A COMPARISON OF THREE CREDIBILITY FORMULAE USING MULTIDIMENSIONAL TECHNIQUES

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Stimulated by the paper of W. S. Jewell *) on multidimensional credibility I should like to show that at level 2 assumptions (Jewell's terminology) one can obtain explicit formulae for forecasting total losses in the future based upon total losses and number of claims observed in the past.

1. THE MODEL

Random variables

		mean	variance	parameter of distribution
number of claims	k	E_k	σ_k^2	$\eta \varepsilon H$
average claim size	$\bar{y} = \sum_{i=1}^{k} y_i \mid k$	$E[\bar{y} \mid k]$	$\mathrm{Var}[\bar{y} \mid k] = \frac{\sigma_y^2}{k}$	$\theta \varepsilon H$

total claims $\quad T = k\bar{y} = \sum_{i=1}^{k} y_i$

Hypotheses

α) η, θ are independent random variables (level 2 assumption)

β) Given (η, θ) the random variables $\{k, y_1, y_2, \ldots, y_n, \ldots\}$ are independent

Given θ the random variables $\{y_1, y_2, \ldots y_n, \ldots\}$ are independent.

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*) William S. Jewell, „Multi-Dimensional Credibility", Operations Research Center Report No. 73-7. Department of Industrial Engineering and Operations Research. University of California, Berkeley. This paper was presented to the 10th ASTIN Colloquium held at the University of Essex U.K. 1973. In agreement with the author it will not be published in the Astin Bulletin.

2. Formulae for Moments and Some Notation

$$E[T \mid (\eta, \theta)] = E_k(\eta) \cdot E_y(\theta)$$
$$\text{Var}\,[T \mid (\eta, \theta)] = E_k(\eta)\,.\,\sigma_y^2(\theta) + E_y^2(\theta)\,\sigma_k^2(\eta).$$

We abbreviate as follows

$$E[E_k(\eta)] = m_k \qquad E[E_y(\theta)] = m_y$$
$$E[\sigma_k^2(\eta)] = v_k \qquad E[\sigma_y^2(\theta)] = v_y$$
$$\text{Var}\,[E_k(\eta)] = w_k \qquad \text{Var}\,[E_y(\theta)] = w_y$$

and obtain for the operators $E[\cdot]$ and Var $[\cdot]$ with respect to the probability over the product space of observations and parameters

$$E\,[E(T \mid (\eta, \theta))] = m_k \cdot m_y$$
$$\text{Var}\,[E(T \mid (\eta, \theta))] = w_k w_y + m_k^2 w_y + m_y^2 w_k$$
$$E\,[\text{Var}\,(T \mid (\eta, \theta))] = m_k v_y + (m_y^2 + w_y)\,v_k.$$

3. The Credibility Formula and its Mean Quadratic Error

We want to estimate $E_k(\eta) \cdot E_y(\theta)$

$$\text{by } \alpha k\bar{y} + \beta k m_y + \gamma m_k m_y$$

where α, β, γ are such that

$$\bar{V}(\alpha, \beta, \gamma) = E[\alpha k\bar{y} + \beta k m_y + \gamma m_k m_y - E_k(\eta)\,E_y(\theta)]^2 = \text{minimum}.$$

A rather tedious but straightforward calculation shows that

$$\left.\begin{aligned} \bar{V}(\alpha, \beta, \gamma) &= \alpha^2\underbrace{[m_k v_y + v_k w_y]}_{A} + (1-\alpha)^2\underbrace{[m^2{}_k w_y + w_k w_y]}_{B} \\ (\alpha+\beta)^2 \underbrace{v_k m_y^2}_{C} &+ (1-\alpha-\beta)^2 \underbrace{w_k m_y^2}_{D} + (\alpha+\beta+\gamma-1)^2 m_k^2 m_y^2 \end{aligned}\right\} \quad (1)$$

4. The Three Cases

I *Using both k and $k\bar{y}$* (multidimensional case)

Then the minimizing parameters $(\alpha^*, \beta^*, \gamma^*)$ are

$$\alpha^* = \frac{m_k^2 w_y + w_k w_y}{m_k^2 w_y + w_k w_y + m_k v_y + v_k w_y} = \frac{B}{A+B} \quad (2)$$

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$$\left.\begin{aligned} \alpha^* + \beta^* &= \frac{w_k}{v_k + w_k} = \frac{D}{C + D} \\ \alpha^* + \beta^* + \gamma^* &= 1. \end{aligned}\right\} \qquad (2)$$

Observe that β^* may be negative.

II *Using only k* (case of auxiliary variable)

$$\left.\begin{aligned} \alpha^* &= 0 \\ \beta^* &= \frac{w_k}{v_k + w_k} = \frac{D}{C + D} \\ \gamma^* &= 1 - \beta^*. \end{aligned}\right\} \qquad (3)$$

III *Using only* $k\bar{y}$ (onedimensional case)

$$\left.\begin{aligned} \alpha^* &= \frac{m_k^2 w_y + w_k w_y + w_k m_y^2}{m_k^2 w_y + w_k w_y + m_k v_y + v_k w_y + m_y^2 v_k + m_y^2 w_k} = \frac{B + D}{A+B+C+D} \\ \beta^* &= 0 \\ \gamma^* &= 1 - \alpha^*. \end{aligned}\right\} \qquad (4)$$

5. Comparison of the Three Formulae

$\bar{V}(\alpha, \beta, \gamma)$ measures the mean quadratic error which now can be used to decide which formula might be appropriate

Case I

	increase to case I	relative increase
From 1) and 2) we obtain		
$\bar{V}(\alpha^*, \beta^*, \gamma^*) = \frac{AB}{A + B} + \frac{CD}{C + D}$	0	0

Case II

From 1) and 3) we obtain		
$\bar{V}(0, \beta^*, \gamma^*) = B + \frac{CD}{C + D}$	$\frac{B^2}{A + B}$	Δ_{II}

Case III

From 1) and 4) we obtain		
$\bar{V}(\alpha^*, 0, \gamma^*) = \frac{(A + C)(B + D)}{A + B + C + D}$	$\frac{(BC - AD)\left(\frac{B}{A + B} - \frac{D}{C + D}\right)}{A + B + C + D}$	Δ_{III}

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It is easy to show that both Δ_I and Δ_{II} are always non-negative (as we know already by construction)

Observe that $\Delta_{III} < \Delta_{II}$ if $A = B = C = D$
however $\Delta_{II} < \Delta_{III}$ if B sufficiently small
(keep A, C, D constant and let $B \to 0$)

This shows that the formula of case III is not necessarily better than the formula of case II, but certainly the formula of case I is. Whether to choose it in any practical situation should be decided after consideration of the numerical values of Δ_{II} and Δ_{III}.

6. The Formulae for n Years

By changing from k (number of claims in one year)
to $k^{(n)}$ (number of claims in n years)
we get the credibility formulae for n years.

The following identities are used for substitutions

$$m_k^{(n)} = nm_k \qquad E_k^{(n)}(\eta) = nE_k(\eta)$$
$$v_k^{(n)} = nv^k \qquad \sigma_k^{2(n)}(\eta) = n\sigma_k^2(\eta)$$
$$w_k^{(n)} = n^2 w_k.$$

Case I

Formulae for one year $\alpha^* = \dfrac{1}{1+Z}$ $\qquad Z = \dfrac{m_k v_y + v_k w_y}{m_k^2 w_y + w_k w_y}$

$$(\alpha^* + \beta^*) = \frac{1}{1+Z_k} \qquad Z_k = \frac{v_k}{w_k}$$

Formulae for n years One checks that $\quad Z \to \dfrac{Z}{n}$

$$Z_k \to \frac{Z_k}{n}$$

hence $$(\alpha^*)_n = \frac{n}{n+Z}$$

$$(\alpha^* + \beta^*)_n = \frac{n}{n+Z_k}.$$

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Case II

Formula for one year $\beta^* = \frac{1}{1 + Z_k} \quad Z_k = \frac{v_k}{w_k}$

Formula for n years $(\beta^*)_n = \frac{n}{n + Z_k}$

Case III

Formula for one year $\alpha^* = \frac{1}{1 + Z_{z\bar{y}}} \quad Z_{k\bar{y}} = \frac{m_k v_y + v_k w_y + m_y^2 v_k}{m_k^2 w_y + w_k w_y + w_k m_y^2}$

Formula for n years $Z_{k\bar{y}} \to \frac{Z_{k\bar{y}}}{n}$

hence $$(\alpha^*)_n = \frac{n}{n + Z_{k\bar{y}}}.$$

In order to compute also the expected error $\bar{V}(\alpha, \beta, \gamma)$ the following substitutions must be made for the n year formulae

$$A \to nA$$
$$B \to n^2B$$
$$C \to nC$$
$$D \to n^2D.$$

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