BULL. AUSTRAL. MATH. SOC. 47B20
VOL. 24 (1981), 61-67. (47B15, 47A10)

ON k-QUASIHYPONORMAL OPERATORS II

B.C. GUPTA AND P.B. RAMANUJAN

An cperator T on a Hilbert space is in the class of k-quasi-hyponormal operators $Q(k)$, if $T^{*k}(T^*T-TT^*)T^k \geq 0$. It is shown that if T is in $Q(k)$ and S is normal such that $TX = XS$, where X is one to one with dense range, then T is normal; and is unitarily equivalent to S. It is proved that S can be replaced by a cohyponormal operator, if T in $Q(1)$ is one to one. It is also shown that two quasisimilar operators in $Q(k)$ have equal spectra, and every reductive operator quasi-similar to a normal operator is normal.

A bounded linear operator T on a Hilbert space H is called k-quasihyponormal if $T^{*k}(T^*T-TT^*)T^k \geq 0$, or equivalently, $\|T^*T^k x\| \leq \|T^{k+1}x\|$ for every $x \in H$, where k is a positive integer. Clearly, the class $Q(k)$ of all k-quasihyponormal operators on H contains all hyponormal operators and forms a strictly increasing sequence in k. The class $Q(1)$ is the class of quasihyponormal operators [*10*]. For an operator $T \in Q(k)$ the following representation was obtained in [*4*].

THEOREM A. *An operator T is in $Q(k)$ if and only if T has matrix representation*

$$(*) \qquad T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

Received 16 February 1981. The authors gratefully acknowledge the support of the University Grants Commission of India.

https://doi.org/10.1017/S0004972700007437 Published online by Cambridge University Press

with respect to a pair of complementary orthogonal subspaces of the Hilbert space H *, where*

(a) $T_1^*T_1 - T_1T_1^* \geq T_2T_2^*$, *and*

(b) $T_3^k = 0$.

The representation (*) *is not unique; however, we can always take* $T_1 = T|\overline{R(T^k)}$, T *restricted to* $\overline{R(T^k)}$.

Using the representation (*) it was shown that eigenspaces corresponding to non-zero eigenvalues are reducing and several structure theorems for operators in $Q(k)$ were proved. Further, it was shown that there is a non-hyponormal operator in $Q(1)$ with reducing kernel; and since restriction of an operator $T \in Q(1)$ to an invariant subspace is again in $Q(1)$, this also gives a one to one non-hyponormal operator in $Q(1)$. In this paper, we continue the study of operators in $Q(k)$.

We denote the kernel, the range, the spectrum and the closure of the numerical range of an operator T by $N(T)$, $R(T)$, $\sigma(T)$ and $\overline{W(T)}$ respectively. The norm closure of a subspace M of H is denoted by $\overline{M}$ and the Banach algebra of all operators on a Hilbert space H by $B(H)$.

It is shown in this paper that if $T \in Q(k)$ and S is normal such that $TX = XS$ where $N(X) = N(X^*) = \{0\}$ then T is normal, and is unitarily equivalent to S . If in addition, T is in $Q(1)$ with $N(T) = \{0\}$ then the normal operator S can be replaced by a cohyponormal operator without affecting the conclusion. In case T is an arbitrary hyponormal operator these results are due to Stampfli and Wadhwa [*11*] and Radjabalipour [*8*].

It is known that two quasimilar hyponormal operators have equal spectra [5], and every reductive operator similar to a normal operator is normal [*6*]. We show that two quasisimilar operators in $Q(k)$ have equal spectra, and every reductive operator which is quasisimilar to a normal operator is normal.

For our purpose, we mention the following which is an easy modification of Theorem 1 in [*11*].

THEOREM B. *Let* $T \in B(H)$ *be hyponormal and let* $S \in B(K)$ *be* *normal. If* $TX = XS$ *where* $X : K \to H$ *is a one to one bounded linear operator with dense range then* T *is normal and is unitarily equivalent to* S .

THEOREM 1. *Let* $T \in Q(k)$, S *a normal operator and let* $TX = XS$ *where* X *is a one to one operator with dense range. Then* T *is a normal operator unitarily equivalent to* S .

Proof. Let $T_1 = T|\overline{R(T^k)}$ and $S_1 = S|\overline{R(S^k)}$. Then by Theorem A, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ and } S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

where S_1 is normal, $T_3^k = 0$ and $T_1^*T_1 - T_1T_1^* \geq T_2T_2^*$. Since $T^kX = XS^k$ and X has dense range, $\overline{X\left(\overline{R(S^k)}\right)} = \overline{R(T^k)}$. If we denote the restriction of X to $\overline{R(S^k)}$ by X_1 then $X_1 : \overline{R(S^k)} \to \overline{R(T^k)}$ is one to one and has dense range and for every $x \in \overline{R(S^k)}$, $X_1S_1x = XSx = TXx = T_1X_1x$ so that $X_1S_1 = T_1X_1$. Now since T_1 is hyponormal it follows from Theorem B that T_1 is a normal operator unitarily equivalent to S_1 . But then $T_2T_2^* = 0$, which implies that $T_2 = 0$ and therefore $\overline{R(T^k)}$ reduces T . Since $X^*\left(N\left(T^{*k}\right)\right) \subset N\left(S^{*k}\right) = N(S^*)$, for each $x \in N\left(T^{*k}\right)$, we have $X^*T_3^*x = X^*T^*x = S^*X^*x = 0$. But X has dense range and so X^* is one to one. Therefore $T_3^*x = 0$ for every $x \in N\left(T^{*k}\right)$. Thus $T_3 = 0$. Hence $T = T_1 \oplus 0$. This completes the proof.

As an application, we get the following version of [8, Corollary 1] for operators in $Q(1)$.

THEOREM 2. *Let* $T \in Q(1)$ *be one to one,* S *a cohyponormal operator and let* X *be a one to one operator with dense range such that* $TX = XS$. *Then* T *and* S *are unitarily equivalent normal operators.*

Proof. Suppose S is not normal. Then by Theorem 1 of [7] there exists a non-zero vector $x \in H$ and a bounded function $f : \mathbb{C} \to H$ such

https://doi.org/10.1017/S0004972700007437 Published online by Cambridge University Press

that $(S-\lambda I)f(\lambda) \equiv x$. Then it follows that $Xf : \mathbb{C} \to H$ is a bounded function such that $(T-\lambda I)Xf(\lambda) = Xx$. Let $Xf(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$ and $Xx = x_1 \oplus x_2$ be the decompositions of $Xf(\lambda)$ and Xx relative to the decomposition $H = \overline{R(T)} \oplus N(T^*)$. Then Theorem A gives

$$\left(T_1-\lambda I\right)f_1(\lambda) + T_2f_2(\lambda) = x_1$$

and

$$-\lambda f_2(\lambda) = x_2$$

for all $\lambda \in \mathbb{C}$.

In particular, if $\lambda = 0$ then $x_2 = 0$. Therefore $f_2(\lambda) = 0$ if $\lambda \neq 0$ and x_1 is a non-zero vector. So $\left(T_1-\lambda I\right)f_1(\lambda) = x_1$ for all $\lambda \neq 0$. Now Theorem 1 and Proposition 2 of [8] imply that

$$X_{T_1}(0) = \left\{x \in H : \text{there exists an analytic function } f_x : \mathbb{C}\backslash\{0\} \to H \text{ such that } \left(T_1-\lambda I\right)f_x(\lambda) = x\right\}$$

is a closed invariant subspace of T_1 containing the non-zero vector x_1 and $\sigma\left(T_1 | X_{T_1}(0)\right) = \{0\}$. Since T_1 is hyponormal, $T_1 | X_{T_1}(0) = 0$. So $Tx_1 = T_1x_1 = 0$. But T is one to one and therefore $x_1 = 0$, a contradiction. Hence S must be normal and the result follows from Theorem 1. //

If T and T^* both are hyponormal then T is normal. For $T \in Q(k)$, we have the following:

THEOREM 3. *If* $T \in Q(k)$ *is cohyponormal then* T *is normal.*

Proof. Since T is cohyponormal, $N(T^*) = N\left(T^{*k}\right)$ reduces T . Therefore $T = T_1 \oplus T_3$, where $T_1 = T|\overline{R\left(T^k\right)}$ and T_1^* both are hyponormal. Also T_3^* is nilpotent and hyponormal. Hence $T_3 = 0$ and T is normal.//

Let Θ be the class of all operators T on H for which T^*T and $T + T^*$ commute. In [2] Campbell proved the following:

THEOREM C. *If* $T \in \Theta$ *and* T^* *is hyponormal then* T *is normal.*

https://doi.org/10.1017/S0004972700007437 Published online by Cambridge University Press

THEOREM D. *If* $T \in \Theta$ *is hyponormal then* T *is subnormal.*

Theorem C remains valid even if $T^* \in Q(1)$.

THEOREM 4. *If* $T^* \in Q(1)$ *and* $T \in \Theta$ *then* T *is normal.*

Proof. Let $R = T^*|\overline{R(T^*)}$. Then R is hyponormal. Also since $T \in \Theta$, $N(T)$ is reducing [3] and therefore Theorem A implies that $T^* = R \oplus 0$ is hyponormal. Hence by Theorem C, T is normal. //

Question. If $T \in \Theta \cap Q(1)$, must it be subnormal?

THEOREM 5. *If* $A, B \in Q(k)$ *are quasisimilar then they have equal spectra.*

Proof. Suppose X and Y are one to one operators on H with dense range such that $XA = BX$ and $YB = AY$. Let $A_1 = A|\overline{R(A^k)}$ and $B_1 = B|\overline{R(B^k)}$. Then

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$$

where A_1, B_1 are hyponormal and A_3, B_3 are nilpotents. Therefore $\sigma(A) = \sigma(A_1) \cup \{0\}$ and $\sigma(B) = \sigma(B_1) \cup \{0\}$. In view of the fact that quasisimilar hyponormal operators have equal spectra [5], it suffices to show that A_1 and B_1 are quasisimilar.

Since $XA^k = B^kX$ and $YB^k = A^kY$, it follows that the restrictions $X : \overline{R(A^k)} \to \overline{R(B^k)}$ and $Y : \overline{R(B^k)} \to \overline{R(A^k)}$ are one to one and have dense range. Now for any $x \in \overline{R(A^k)}$, $XA_1x = XAx = BXx = B_1Xx$ and similarly for any $y \in \overline{R(B^k)}$, $YB_1y = A_1Yy$. Thus A_1 and B_1 are quasisimilar. Hence the result. //

In [9], Sheth proved that if T is hyponormal and $S^{-1}TS = T^*$ where $0 \notin \overline{W(S)}$ then T is self-adjoint. We prove the following:

THEOREM 6. *If* $T \in Q(k)$ *is such that* $S^{-1}TS = T^*$ *where* $0 \notin \overline{W(S)}$ *then* T *is direct sum of a self-adjoint operator and a nilpotent operator.*

https://doi.org/10.1017/S0004972700007437 Published online by Cambridge University Press

Proof. Write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

as usual. Then $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, by Theorem 1 of [12], $\sigma(T)$ and hence $\sigma(T_1)$ is real. Thus T_1 is self-adjoint. But then $T_2 = 0$, and we are done. //

Recall that an operator T is reductive if every invariant subspace of T is reducing. Every reductive operator similar to a normal operator is normal [6, Lemma 2.4]. The following shows that in this result similarity condition can be weakened to quasisimilarity.

THEOREM 7. *If T is reductive and quasisimilar to a normal operator then T is normal.*

Proof. Since T is reductive and quasisimilar to a normal operator by a result of Apostol [1] there exists a basic system $\{X_n\}$ of reducing subspaces such that each $S_n = T|X_n$ is reductive and similar to a normal operator and therefore S_n itself is normal for each n . Since $\bigvee_{n=1}^{\infty} X_n = H$, for each $x \in H$, we have $x = \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} x_{mn} \right)$, where $x_{mn} \in X_n$ and for each m , $x_{mn} = 0$ for all but finitely many n's . Therefore for each $x \in H$,

$$\begin{aligned} TT^*x &= \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} TT^*x_{mn} \right) \\ &= \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} S_n S_n^* x_{mn} \right) \\ &= \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} S_n^* S_n x_{mn} \right) \\ &= \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} T^*Tx_{mn} \right) \\ &= T^*Tx . \end{aligned}$$

Thus T is normal. //

https://doi.org/10.1017/S0004972700007437 Published online by Cambridge University Press

References

[1] Constantin Apostol, "Operators quasisimilar to a normal operator", *Proc. Amer. Math. Soc.* **53** (1975), 104-106.

[2] Stephen L. Campbell, "Linear operators for which T^*T and $T + T^*$ commute III", *Pacific J. Math.* **76** (1978), 17-19.

[3] Stephen L. Campbell, "Operator-valued inner functions analytic on the closed disc II", *Pacific J. Math.* **60** (1975), no. 2, 37-49.

[4] S.L. Campbell and B.C. Gupta, "On *k*-quasihyponormal operators", *Math. Japon.* **23** (1978), 185-189.

[5] Stuart Clary, "Equality of spectra of quasi-similar hyponormal operators", *Proc. Amer. Math. Soc.* **53** (1975), 88-90.

[6] Eric Nordgren, Heydar Radjavi, and Peter Rosenthal, "On operators with reducing invariant subspaces", *Amer. J. Math.* **97** (1975), 559-570.

[7] C.R. Putnam, "Hyponormal contractions and strong power convergence", *Pacific J. Math.* **57** (1975), 531-538.

[8] M. Radjabalipour, "Ranges of hyponormal operators", *Illinois J. Math.* **21** (1977), 70-75.

[9] I.H. Sheth, "On hyponormal operators", *Proc. Amer. Math. Soc.* **17** (1966), 998-1000.

[10] I.H. Sheth, "Quasi-hyponormal operators", *Rev. Roumaine Math. Pures Appl.* **19** (1976), 1049-1053.

[11] Joseph G. Stampfli & Bhushan L. Wadhwa, "An asymmetric Putnam-Fugledge theorem for dominant operators", *Indiana Univ. Math. J.* **25** (1976), 359-365.

[12] James P. Williams, "Operators similar to their adjoints", *Proc. Amer. Math. Soc.* **20** (1969), 121-123.

Department of Mathematics,
Sadar Patel University,
Vallabh Vidyanagar 388 120,
Gujarat,
India.

Department of Mathematics,
Saurashtra University,
Rajkot 360 005,
Gujarat,
India.