A CHARACTERIZATION OF PRÜFER DOMAINS

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The purpose of this note is to give a new characterization of Prüfer domains using the concept of ring epimorphism, and to indicate some connections with well-known properties of Prüfer domains. All rings are commutative and have a unit element.

An extension $R \subseteq S$ of rings is called epimorphic, if the injection map is an epimorphism in the category of commutative rings, i.e. cancellable on the right. $R \subseteq S$ is epimorphic if and only if the natural map $S \otimes_R S \to S$ is an isomorphism (see e.g. [8]).

We will say, that $R \subseteq S$ is a completely epimorphic extension, if $R \subseteq T$ is epimorphic for all rings T "between" R and S, i.e. $R \subseteq T \subseteq S$. This is clearly equivalent to the following condition: for all $s \in S$, we have $s \otimes 1 = 1 \otimes s$ in $R[s] \otimes_R R[s]$.

A homomorphism $R \to S$ is called flat, if S is a flat R-module, and an extension $R \subseteq S$ is called essential, if each non-zero ideal of S has non-zero intersection with R.

LEMMA 1. *A completely epimorphic extension $R \subseteq S$ is essential.*

Proof. If $0 \neq s \in S$, then $s \otimes 1 = 1 \otimes s$ in $R[s] \otimes_R R[s]$. By [1, Chapter 1, §2, No. 11, Lemma 10], there exist elements $x_j \in R[s]$ and $a_{jk} \in R$ $(j = 0, \ldots, m,\ k = 0, \ldots, n)$ such that, among other relations,

$$s = \sum_{j=0}^{m} x_j a_{j0} \quad \text{and}$$

$$\sum_{k=0}^{n} a_{jk} s^k = 0 \quad (j = 0, \ldots, m).$$

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Since $s \neq 0$, there is at least one index i, such that $a_{i0} \neq 0$ and we can write

$$-a_{i0} = s\left(\sum_{k=1}^{n} a_{ik} s^{k}\right),$$

which implies the statement of the lemma.

LEMMA 2 (Lazard [5]). *If* $R \subseteq S$ *is epimorphic and flat, and if* T *is a ring between* R *and* S, *such that* $R \subseteq T$ *is flat, then* $R \subseteq T$ *is epimorphic*.

Proof. The following diagram is commutative:

$$\begin{array}{ccc} T \otimes_R T & \longrightarrow & T \\ {\scriptstyle i \otimes i}\downarrow & & \downarrow{\scriptstyle i} \\ S \otimes_R S & \xrightarrow{\ \cong\ } & S \end{array}$$

Since T and S are flat, $i \otimes i$ is a monomorphism, hence $T \otimes_R T \to T$ is an isomorphism.

LEMMA 3. *If* $R \subseteq S$ *is completely epimorphic, then every* T *between* R *and* S *is integrally closed*.

Proof. Suppose there exists a T which is not integrally closed and let $s \notin T$ be integral over T. Then $T[s]$ is a finitely generated T-module and $T \subseteq T[s]$ is not epimorphic by [4, Proposition 1.5]. This implies of course, that $R \subseteq T[s]$ is not epimorphic.

If R is an integral domain with quotient field K, then a ring between R and K is called an overring of R. Every essential epimorphic extension, and in particular every completely epimorphic extension of an integral domain is isomorphic to an overring [8, Corollary 9.11].

However, not every overring of an integral domain is an epimorphic extension, as it will be shown in the next proposition.

A Prüfer domain is an integral domain, such that every finitely generated ideal is invertible. A Dedekind domain is a noetherian Prüfer domain.

PROPOSITION. Let R be an integral domain with quotient field K. Then R is a Prüfer domain if and only if $R \subseteq K$ is completely epimorphic.

Proof. If R is Prüfer, then the tensor product of torsion-free modules is torsion-free [2, VII, Proposition 4.5], hence, for any $s = a/b \in K$, $b(s \otimes 1 - 1 \otimes s) = 0$ in $R[s] \otimes_R R[s]$ implies $s \otimes 1 = 1 \otimes s$. The converse follows from Lemma 3 and a result of Davis [3].

The proof gives us a little bit more information.

COROLLARY 1. If R is an integral domain with quotient field K, then the following conditions are equivalent:

(i) R is a Prüfer domain;

(ii) the tensor-product of torsion-free modules is torsion-free;

(iii) $R[s] \otimes_R R[s]$ is torsion-free for all $s \in K$.

COROLLARY 2. If R is a Prüfer domain, then the following conditions for a ring $T \supseteq R$ are equivalent:

(i) T is isomorphic (over R) to an overring of R;

(ii) $R \subseteq T$ is essential and epimorphic;

(iii) $R \subseteq T$ is completely epimorphic.

This follows from the proposition and [8, Corollary 9.11]. With the appropriate definition of "overring", this corollary generalizes to semihereditary rings.

Some well-known properties of Prüfer domains can now be deduced from the preceding results; we list two examples.

COROLLARY 3. Every overring of a Prüfer domain is a Prüfer domain.

COROLLARY 4 (Richman [7]). An integral domain is Prüfer if and only if every overring is flat.

This is a consequence of Lemma 2 and the fact, that a torsion-free module over a Prüfer domain is flat [2, VII, Proposition 4.2.].

We finally give a new proof of a result, which was first established in [6].

COROLLARY 5. Every overring of a Dedekind domain is a Dedekind domain.

Proof. By Corollary 3, every overring T is a Prüfer domain, and it remains to show, that every T is noetherian. This follows from [4, Corollary 2.3.], since R is noetherian and $R \subseteq T$ is a flat epimorphism.

REFERENCES

1. N. Bourbaki, Algèbre commutative, Chapitres 1 et 2. (Act. Sci. Ind. 1290, Hermann, Paris, 1961).

2. H. Cartan, and S. Eilenberg, Homological algebra. (Princeton University Press, 1956).

3. E.D. Davis, Overrings of commutative rings II. Integrally closed overrings. Trans. Amer. Math. Soc. 110 (1964) 196-212.

4. D. Lazard, Epimorphismes plats d'anneaux. C.R. Acad. Sci. Paris 266 (1968) 314-317.

5. D. Lazard, Epimorphismes plats. Séminaire P. Samuel (Algèbre commutative) 1967/68, Exposé No. 4. (Secrétariat mathématique, Paris 1968).

6. S. MacLane and O.F.G. Schilling, Infinite number fields with Noether ideal theories. Amer. J. Math. 61 (1939) 771-782.

7. F. Richman, Generalized quotient rings. Proc. Amer. Math. Soc. 19 (1965) 794-799.

8. H.H. Storrer, Epimorphismen von kommutativen Ringen. Comm. Math. Helv. 43 (1968) 378-401.

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