\zeta(s) \neq 0$ for $1 - \epsilon \leqq \sigma \leqq 1$. Whether a proof for the example of Theorem 2, or for a different example, can be given which does not depend on an unproved hypothesis, is not known to us.

THEOREM 2. Let $g(n)$ be defined by $\sum_{n=1}^{\infty} g(n)n^{-s} = -\zeta'(s)/\zeta^3(s) = \mathcal{G}(s)$ where $\zeta(s)$ is the Riemann zeta-function. Suppose there exists a fixed $\epsilon > 0$ such that $\zeta(s) \neq 0$, for $1 - \epsilon \leqq \sigma \leqq 1$.

If $\lim_{n\to\infty} h(n) = L$, $L$ finite, then $\sum_{n=1}^{\infty} 1/n \sum_{d|n} g(d)h(n/d) = L$;

$\sum_{n=1}^{\infty} 1/n \sum_{d|n} g(d) = 1$, but convergence is not absolute.

<u>Proof.</u> The last clause is obvious; $\zeta(s)\mathcal{G}(s) = -\zeta'(s)/\zeta^2(s) = -\sum_{n=1}^{\infty} \mu(n) \log n \, n^{-s}$, $\sigma \geqq 1$, and by a classical result of Landau, $\sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1$, and the convergence is clearly not absolute.

To prove the theorem, it only remains to show that

$$(2) \qquad \sum_{n=1}^{\infty} \frac{g(n)}{n} = 0$$

$$(3) \qquad \sup_{r} \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{m \le r/n} \frac{g(m)}{m} \right| < \infty$$

since one then simply applies the Silverman-Toeplitz conditions to the matrix $(C_{r,n}) = \frac{1}{n} \sum_{m \le r/n} \frac{g(m)}{m}$, and in fact it suffices to prove

$$(4) \qquad \sum_{m \le t} \frac{g(m)}{m} = O\left( \frac{1}{\log^{\Delta}(t+1)} \right) \qquad \text{for some } \Delta > 1$$

(cf. [3], [4]).

Equation (4) however follows by standard complex-analytic methods (e.g. any of [1], [2], [7]), on using the $\zeta$-function hypothesis stated above. One merely applies e.g. [7] Lemma 3.12 together with the observation that

$$(5) \qquad |g(n)| = O\left( \sum_{d|n} |\mu(d)| \log d \, |\mu(n/d)| \right) = O\left( \sum_{d|n} \log d \right) = O(n^{\delta})$$

for every $\delta > 0$ to obtain,

$$(6) \qquad \sum_{n \le x} g(n) = -\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{x^s}{s} \frac{\zeta'(s)}{\zeta^3(s)} ds + O\left( \frac{x^b}{T(b-1)} \right) + O\left( \frac{x^{1+\delta} \log x}{T} \right),$$

for every fixed $b > 1$ and every $\delta > 0$ where $x$ is half an odd integer.

Noting that

$$\lim_{s \to 1} \frac{\zeta'(s)}{\zeta^3(s)} = 0;$$



https://doi.org/10.4153/CMB-1966-055-8 Published online by Cambridge University Press

moving the line of integration to the left to $\sigma = 1 - \epsilon'$, $0 < \epsilon' < \epsilon$, $\epsilon'$ fixed, where $\epsilon$ is as above; taking $\delta$ fixed, $0 < \delta < \epsilon$ and $b$ fixed $1 < b < 1 + \delta$ ; applying Cauchy's Theorem; and making the standard estimates (on the assumption of our $\zeta$ - function hypothesis); we get from (6), on taking $T = x^{\delta + \epsilon'}$, that

$$\sum_{n \leq x} g(n) = O(x^{1-\epsilon'} \log^A x)$$

where $A$ is a positive constant. A partial summation and the continuity theorem for Dirichlet series, now gives

$$\sum_{n \leq x} \frac{g(n)}{n} = O(x^{-\epsilon'} \log^A x)$$

which is considerably more than the modest requirement (4).

The difficulty in proving (4) without any unproved hypothesis lies in the use of [7], Lemma 3.12, or an equivalent estimate, and the fact that $|g(n)|$ is frequently as large as $\tau(n) = \sum_{d|n} 1$, and hence the estimate (5) cannot be substantially improved. Of course, the non-zero terms of $\sum \frac{g(n)}{n}$ have variable sign and presumably cancel each other to a certain extent. If it were possible to determine the nature of this cancellation more precisely one could perhaps prove at least the estimate (4) without using any unproved hypothesis.

If we reformulate the above results in terms of the Silverman-Toeplitz conditions for regularity of a summation method, we have the result that:

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{d|n} f(d) \right|$$

converges implies that

$$\sup_{r} \sum_{n \leq r} \frac{1}{n} \left| \sum_{m \leq r/n} \frac{f(m)}{m} \right| < \infty ,$$

but the converse (at least on the $\zeta$-function hypothesis assumed above) is false.



https://doi.org/10.4153/CMB-1966-055-8 Published online by Cambridge University Press

It is perhaps worth noting here that Rubel's result [3] also provides a Tauberian condition under which convergence can be deduced from Ingham-Wintner summability defined by:

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \sum_{d|n} d a_d .$$

(Tauberian conditions for the deduction of Ingham-Wintner summability from Abel or Cesaro summability have been discussed in [5]. The Ingham-Wintner method lies between $(C, -\delta)$ and $(C, \delta)$ but is not comparable with convergence.) In fact, application of the Möbius inversion formula, combined with [3] gives the proposition:

(*) If $$\frac{1}{x} \sum_{n\leq x} \sum_{d|n} d a_d = L + o(1/x) \text{ as } x \to \infty,$$

then $\Sigma a_n$ converges to $L$.

A similar connection exists between Theorem 1 and a subclass of the $(\mathfrak{D}, h(n))$-summability methods recently introduced [6].

## REFERENCES


1. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, reprinted with an appendix by Paul Bateman, Chelsea, (1953).

2. K. Prachar, Primzahlverteilung, Springer, Berlin, (1957).

3. L.A. Rubel, An Abelian Theorem for Number-Theoretic Sums. Acta Arith. 6, (1960) pages 175-177, correction Acta Arith. 6, (1961), page 523.

4. S.L. Segal, Dirichlet convolutions and the Silverman-Toeplitz conditions. Acta Arith. 10, (1964), pages 287-291.

5. S.L. Segal, On Ingham's Summation Method. Can. Journ. Math. 18, (1966), pages 97-105.

6. S.L. Segal, Summability by Dirichlet Convolutions. Submitted for publication.

7. E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, Oxford, (1951).

University of Rochester



https://doi.org/10.4153/CMB-1966-055-8 Published online by Cambridge University Press