SUBSPACES OF RIEMANNIAN SPACES

RICHARD BLUM

Summary. In this paper, results obtained by the author for Riemannian Spaces V_n imbedded in Euclidean Spaces E_N **(3; 4)** are extended to V_n imbedded in V_N.

The first section is introductory. In §2 the general result is obtained. This is the establishment of a certain dependency among the three basic sets of equations of the V_n with respect to the V_N, namely the equations of Gauss, Codazzi and Kuehne. In §3 it is assumed that V_N is of constant curvature with $N = n + 1$. This case is discussed with the help of a generalization of the type number τ introduced by Thomas **(10)**.

Throughout the paper the conventional tensor notation has been adopted. Capital latin indices vary from 1 to N, small latin indices from 1 to n, and small greek indices from 1 to $\nu = N - n$. Whenever an index occurs twice in an expression, the summation with respect to that index has to be performed, except when otherwise stated. This summation convention is not restricted to indices with opposite (i.e. one of covariant and the other of contravariant) character.

1. Introduction. We consider a V_N given by the positive definite metric:

$$dS^2 = A_{IJ}dX^I dX^J \quad |A_{IJ}| \neq 0 \tag{1}$$

in which the A_{IJ} are continuous functions of the X^I, having continuous partial derivatives up to the third order; and a V_n whose metric

$$ds^2 = a_{ij}dx^i dx^j \quad (n < N) \tag{2}$$

satisfies similar conditions with respect to the x^i.

A set of necessary conditions for the V_n to be imbedded[1] in the V_N is given by the following equations **(8**, no. 47**)**, known respectively as the equations of Gauss, Codazzi and Kuehne:

$$\text{(I)} \quad G_{ijkl} \equiv r_{ijkl} - (b_{\alpha|ik}b_{\alpha|jl} - b_{\alpha|il}b_{\alpha|jk}) - R_{IJKL}X^I_{,i}X^J_{,j}X^K_{,k}X^L_{,l} = 0,$$

$$\text{(II)} \quad C_{\alpha|ijk} \equiv b_{\alpha|ij,k} - b_{\alpha|ik,j} - (t_{\beta\alpha|k}b_{\beta|ij} - t_{\beta\alpha|j}b_{\beta|ik}) + R_{IJKL}\xi^I_\alpha X^J_{,i}X^K_{,j}X^L_{,k} = 0,$$

$$\text{(III)} \quad K_{\alpha\beta|ij} \equiv t_{\alpha\beta|i,j} - t_{\alpha\beta|j,i} + (t_{\gamma\alpha|i}t_{\gamma\beta|j} - t_{\gamma\alpha|j}t_{\gamma\beta|i}) + a^{kl}(b_{\alpha|ki}b_{\beta|lj} - b_{\alpha|kj}b_{\beta|li}) + R_{IJKL}\xi^I_\alpha\xi^J_\beta X^K_{,i}X^L_{,j} = 0.$$

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[1]Throughout this paper, by "imbedding" is meant *local* and *isometrical* imbedding.

Here, r_{ijkl} and R_{IJKL} are the components of the covariant curvature tensor in V_n and V_N respectively. The ξ^I_α are $N - n$ contravariant vectors in V_N of unit length, perpendicular to one another and to the V_n. The $b_{\alpha|ij} = b_{\alpha|ji}$ are coefficients of the fundamental forms of the second kind and $t_{\alpha\beta|i} = -t_{\beta\alpha|i}$ are the "torsions". The index after the comma denotes covariant differentiation with respect to the tensor a_{ij} given by (2).

If V_N is of constant curvature K_0 we have

$$R_{IJKL} = K_0(A_{IK}A_{JL} - A_{IL}A_{JK}),$$

and equations (I), (II), (III) become:

$$(\mathrm{I}') \quad G_{ijkl} \equiv r_{ijkl} - (b_{\alpha|ik}b_{\alpha|jl} - b_{\alpha|il}b_{\alpha|jk}) - K_0(a_{ik}a_{jl} - a_{il}a_{jk}) = 0,$$

$$(\mathrm{II}') \quad C_{\alpha|ijk} \equiv b_{\alpha|ij,k} - b_{\alpha|ik,j} - (t_{\beta\alpha|k}b_{\beta|ij} - t_{\beta\alpha|j}b_{\beta|ik}) = 0,$$

$$(\mathrm{III}') \quad K_{\alpha\beta|ij} \equiv t_{\alpha\beta|i,j} - t_{\alpha\beta|j,i} + (t_{\gamma\alpha|i}t_{\gamma\beta|j} - t_{\gamma\alpha|j}t_{\gamma\beta|i}) + a^{kl}(b_{\alpha|ki}b_{\beta|lj} - b_{\alpha|kj}b_{\beta|li}) = 0.$$

It can be shown that in this case the equations (I′), (II′), (III′) are both *necessary and sufficient* conditions for the V_n to be imbedded in the V_N. For a similar problem, see (**10**, pp. 178–182).

As in the case of a V_n in an E_N (**3**; **4**) the question arises at this point whether all the equations (I), (II), (III) are independent. The following section is devoted to answering this question.

2. Independence considerations. The lefthand sides of (I), (II), (III) are obviously tensors in the V_n, which depend in general also upon the V_N. They are denoted by G_{ijkl}; $C_{\alpha|ijk}$; $K_{\alpha\beta|ij}$ and named respectively the tensors of Gauss, Codazzi and Kuehne of the V_n with respect to the V_N (**3**, p. 167f). We can now reformulate the statement in §1 in the following way:

A necessary condition for the V_n to be imbedded in the V_N is that the tensors of Gauss, Codazzi and Kuehne of the V_n with respect to the V_N should vanish.

We are thus able to consider directly the tensors just introduced and certain combinations of their covariant derivatives. This will lead us to discover in certain cases how many of the conditions (I), (II), (III) are independent.

We define (**4**; **6**):

$$(A) \qquad G_{ijklm} = G_{ijkl,m} + G_{ijlm,k} + G_{ijmk,l},$$

$$(B) \qquad C_{\alpha|ijkl} = C_{\alpha|ijk,l} + C_{\alpha|ikl,j} + C_{\alpha|ilj,k},$$

$$(C) \qquad K_{\alpha\beta|ijk} = K_{\alpha\beta|ij,k} + K_{\alpha\beta|jk,i} + K_{\alpha\beta|ki,j}.$$

These tensors will be appropriately named the "derived tensors" of Gauss, Codazzi and Kuehne of the V_n with respect to the V_N. If we perform the indicated calculations, we obtain:

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$$\text{(A}'\text{)} \quad G_{ijklm} = \quad - b_{\alpha|ik}C_{\alpha|jlm} - b_{\alpha|il}C_{\alpha|jmk} - b_{\alpha|im}C_{\alpha|jkl} + b_{\alpha|jk}C_{\alpha|ilm} + b_{\alpha|jl}C_{\alpha|imk} + b_{\alpha|jm}C_{\alpha|ikl},$$

$$\text{(B}'\text{)} \quad C_{\alpha|ijkl} = \quad t_{\beta\alpha|j}C_{\beta|ikl} + t_{\beta\alpha|k}C_{\beta|ilj} + t_{\beta\alpha|l}C_{\beta|ijk} - b_{\beta|ij}K_{\beta\alpha|kl} - b_{\beta|ik}K_{\beta\alpha|lj} - b_{\beta|il}K_{\beta\alpha|jk} + a^{mp}(b_{\alpha|mj}G_{pikl} + b_{\alpha|mk}G_{pilj} + b_{\alpha|ml}G_{pijk}),$$

$$\text{(C}'\text{)} \quad K_{\alpha\beta|ijk} = \quad t_{\gamma\alpha|i}K_{\gamma\beta|jk} + t_{\gamma\alpha|j}K_{\gamma\beta|ki} + t_{\gamma\alpha|k}K_{\gamma\beta|ij} - t_{\gamma\beta|i}K_{\gamma\alpha|jk} - t_{\gamma\beta|j}K_{\gamma\alpha|ki} - t_{\gamma\beta|k}K_{\gamma\alpha|ij} + a^{mp}(b_{\alpha|mi}C_{\beta|pjk} + b_{\alpha|mj}C_{\beta|pki} + b_{\alpha|mk}C_{\beta|pij}) - a^{mp}(b_{\beta|mi}C_{\alpha|pjk} + b_{\beta|mj}C_{\alpha|pki} + b_{\beta|mk}C_{\alpha|pij}).$$

We notice that these derived tensors[2] do not depend explicitly upon the V_N, the last terms from (I), (II), (III) having disappeared. They have therefore the same form as the corresponding derived tensors of the V_n with respect to an E_N **(6**, p. 90**)**. This remarkable fact enables us to extend the results of **(3**; **4**; **6)** to the present case.

These results are essentially based upon the consideration of the tensors (A′), (B′), (C′) and the number of their components. Thus if, for instance, the Gauss tensor G_{ijkl} vanishes in V_n, then the derived Gauss tensor G_{ijklm} is also zero and (A′) becomes a system of linear and homogenous equations in the $C_{\alpha|ijk}$ which reduces, of course, the number of independent components of the Codazzi tensors $C_{\alpha|ijk}$. It is thus possible that, under conditions to be specified below, all the components of the Codazzi tensors $C_{\alpha|ijk}$ vanish *as a result* of the vanishing of Gauss' tensor G_{ijkl}. Similar considerations are valid with respect to (B′) and (C′).

It is necessary at this point to list the number of components of the different tensors introduced so far **(2**; **3**, pp. 170, 174; **6**, p. 91**)**:

$$G_{ijkl} \ldots n^2(n^2 - 1)/12$$
$$C_{\alpha|ijk} \ldots \nu n(n^2 - 1)/3$$
$$K_{\alpha\beta|ij} \ldots \nu(\nu - 1)\, n(n - 1)/4$$
$$G_{ijklm} \ldots n^2(n^2 - 1)(n - 2)/24$$
$$C_{\alpha|ijkl} \ldots \nu n(n^2 - 1)(n - 2)/8$$
$$K_{\alpha\beta|ijk} \ldots \nu(\nu - 1)\, n(n - 1)(n - 2)/12$$

We mention also the result by Burstin **(7)** that, under our assumptions, every V_n can be imbedded in every V_N provided that $N \geqslant \frac{1}{2}n(n + 1)$ or $\nu = N - n \geqslant \frac{1}{2}n(n - 1)$. It is therefore sufficient to choose $n(n - 1)/2$ as the upper limit for ν.

We are now in the position to enunciate the following two theorems:

[2]The equations obtained by equating (A') and (B') to zero were first used by Allendoerfer **(1)** in the case of a V_n in an E_N to reduce the number of independent equations of (II) and (III).

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THEOREM 2.1 *If the equations* (I) *are satisfied by a set of solutions for* $b_{\alpha|ij}$, *for which the ranks of the matrices of the linear systems:*

$$(\mathrm{A}'') \quad G_{ijklm} \equiv - b_{\alpha|ik}C_{\alpha|jlm} - b_{\alpha|il}C_{\alpha|jmk} - b_{\alpha|im}C_{\alpha|jkl} + b_{\alpha|jk}C_{\alpha|ilm} + b_{\alpha|jl}C_{\alpha|imk} + b_{\alpha|jm}C_{\alpha|ikl} = 0,$$

$$(\mathrm{B}'') \quad C_{\alpha|ijkl} \equiv - b_{\beta|ij}K_{\beta\alpha|kl} - b_{\beta|ik}K_{\beta\alpha|lj} - b_{\beta|il}K_{\beta\alpha|jk} = 0,$$

have maximum value,[3] *then*

(a) *for* $0 \leqslant \nu \leqslant \frac{1}{8}n(n-2)$, *all equations* (II) *and* (III) *are a consequence of equations* (I);

(b) *for* $\frac{1}{8}n(n-2) \leqslant \nu \leqslant \frac{1}{2}n(n-1)$, *a system of*

$$\tfrac{1}{3}n(n^2-1)[\nu - \tfrac{1}{8}n(n-2)]$$

of equations (II) *are independent. The remainder of equations* (II) *and all the equations* (III) *are a consequence of this system and equations* (I).

THEOREM 2.2. *If the equations* (I) *are satisfied by a set of solutions for* $b_{\alpha|ij}$, *for which the ranks,* r *and* r', *of the matrices of* (A$''$) *and* (B$''$) *have not both maximum values, then*

$$\tfrac{1}{3}\nu n(n^2-1) - r$$

of equations (II) *and*

$$\tfrac{1}{4}\nu(\nu-1)\,n(n-1) - r'$$

of equations (III) *remain independent.*

From the table on the previous page it is seen that the matrix of (A$''$) has $\frac{1}{24}n^2(n^2-1)(n-2)$ rows and $\frac{1}{3}\nu n(n^2-1)$ columns and the matrix of (B$''$) has $\frac{1}{8}\nu n(n^2-1)(n-2)$ rows and $\frac{1}{4}\nu(\nu-1)\,n(n-1)$ columns. By comparing the two sets of numbers, Theorems 2.1 and 2.2 are readily verified.

In view of this theorem it would seem important to determine the actual ranks of the matrices of (A$''$) and (B$''$) in terms of certain numerical invariants of the V_n. Except for the particular case treated in the next section, the author has not succeeded in this task.

In the formation of the tensor G_{ijklm} we made use of Bianchi's identities:

$$r_{ijkl,m} + r_{ijlm,k} + r_{ijmk,l} = 0.$$

(Because of this, of course, the number of components of G_{ijklm} equals the number of Bianchi's identities **(2)**.)

But Bianchi's identities are a complete set of identities of order one of the tensor of curvature **(5)**. We have therefore the result:

Equations (A$''$) *are the only ones between the components of* $C_{\alpha|ijk}$, *which can be obtained as a consequence of the validity of equations* (I).

[3]A rectangular matrix with s rows and t columns has maximum rank r if r equals the smaller of the two numbers s, t.

III. It can be seen from equations (I′), (II′), (III′) that the problem of imbedding a V_n in a V_N with constant curvature K_0 is equivalent to the problem of imbedding a V_n in a Euclidean E_N, provided that we substitute for the curvature tensor r_{ijkl} of the V_n the tensor:

$$r_{ijkl} - K_0(a_{ik}a_{jl} - a_{il}a_{jk}).$$

We shall also assume $N = n + 1$ in which case (I′), (II′), (III′) reduce to:

$$\text{(I}''\text{)} \quad G_{ijkl} \equiv r_{ijkl} - (b_{ik}b_{jl} - b_{il}b_{jk}) - K_0(a_{ik}a_{jl} - a_{il}a_{jk}) = 0,$$
$$\text{(II}''\text{)} \quad C_{ijk} \equiv b_{ij,k} - b_{ik,j} = 0,$$

and (A″), (B″) to

$$\text{(A}'''\text{)} \quad G_{ijklm} \equiv \begin{array}{l} - b_{ik}C_{jlm} - b_{il}C_{jmk} - b_{im}C_{jkl} \\ + b_{jk}C_{ilm} + b_{jl}C_{imk} + b_{jm}C_{ikl} = 0. \end{array}$$

Let τ be the rank of the matrix

$$||r_{ijkl} - K_0(a_{ik}a_{jl} - a_{il}a_{jk})||$$

where one of the indices, say i, indicates the rows and the other three indices the columns of the matrix.

It can then be shown that, because of (I″), τ is also the rank of the matrix $||b_{ij}||$ (**10**, p. 184).

The integer τ can be considered as an invariant of the V_n with respect to a V_{n+1} of constant curvature K_0 (in the neighbourhood of the point under consideration). It was introduced by Thomas (**10**, loc. cit.) for a V_n with respect to an E_{n+1}.

For $\tau = 0$, it follows from (I″) that the V_n is of constant curvature K_0. $\tau = 1$ is impossible. For the remaining values of τ we shall prove

THEOREM 3.1. $\tau \geqslant 4$. *All the equations* (II″) *are a consequence of equations* (I″).

THEOREM 3.2. $\tau = 3$. *Of the equations* (II″), *five remain independent. The remainder of the equations* (II″) *are a consequence of these and equations* (I″).

THEOREM 3.3. $\tau = 2$. *Of the equations* (II″), $3n - 4$ *remain independent. The remainder of the equations* (II″) *are a consequence of these and equations* (I″).

Proof. For the values of b_{ij} in the point under consideration we can, by a suitable coordinate transformation, obtain:[4]

$$(3) \qquad \begin{array}{ll} b_{ij} = 0 & (i \neq j), \\ b_{ii} \neq 0 & (i = 1, 2, \ldots, \tau), \\ b_{ii} = 0 & (i = \tau + 1, \ldots, n). \end{array}$$

[4]From here to the end of this section, a repeated index does *not* indicate a summation.

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For these values, and taking in consideration the basic identities of the tensor G_{ijklm} (which follow readily from its definition):

$$G_{ijklm} = - G_{jiklm}$$
$$G_{ijklm} = G_{ijlmk} = G_{ijmkl} = - G_{ijkml} = - G_{ijmlk} = - G_{ijlkm}$$

the system (A‴) reduces to:

$$G_{ijkli} \equiv - b_{ii}C_{jkl} = 0, \tag{4}$$
$$G_{ijijl} \equiv - b_{ii}C_{jjl} - b_{jj}C_{iil} = 0, \tag{5}$$

where i, j, k, l are all distinct, $i = 1, 2, \ldots \tau$ and j, k, $l = 1, 2, \ldots n$.

Proof of 3.1. $\tau \geqslant 4$. From (4) we find then

$$C_{jkl} = 0,$$

and from (5) follows

$$C_{jjl} = 0 \qquad (j = \tau + 1, \tau + 2, \ldots, n).$$

For $j \leqslant \tau$, we obtain from (5), by substituting first i for k and then j for k:

$$\begin{aligned} 0 + b_{kk}C_{jjl} + b_{jj}C_{kkl} &= 0,\\ b_{kk}C_{iil} + 0 + b_{ii}C_{kkl} &= 0,\\ b_{jj}C_{iil} + b_{ii}C_{jjl} + 0 &= 0,\end{aligned} \tag{6}$$

for distinct i, j, k, l with $i, j, k = 1, 2, \ldots, \tau$ and $l = 1, 2, \ldots, n$.

The determinant of (6) being different from zero it follows that

$$C_{iil} = 0 \qquad (i \neq l; i = 1, 2, \ldots, \tau; l = 1, 2, \ldots, n).$$

This completes the proof for the case $\tau \geqslant 4$ because $C_{ijj} \equiv 0$ follows from (II″).

Proof of 3.2. $\tau = 3$. From (4) we obtain

$$C_{jkl} = 0 \qquad (j \neq k \neq l = 1, 2, \ldots, n)$$

provided that at least one of the indices j, k, l, is larger than 3. But the three remaining components of this type, namely C_{123}, C_{231}, and C_{312}, satisfy an identity (which follows easily from (II″)):

$$C_{123} + C_{231} + C_{312} = 0.$$

Thus only two of these components (e.g., C_{123}, C_{231}) remain independent.

From (5) we have

$$C_{jjl} = 0 \qquad (j = 4, 5, \ldots, n; l = 1, 2, \ldots, n).$$

From (6) we obtain for $i = 1, j = 2, k = 3$:

$$\begin{aligned} 0 + b_{33}C_{22l} + b_{22}C_{33l} &= 0,\\ b_{33}C_{11l} + 0 + b_{11}C_{33l} &= 0,\\ b_{22}C_{11l} + b_{11}C_{33l} + 0 &= 0.\end{aligned} \qquad (l = 4, 5, \ldots, n). \tag{7}$$

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We have therefore as above:

$$C_{iil} = 0 \qquad (i = 1, 2, 3; l = 4, 5, \ldots, n).$$

From (5) we obtain for $i, j, l = 1, 2, 3$ in turn:

$$b_{33}C_{221} + b_{22}C_{331} = 0,$$

$$b_{33}C_{112} + b_{11}C_{332} = 0,$$

$$b_{22}C_{113} + b_{11}C_{223} = 0.$$

It follows from these equations that three components of the type C_{iil} $(i \neq l = 1, 2, 3)$ are independent (e.g., C_{112}, C_{223}, C_{331}).

Therefore, in the case under consideration *five* components of the tensor C_{ijk} remain independent, namely C_{123}, C_{231}, C_{112}, C_{223}, C_{331}.

Proof of 3.3. $\tau = 2$. From (4) we obtain

$$C_{jkl} = 0, \qquad (j \neq k \neq l = 1, 2, \ldots, n),$$

provided that at least two of the indices are larger than 2. The remaining components of this type are C_{12l}, C_{2l1}, C_{l12} among which we have (as before) the identity

$$C_{12l} + C_{2l1} + C_{l12} = 0, \qquad (l = 3, 4, \ldots, n).$$

It follows therefore that $2n - 4$ of these components (e.g., C_{12l}, C_{2l1}; $l = 3, 4, \ldots, n$) remain independent.

From (5) we have

$$C_{jjl} = 0, \qquad (j = 3, 4, \ldots, n;\ l = 1, 2, \ldots, n),$$

and

$$b_{22}C_{11l} + b_{11}C_{22l} = 0, \qquad (l = 3, 4, \ldots, n).$$

Thus the $n - 2$ components C_{11l} (or C_{22l}) $(l = 3, 4, \ldots, n)$ are independent.

The two components C_{112} and C_{221} are also independent because they do not occur in any of the equations (5).

Therefore, in the case under consideration, $3(n - 2) + 2 = 3n - 4$ components of the tensor C_{ijk} remain independent, namely, C_{112}, C_{221}, C_{12l}, C_{2l1}, C_{11l} $(l = 3, 4, \ldots, n)$.

The case $\tau \geqslant 4$ of Theorem 3.1 was proved by Thomas (**10**, §5) for a V_n in an E_{n+1}. In its general form but also for a V_n in an E_{n+1}, it was established by the present author (**3**, pp. 196ff). Here the same line of proof has been adopted.

It is remarkable that in the case $\tau = 3$, the number of independent components of the tensor C_{ijk} does not depend upon the number of dimensions of V_n.

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University of Saskatchewan

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